

These are the *Mathematica* solutions for the paper:

“On Pentagon, Ten-Term, and Tetrahedron Relations” by R. M. Kashaev S. M. Sergeev2

computations by Dara O Shayda

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<< **Notation`**

This formulation is incomprehensible and I guessed how it works:

1. Introduction

The Yang–Baxter equation (YBE) [25, 3] can be considered as a tool for both constructing and solving integrable two-dimensional models of statistical mechanics and quantum field theory [2, 9]. Recent progress in understanding the algebraic structure, lying behind the YBE, has led to the theory of quasi-triangular Hopf algebras [7].

The tetrahedron (or three-simplex) equation (TE) [26] has been introduced as a three-dimensional generalization of the YBE. Before describing it in the algebraic form, first consider an associative unital algebra A , and define an important notation to be used throughout the paper. Namely, for each set of distinct integers $\{i_1, i_2, \dots, i_m\} \subset \{1, 2, \dots, n\}$, for $m < n$, define the algebra homomorphism

$$\tau_{i_1, i_2, \dots, i_m} : A^{\otimes m} \rightarrow A^{\otimes n}$$

such that

$$a \otimes b \otimes \dots \otimes c \mapsto 1 \otimes \dots \otimes a \otimes \dots \otimes b \otimes \dots \otimes c \otimes \dots \otimes 1,$$

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where a, b, \dots, c in the r.h.s. stand on $i_1, i_2, \dots, i_m^{\text{th}}$ positions, respectively, and unit elements, on the others. The notation to be used follows:

$$u_{i_1 i_2 \dots i_m} := \tau_{i_1, i_2, \dots, i_m}(u), \quad u \in A^{\otimes m}, \quad (1.1)$$

i.e. the subscripts indicate the way an element of the algebra $A^{\otimes m}$ is interpreted as an element of the algebra $A^{\otimes n}$.

Next, we shall find it convenient to use the “permutation operator” P , which is an (additional) element in $A^{\otimes 2}$, defined by

$$Pa \otimes b = b \otimes aP, \quad P^2 = 1 \otimes 1, \quad a, b \in A. \quad (1.2)$$

The (constant) TE is a nonlinear relation in $A^{\otimes 6}$ on an invertible element $R \in A^{\otimes 3}$:

```

Notation[  $R_{i,j}$   $\Rightarrow$  makeA[A, {i_, j_}, 3] ]
Notation[  $R_{i,j,k}$   $\Rightarrow$  makeA[A, {i_, j_, k_}, 6] ]
Notation[  $R_{i,j,k,h}$   $\Rightarrow$  makeA[A, {i_, j_, k_, h_}, 10] ]

makeMatrix[dim_, a_] := Array[Subscript[a, #1, #2] &, dim]

(* makeA generates the A matrices in Hietarinta paper *)

makeA[A_, list_, n_] := Module[{res, i, j, k},

  res = ConstantArray[-1, {n, n}];

  Table[If[MemberQ[list, i] != True, res[[i]][[j]] = KroneckerDelta[i, j];
    res[[j]][[i]] = KroneckerDelta[j, i]], {i, 1, n}, {j, 1, n}];

  Table[k = 1; Table[
    If[res[[list[[i]]]][[j]] == -1, res[[list[[i]]]][[j]] = A[[i]][[k]]; k = k + 1];,
    {j, 1, n}], {i, 1, Length[list]}];
  res

]

```

4-Simplex: Kashaev, Sergeev

```

A = makeMatrix[{4, 4}, a];
A // MatrixForm

```

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

Make the R matrices:


```
R3,6,8,9 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1,1} & 0 & 0 & a_{1,2} & 0 & a_{1,3} & a_{1,4} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,1} & 0 & 0 & a_{2,2} & 0 & a_{2,3} & a_{2,4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_{3,1} & 0 & 0 & a_{3,2} & 0 & a_{3,3} & a_{3,4} & 0 \\ 0 & 0 & a_{4,1} & 0 & 0 & a_{4,2} & 0 & a_{4,3} & a_{4,4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Too large set of equations of 3rd degree to printout:

```
(R10,1,2,3·R10,4,5,6·R1,4,7,8·R2,5,7,9·R3,6,8,9 - R3,6,8,9·R2,5,7,9·R1,4,7,8·R10,4,5,6·R10,1,2,3) //
MatrixForm
```

$$\begin{pmatrix} -a_{1,1} a_{1,2} a_{2,4} + a_{1,1} a_{2,1} a_{2,4} \\ -a_{1,3} a_{2,1} a_{3,1} + a_{1,1} a_{2,1} a_{3,4} - a_{1,1} (a_{1,3} a_{2,4} a_{3,2} + a_{1,2} a_{3,4}) \\ -a_{1,4} a_{3,1} a_{4,1} - a_{2,1} (a_{1,3} a_{4,1} + a_{1,4} a_{3,1} a_{4,3}) + a_{1,1} a_{2,1} a_{4,4} - a_{1,1} (a_{1,4} a_{3,4} a_{4,2} + a_{2,4} (a_{1,3} a_{4,2} + \\ -a_{1,1} a_{2,2} a_{2,4} \\ -a_{2,1} a_{2,3} a_{3,1} - a_{1,1} (a_{2,3} a_{2,4} a_{3,2} + a_{2,2} a_{3,4}) \\ -a_{2,4} a_{3,1} a_{4,1} - a_{2,1} (a_{2,3} a_{4,1} + a_{2,4} a_{3,1} a_{4,3}) - a_{1,1} (a_{2,4} a_{3,4} a_{4,2} + a_{2,4} (a_{2,3} a_{4,2} + a_{2,4} a_{3,1} \\ a_{3,1} - a_{3,1}^2 - a_{2,1} a_{3,1} a_{3,3} - a_{1,1} (a_{2,4} a_{3,2} a_{3,3} + a_{3,2} a_{3,4}) \\ a_{4,1} - a_{3,1} a_{4,1} - a_{3,1} a_{3,4} a_{4,1} - a_{2,1} (a_{3,3} a_{4,1} + a_{3,1} a_{3,4} a_{4,3}) - a_{1,1} (a_{3,4}^2 a_{4,2} + a_{2,4} (a_{3,3} a_{4,2} + a_{3,1} \\ -a_{4,1}^2 - a_{3,1} a_{4,1} a_{4,4} - a_{2,1} (a_{4,1} a_{4,3} + a_{3,1} a_{4,3} a_{4,4}) - a_{1,1} (a_{4,2} a_{4,4} + a_{3,4} a_{4,2} a_{4,4} + a_{2,4} (a_{4,2} \\ a_{1,1}^2 - a_{1,1} a_{1,4} + a_{1,1} a_{1,4} a_{2,1} \end{pmatrix}$$

