Haar Wavelets: Symbolic Computation

Symbolic computation of wavelets is of paramount importance and effectiveness for teaching the Wavelet theory for both programming purposes as well as more theoretical mathematical treatments.

Visualization of the symbolic equations accompanied by the graphical plots empowers the novice reader to acquire speedy expertise and relevant insights into the theory.

ϕ: Scaling Function, Father Wavelet

Let’s define a simple box function:

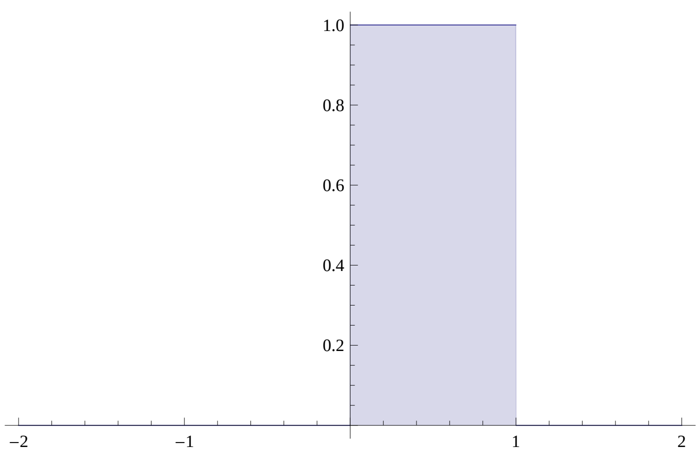
```
BoxFunction[x0_] := Module[{x = x0},
  Boole[0 ≤ x && x < 1]
 ]
```

Visualize the symbolic form:

$$\text{BoxFunction}(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Visualize the plot:

```
Plot[BoxFunction[x], {x, -2, 2}, PlotRange → All, Filling → Axis]
```



Symbol ϕ (Phi) represents the above function which in the Wavelet theory is coined as Scaling Function:

$$\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

```
ϕHaar[x0_] := Module[{x = x0},
  Boole[0 ≤ x && x < 1]
 ]
```

For scaling purposes to finer and finer details let’s define a variation of ϕ :

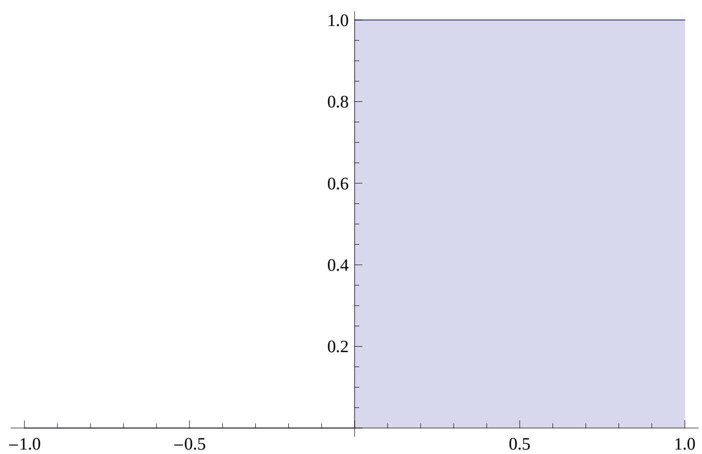
$$\phi_{J,k}(x) = 2^{J/2}\,\phi\left(2^J x - k\right)$$

```
ϕ[x_, J_, k_] := 2^(J/2) ϕHaar[(2^J x) - k]
```

Now let’s keep k = 0 and vary J i.e. scaling.

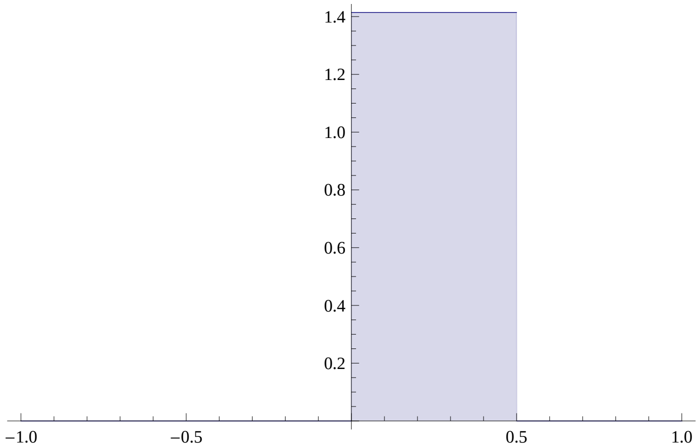
$\phi_{0,0}$ which is the BoxFunction

```
Plot[ϕ[x, 0, 0], {x, -1, 1}, Filling → Axis]
```



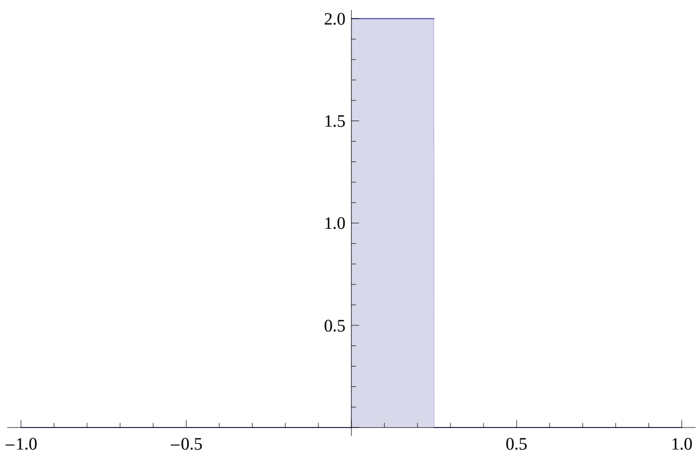
$\phi_{1,0}$

```
Plot[φ[x, 1, 0], {x, -1, 1}, Filling → Axis]
```



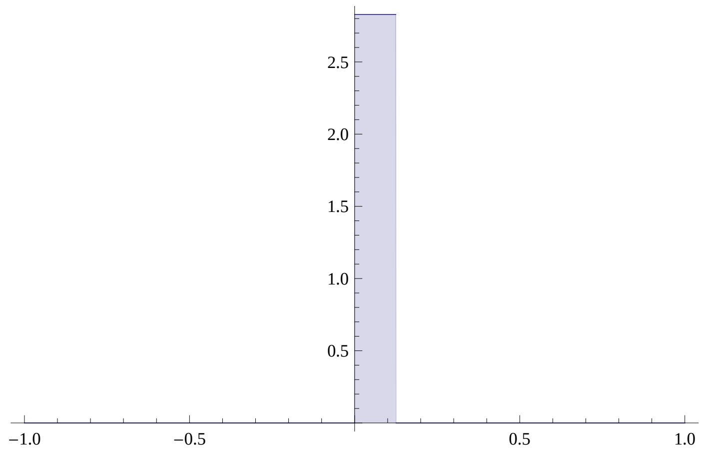
$\phi_{2,0}$

```
Plot[φ[x, 2, 0], {x, -1, 1}, Filling → Axis]
```



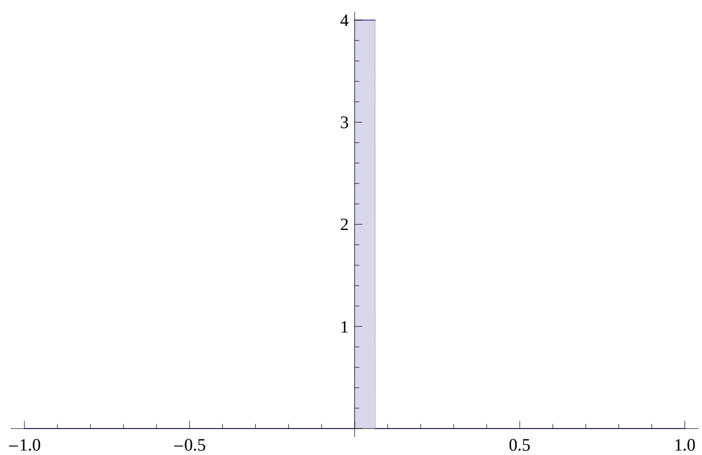
$\phi_{3,0}$

```
Plot[ϕ[x, 3, 0], {x, -1, 1}, Filling → Axis]
```



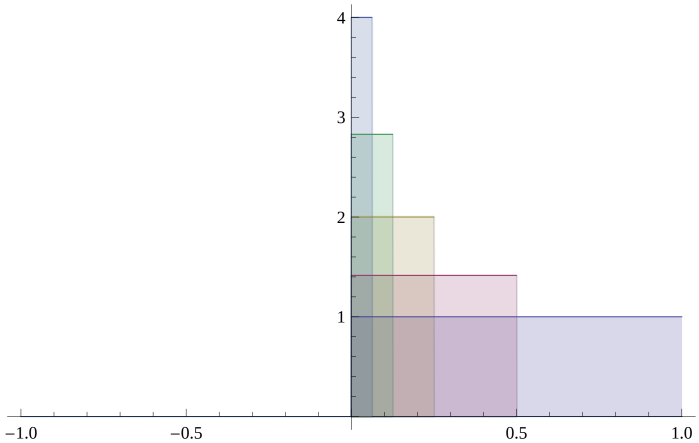
$\phi_{4,0}$

```
Plot[ϕ[x, 4, 0], {x, -1, 1}, Filling → Axis]
```



All together:

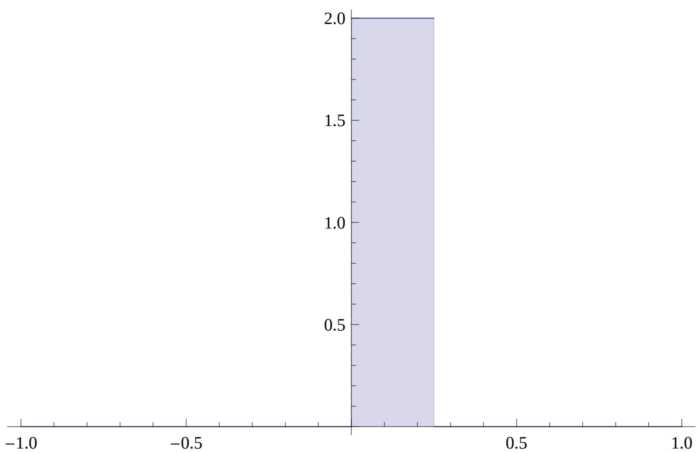
```
Plot[Evaluate[Table[ϕ[x, j, 0], {j, 0, 4}]],
 {x, -1, 1}, Filling → Axis, PlotRange → All]
```



Let’s keep J = 2 and vary k i.e. translate to righ:

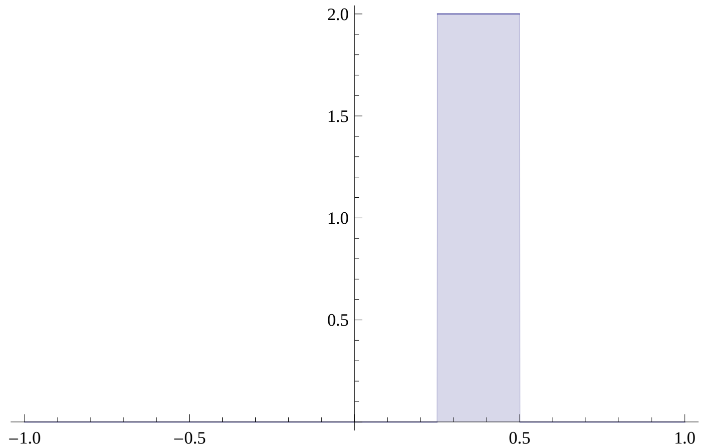
$\phi_{2,0}$

```
Plot[ϕ[x, 2, 2^0 - 1], {x, -1, 1}, Filling → Axis]
```



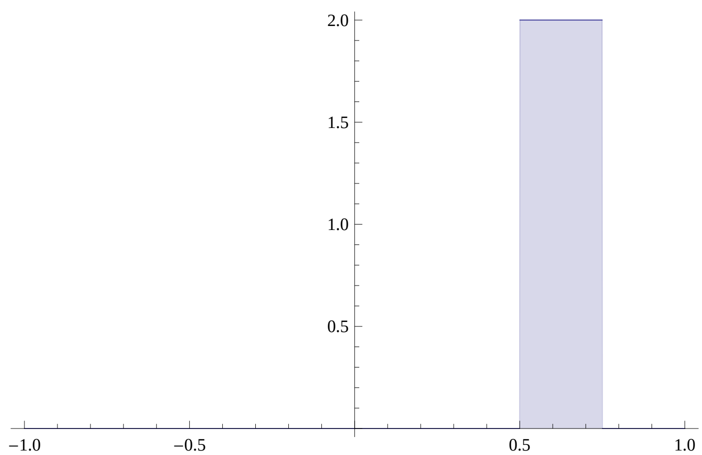
$\phi_{2,1}$

```
Plot[ϕ[x, 2, 1], {x, -1, 1}, Filling → Axis]
```



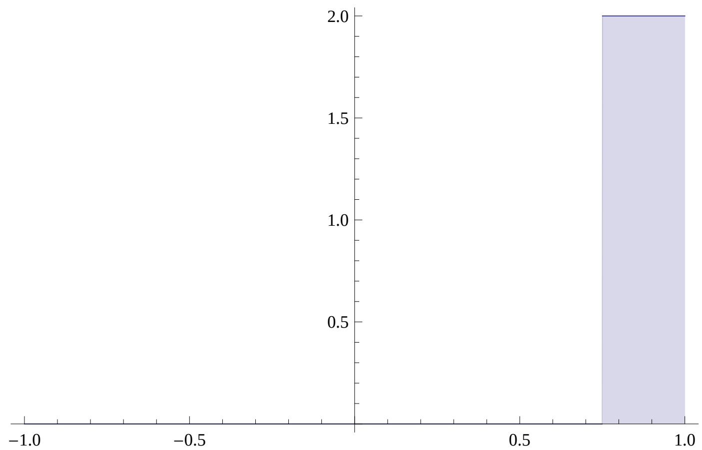
$\phi_{2,2}$

```
Plot[ϕ[x, 2, 2], {x, -1, 1}, Filling → Axis]
```



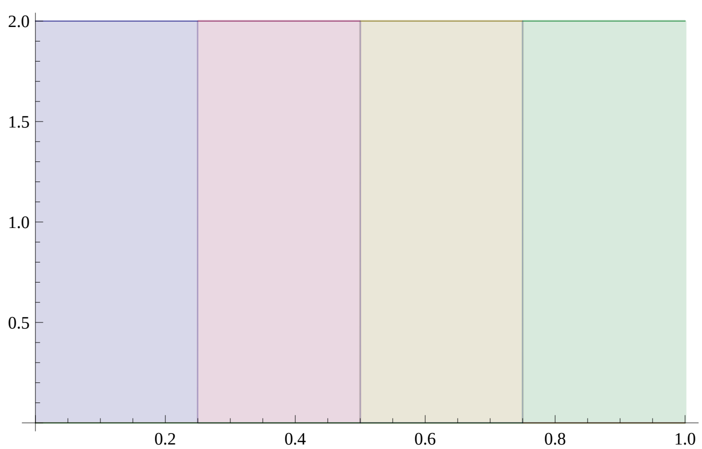
$\phi_{2,3}$

```
Plot[ϕ[x, 2, 3], {x, -1, 1}, Filling → Axis]
```



All together:

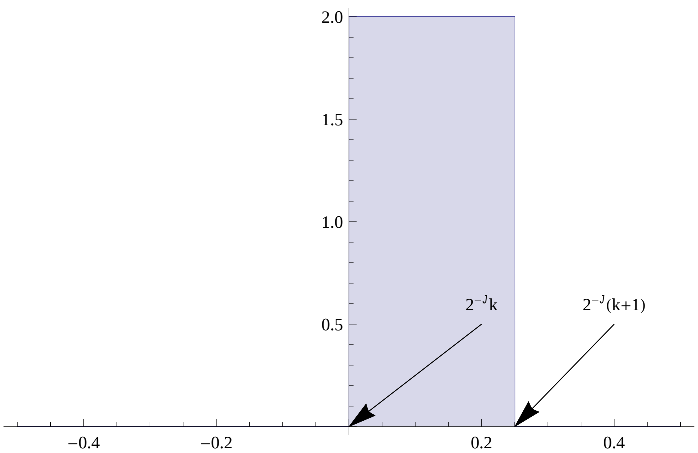
```
Plot[Evaluate[Table[ϕ[x, 2, k], {k, 0, 2^2 - 1}]], {x, 0, 1}, Filling → Axis]
```



Let’s study the support of the function $\phi_{J,k}$ (where it is nonzero), first plot for J = 2, k = 0 and the second plot for J = 3, k = 1:

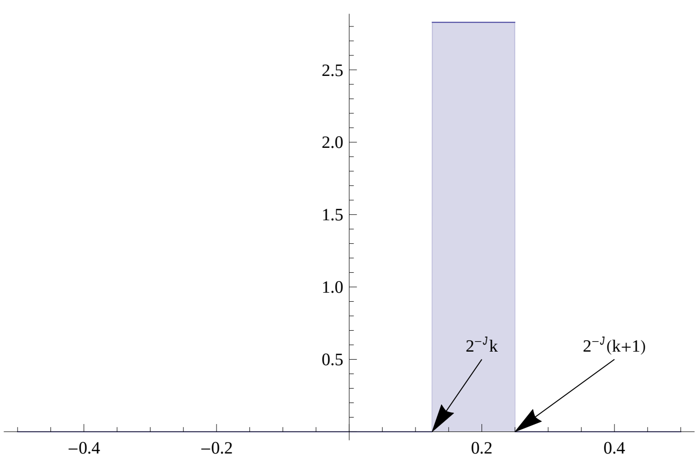
```
g1 = Plot[ϕ[x, 2, 0], {x, -0.5, 0.5}, Filling → Axis];
J = 2; k = 0;
g2 = Graphics[
   {Arrow[{{0.4, 0.5}, {2^-J (k + 1), 0}}], Text["2^-J (k+1)", {0.4, 0.5 + 0.1}]} ];
g3 = Graphics[{Arrow[{{0.2, 0.5}, {2^-J (k + 0), 0}}],
    Text["2^-J k", {0.2, 0.5 + 0.1}]} ];
```

```
Show[g1, g2, g3]
```




```
g1 = Plot[ϕ[x, 3, 1], {x, -0.5, 0.5}, Filling → Axis];
J = 3; k = 1;
g2 = Graphics[
   {Arrow[{{0.4, 0.5}, {2^-J (k + 1), 0}}], Text["2^-J (k+1)", {0.4, 0.5 + 0.1}]} ];
g3 = Graphics[{Arrow[{{0.2, 0.5}, {2^-J (k + 0), 0}}],
    Text["2^-J k", {0.2, 0.5 + 0.1}]} ];
```

```
Show[g1, g2, g3]
```



Therefore the base of the $\phi_{J,k}$ can be characterized as follows:

$$\phi_{J,k}(x) = \begin{cases} 2^{J/2} & 2^{-J}\,k \le x \le 2^{-J}\,(k+1) \\ 0 & \text{otherwise} \end{cases}$$

The Inner Product & Reconstruction

The following integral equation forms an inner poduct over a given Real function space:

$$c_{J,k} = \int_0^1 f(x)\,\phi_{J,k}(x)\,dx = \langle f, \phi_{J,k} \rangle$$

If we treat the $\phi_{J,k}$ as basis for a functional vector space the coefficients $\{c_{J,k}\}_{k=0}^{2^J-1}$ can linearly sum up a new function f_J:

$$f_J(x) = \sum_{k=0}^{2^J-1} c_{J,k}\,\phi_{J,k}(x)$$

Example 1

```
(* Make up a new function *)
f[t_] := Sin[8 * t];
```

```
(* Let's look at J = 4 and k = 0 *)
J = 4;
k = 0;

Integrate[f[x] * ϕ[x, J, k], {x, 0, 1}]
```

$$\text{Sin}\left[\frac{1}{4}\right]^2$$

```
(* Let's look at J = 4 and k = 1 *)
J = 4;
k = 1;

Integrate[f[x] * ϕ[x, J, k], {x, 0, 1}]
```

$$\frac{1}{2}\left(\text{Cos}\left[\frac{1}{2}\right] - \text{Cos}[1]\right)$$

Let’s calculate the sum $\sum_{k=0}^{2^J-1} c_{J,k}\,\phi_{J,k}(x)$:

```
(* Make a list of all ϕJ, k(x) functions,
however make sure the variable is changed from x to t to avoid confusion *)
list = Table[ϕ[t, J, k] * Integrate[f[x] * ϕ[x, J, k], {x, 0, 1}], {k, 0, 2^J - 1}]
```

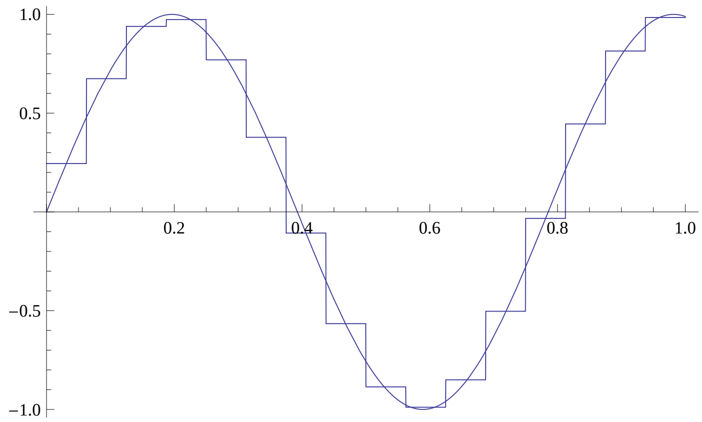
$$\left\{4\,\text{Boole}[0 \le 16\,t \,\&\&\, 16\,t < 1]\,\text{Sin}\left[\frac{1}{4}\right]^2,\right.$$
$$2\,\text{Boole}[0 \le -1 + 16\,t \,\&\&\, -1 + 16\,t < 1]\left(\text{Cos}\left[\frac{1}{2}\right] - \text{Cos}[1]\right),$$
$$2\,\text{Boole}[0 \le -2 + 16\,t \,\&\&\, -2 + 16\,t < 1]\left(\text{Cos}[1] - \text{Cos}\left[\frac{3}{2}\right]\right),$$
$$2\,\text{Boole}[0 \le -3 + 16\,t \,\&\&\, -3 + 16\,t < 1]\left(\text{Cos}\left[\frac{3}{2}\right] - \text{Cos}[2]\right),$$
$$2\,\text{Boole}[0 \le -4 + 16\,t \,\&\&\, -4 + 16\,t < 1]\left(\text{Cos}[2] - \text{Cos}\left[\frac{5}{2}\right]\right),$$
$$2\,\text{Boole}[0 \le -5 + 16\,t \,\&\&\, -5 + 16\,t < 1]\left(\text{Cos}\left[\frac{5}{2}\right] - \text{Cos}[3]\right),$$
$$2\,\text{Boole}[0 \le -6 + 16\,t \,\&\&\, -6 + 16\,t < 1]\left(\text{Cos}[3] - \text{Cos}\left[\frac{7}{2}\right]\right),$$
$$2\,\text{Boole}[0 \le -7 + 16\,t \,\&\&\, -7 + 16\,t < 1]\left(\text{Cos}\left[\frac{7}{2}\right] - \text{Cos}[4]\right),$$
$$2\,\text{Boole}[0 \le -8 + 16\,t \,\&\&\, -8 + 16\,t < 1]\left(\text{Cos}[4] - \text{Cos}\left[\frac{9}{2}\right]\right),$$
$$2\,\text{Boole}[0 \le -9 + 16\,t \,\&\&\, -9 + 16\,t < 1]\left(\text{Cos}\left[\frac{9}{2}\right] - \text{Cos}[5]\right),$$
$$2\,\text{Boole}[0 \le -10 + 16\,t \,\&\&\, -10 + 16\,t < 1]\left(\text{Cos}[5] - \text{Cos}\left[\frac{11}{2}\right]\right),$$
$$2\,\text{Boole}[0 \le -11 + 16\,t \,\&\&\, -11 + 16\,t < 1]\left(\text{Cos}\left[\frac{11}{2}\right] - \text{Cos}[6]\right),$$
$$2\,\text{Boole}[0 \le -12 + 16\,t \,\&\&\, -12 + 16\,t < 1]\left(\text{Cos}[6] - \text{Cos}\left[\frac{13}{2}\right]\right),$$
$$2\,\text{Boole}[0 \le -13 + 16\,t \,\&\&\, -13 + 16\,t < 1]\left(\text{Cos}\left[\frac{13}{2}\right] - \text{Cos}[7]\right),$$
$$2\,\text{Boole}[0 \le -14 + 16\,t \,\&\&\, -14 + 16\,t < 1]\left(\text{Cos}[7] - \text{Cos}\left[\frac{15}{2}\right]\right),$$
$$\left.2\,\text{Boole}[0 \le -15 + 16\,t \,\&\&\, -15 + 16\,t < 1]\left(\text{Cos}\left[\frac{15}{2}\right] - \text{Cos}[8]\right)\right\}$$

```
(* Add all the above functions into a sum, basically the integral sum *)
fJ = Plus @@ list
```

$$
\begin{aligned}
&2\,\text{Boole}[0 \le -1 + 16\,t \,\&\&\, -1 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{1}{2}\right] - \text{Cos}[1]\right) + \\
&2\,\text{Boole}[0 \le -2 + 16\,t \,\&\&\, -2 + 16\,t < 1]\left(\text{Cos}[1] - \text{Cos}\left[\tfrac{3}{2}\right]\right) + \\
&2\,\text{Boole}[0 \le -3 + 16\,t \,\&\&\, -3 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{3}{2}\right] - \text{Cos}[2]\right) + \\
&2\,\text{Boole}[0 \le -4 + 16\,t \,\&\&\, -4 + 16\,t < 1]\left(\text{Cos}[2] - \text{Cos}\left[\tfrac{5}{2}\right]\right) + \\
&2\,\text{Boole}[0 \le -5 + 16\,t \,\&\&\, -5 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{5}{2}\right] - \text{Cos}[3]\right) + \\
&2\,\text{Boole}[0 \le -6 + 16\,t \,\&\&\, -6 + 16\,t < 1]\left(\text{Cos}[3] - \text{Cos}\left[\tfrac{7}{2}\right]\right) + \\
&2\,\text{Boole}[0 \le -7 + 16\,t \,\&\&\, -7 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{7}{2}\right] - \text{Cos}[4]\right) + \\
&2\,\text{Boole}[0 \le -8 + 16\,t \,\&\&\, -8 + 16\,t < 1]\left(\text{Cos}[4] - \text{Cos}\left[\tfrac{9}{2}\right]\right) + \\
&2\,\text{Boole}[0 \le -9 + 16\,t \,\&\&\, -9 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{9}{2}\right] - \text{Cos}[5]\right) + \\
&2\,\text{Boole}[0 \le -10 + 16\,t \,\&\&\, -10 + 16\,t < 1]\left(\text{Cos}[5] - \text{Cos}\left[\tfrac{11}{2}\right]\right) + \\
&2\,\text{Boole}[0 \le -11 + 16\,t \,\&\&\, -11 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{11}{2}\right] - \text{Cos}[6]\right) + \\
&2\,\text{Boole}[0 \le -12 + 16\,t \,\&\&\, -12 + 16\,t < 1]\left(\text{Cos}[6] - \text{Cos}\left[\tfrac{13}{2}\right]\right) + \\
&2\,\text{Boole}[0 \le -13 + 16\,t \,\&\&\, -13 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{13}{2}\right] - \text{Cos}[7]\right) + \\
&2\,\text{Boole}[0 \le -14 + 16\,t \,\&\&\, -14 + 16\,t < 1]\left(\text{Cos}[7] - \text{Cos}\left[\tfrac{15}{2}\right]\right) + \\
&2\,\text{Boole}[0 \le -15 + 16\,t \,\&\&\, -15 + 16\,t < 1]\left(\text{Cos}\left[\tfrac{15}{2}\right] - \text{Cos}[8]\right) + \\
&4\,\text{Boole}[0 \le 16\,t \,\&\&\, 16\,t < 1]\,\text{Sin}\left[\tfrac{1}{4}\right]^2
\end{aligned}
$$

```
(* Plot the original f which was the Sin with the fJ *)
g1 = Plot[fJ /. {t → x}, {x, 0, 1}] ;
g2 = Plot[f[x], {x, 0, 1}];
```

```
Show[g1, g2, PlotRange → All]
```



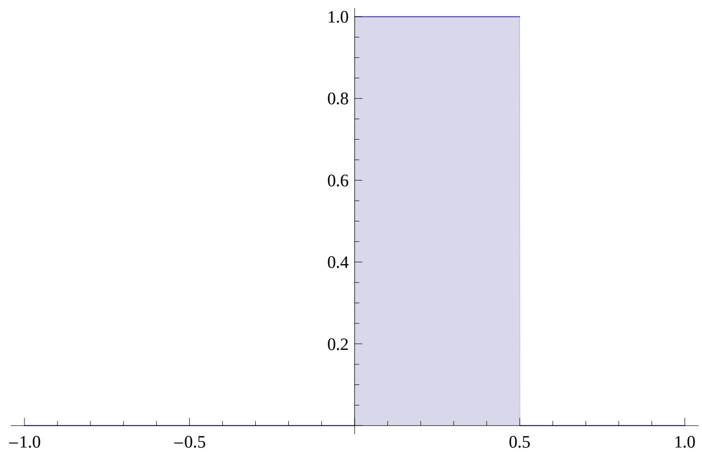
Oh! Wow! Therefore the $f_J(x)$ is some sort of reconstructionfor f based upon the linear vector sum of the form $\sum_{k=0}^{2^J-1} c_{J,k}\,\phi_{J,k}(x)$.

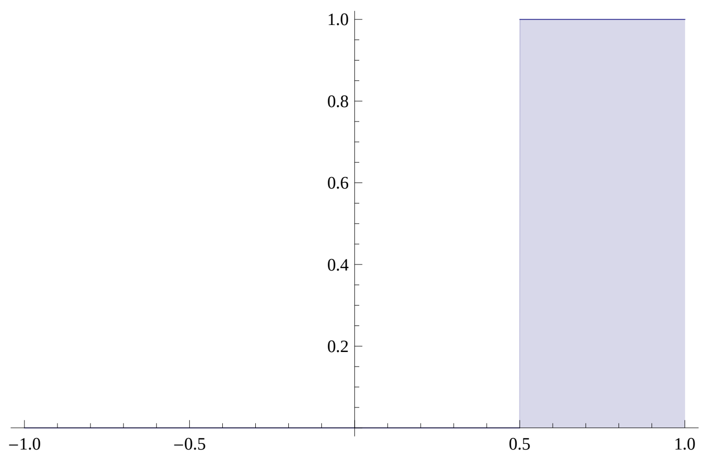
Scaling Equation Or Dilation Equation

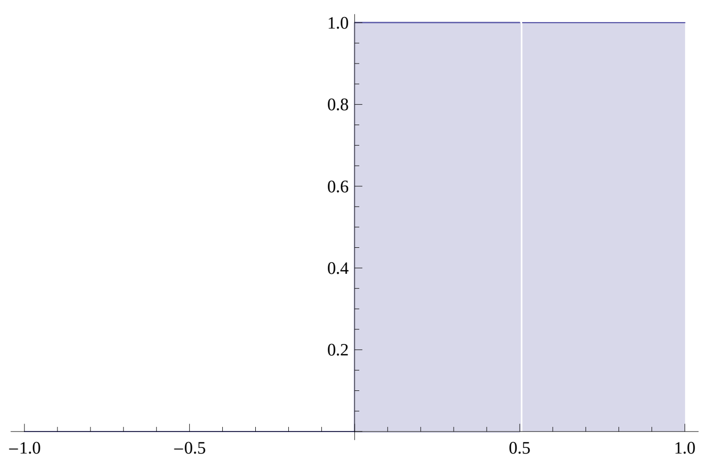
Special case for Haar Wavelets:

$\phi\mathrm{Haar}(x) = \phi\mathrm{Haar}(2\,x) + \phi\mathrm{Haar}(2\,x - 1)$

```
g1 = Plot[ϕHaar[2 x] , {x, -1, 1}, Filling → Axis]
g2 = Plot[ϕHaar[2 x - 1], {x, -1, 1}, Filling → Axis]
g3 = Plot[ϕHaar[2 x] + ϕHaar[2 x - 1], {x, -1, 1}, Filling → Axis]
```





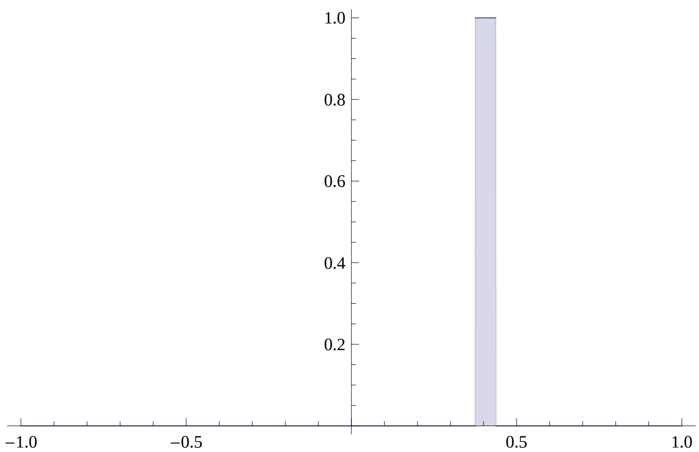


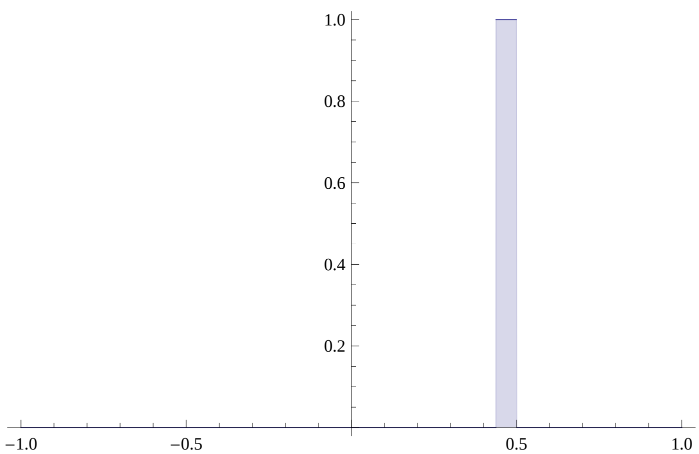
General form of Scaling/Dilation equation for Haar Wavelets:

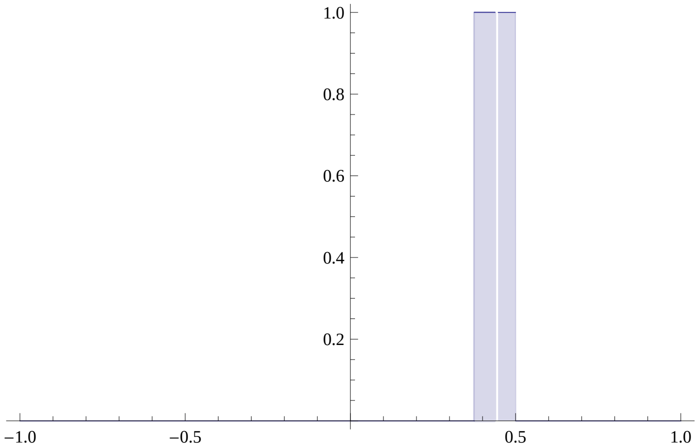
$$\phi\text{Haar}\left(2^{J-1}\,x-k\right)=\ \phi\text{Haar}\left(2^{J}\,x-2\,k\right)+\phi\text{Haar}\left(2^{J}\,x-2\,k-1\right)$$

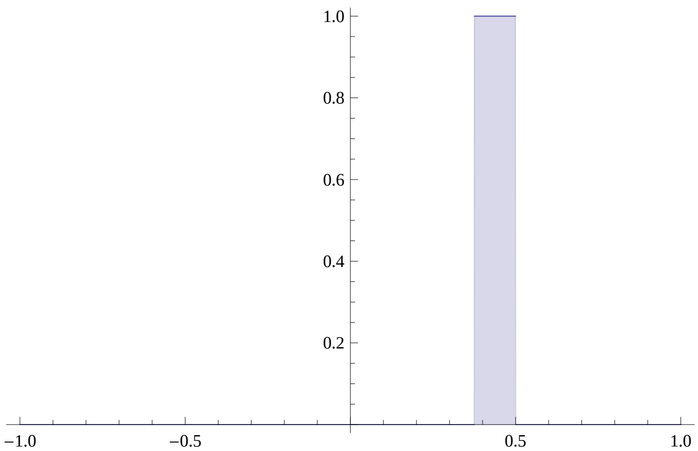
```
J = 4;
k = 3;
```

```
g1 = Plot[ϕHaar[2^J x - 2 k] , {x, -1, 1}, Filling → Axis]
g2 = Plot[ϕHaar[2^J x - 2 k - 1], {x, -1, 1}, Filling → Axis]
g3 = Plot[ϕHaar[2^J x - 2 k] + ϕHaar[2^J x - 2 k - 1], {x, -1, 1}, Filling → Axis]
g4 = Plot[ϕHaar[2^(J-1) x - k], {x, -1, 1}, Filling → Axis]
```









ψ: Wavelet Function, Mother Wavelet

Let’s calculate *f0* and *f1* for an aribitrary function:

```
(* Make up a new function *)
f[t_] := Sin[8 * t];

(* Let's look at J = 0 and k = 0 *)
J = 0;
k = 0;

c0,0 = Integrate[f[x] * ϕ[x, J, k], {x, 0, 1}]
```

$$\frac{\text{Sin}[4]^2}{4}$$

```
(* Let's look at J = 1 and k = 0 *)
J = 1;
k = 0;

c1,0 = Integrate[f[x] * ϕ[x, J, k], {x, 0, 1}]
```

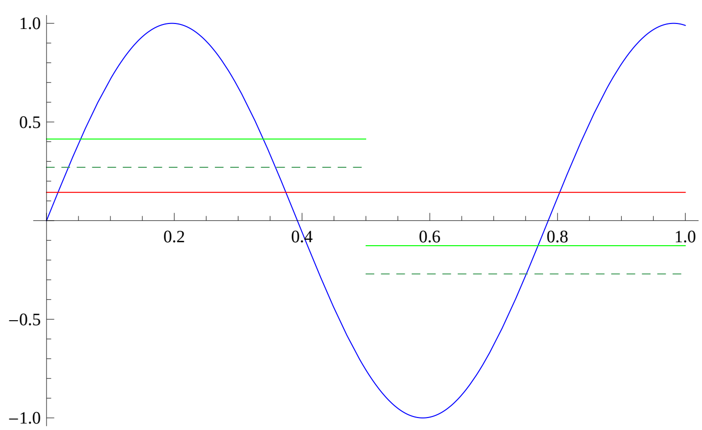
$$\frac{\text{Sin}[2]^2}{2\sqrt{2}}$$

```
(* Let's look at J = 1 and k = 1 *)
J = 1;
k = 1;

c1,1 = Integrate[f[x] * ϕ[x, J, k], {x, 0, 1}]
```

$$\frac{\text{Cos}[4] - \text{Cos}[8]}{4\sqrt{2}}$$

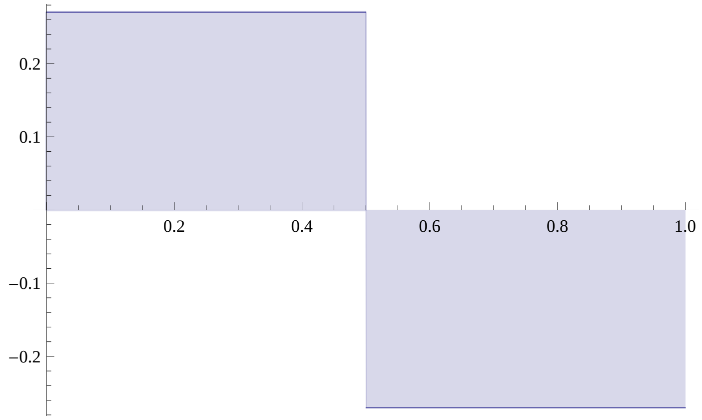
From above construct *f0* and *f1* as the linear sum of the vectors $\phi_{J,k}(x)$:

```
f0 [x_] := c0,0 * ϕHaar[x]

f1[x_] := c1,0 * ϕ[x, 1, 0] + c1,1 * ϕ[x, 1, 1]

Plot[{f[x], f0[x], f1[x], f1[x] - f0[x]} ,
 {x, 0, 1}, PlotStyle → {Blue, Red, Green, Dashed}]
```




```
df[x_] := ((-c1,1 + c1,0) ϕHaar[2 x] - (c1,0 - c1,1) ϕHaar[2 x - 1]) / √2
```

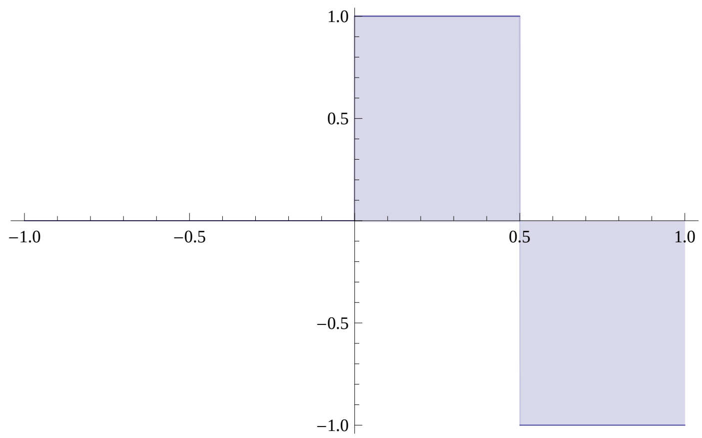
```
Plot[df[x], {x, 0, 1}, Filling → Axis]
```




```
d0,0 = (-c1,1 + c1,0) / √2
```

$$\frac{-\frac{\mathrm{Cos}[4]-\mathrm{Cos}[8]}{4\sqrt{2}}+\frac{\mathrm{Sin}[2]^2}{2\sqrt{2}}}{\sqrt{2}}$$

```
ψ[x_] := ϕHaar[2 x] - ϕHaar[2 x - 1]
Plot[ψ[x], {x, -1, 1}, Filling → Axis]
```

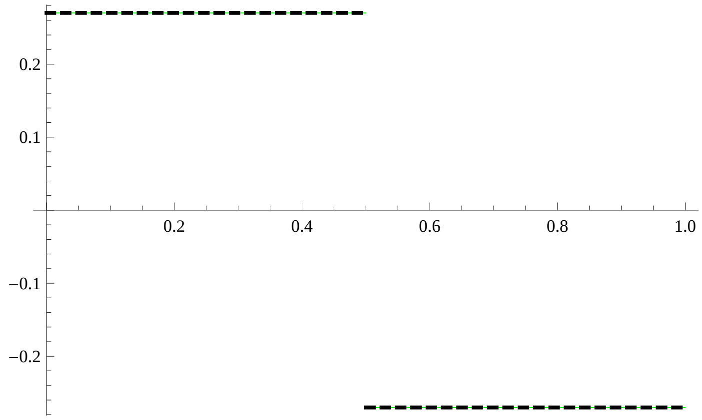


Therefore:

$f_1(x) - f_0(x) = d_{0,0}\,\psi(x)$

As plot below renders the equality:

```
Plot[{d0,0 * ψ[x], f1[x] - f0[x]},
 {x, 0, 1}, PlotStyle → {Green, {Black, Thick, Dashed}}]
```



Or

$f_1(x) = c_{0,0}\,\phi(x) + d_{0,0}\,\psi(x)$

Let’s plot and see:

```
Plot[{c0,0 * ϕHaar[x] + d0,0 * ψ[x], f1[x]},
 {x, 0, 1}, PlotStyle → {Green, {Black, Thick, Dashed}}]
```