```
sol = Solve[
```

```
R10,1,2,3·R10,4,5,6·R1,4,7,8·R2,5,7,9·R3,6,8,9 - R3,6,8,9·R2,5,7,9·R1,4,7,8·R10,4,5,6·R10,1,2,3 ==
ConstantArray[0, {10, 10}], Flatten[A]];
```

```
sol2 = DeleteDuplicates[sol];
Length[sol2]
```

```
83
```

83 solutions, 75 non-invertible 4-Simplex solutions

$$\begin{aligned}
& \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{a_{3,3}} & 0 & 0 & 0 \\ 0 & 0 & a_{3,3} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{a_{3,3}} & 0 & 0 & 0 \\ 0 & 0 & a_{3,3} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & a_{2,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{a_{3,3}} & 1 & 0 & 0 \\ 0 & 0 & a_{3,3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{a_{3,3}} & 1 & 0 & 0 \\ 0 & 0 & a_{3,3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & a_{1,4} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & a_{1,4} \\ 1 & \frac{1}{a_{1,4}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & a_{1,4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 - a_{1,4} & 0 & a_{1,4} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a_{3,3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 - a_{3,2} & 0 & 0 \\ 0 & a_{3,2} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & a_{2,2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
& \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}
\end{aligned}$$

4-Simplex \mathbb{Z}_2 , 75 solutions: Kashaev, Sergeev

```
sol = Solve[
  R10,1,2,3.R10,4,5,6.R1,4,7,8.R2,5,7,9.R3,6,8,9 - R3,6,8,9.R2,5,7,9.R1,4,7,8.R10,4,5,6.R10,1,2,3 ==
  ConstantArray[0, {10, 10}], Flatten[A], Modulus -> 2];
```

```
sol2 = DeleteDuplicates[sol];
Length[sol2]
```

75

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \left. \begin{matrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

4-Simplex \mathbb{Z}_3 , 236 solutions :

Kashaev, Sergeev

```
sol = Solve[
  R10,1,2,3.R10,4,5,6.R1,4,7,8.R2,5,7,9.R3,6,8,9 - R3,6,8,9.R2,5,7,9.R1,4,7,8.R10,4,5,6.R10,1,2,3 ==
  ConstantArray[0, {10, 10}], Flatten[A], Modulus -> 3];
```

```
sol2 = DeleteDuplicates[sol];
Length[sol2]
```

236

```
noninvertible = {};
noninvertible2 = {};
invertible = {};
Table[ If [SameQ[Mod[Det[A /. sol2[[i]]], 3], 0], AppendTo[noninvertible,
  A /. sol2[[i]] // MatrixForm]; AppendTo[noninvertible2, A /. sol2[[i]]],
  AppendTo[invertible, A /. sol2[[i]]]], {i, 1, Length[sol2]};
noninvertible
```

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

4-Simplex \mathbb{Z}_4 : Kashaev, Sergeev ABORTED, TOO LONG TO COMPUTE

```
sol = Solve[
  R10,1,2,3·R10,4,5,6·R1,4,7,8·R2,5,7,9·R3,6,8,9 - R3,6,8,9·R2,5,7,9·R1,4,7,8·R10,4,5,6·R10,1,2,3 ==
  ConstantArray[0, {10, 10}], Flatten[A], Modulus → 4];
$Aborted
```

4-Simplex \mathbb{Z}_5 , 1014 solutions: Kashaev, Sergeev

```
SetSystemOptions["ReduceOptions" → "MaxModularPoints" → 1 000 000 000 000 000];

sol = Solve[
  R10,1,2,3·R10,4,5,6·R1,4,7,8·R2,5,7,9·R3,6,8,9 - R3,6,8,9·R2,5,7,9·R1,4,7,8·R10,4,5,6·R10,1,2,3 ==
  ConstantArray[0, {10, 10}], Flatten[A], Modulus → 5];

sol2 = DeleteDuplicates[sol];
Length[sol2]
1014

noninvertible = {};
noninvertible2 = {};
invertible = {};
Table[ If [SameQ[Mod[Det[A /. sol2[[i]]], 5], 0], AppendTo[noninvertible,
  A /. sol2[[i]] // MatrixForm]; AppendTo[noninvertible2, A /. sol2[[i]]],
  AppendTo[invertible, A /. sol2[[i]]]], {i, 1, Length[sol2]};
noninvertible
```


$$\begin{aligned}
& \begin{pmatrix} 0 & 5 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 & 2 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 & 4 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 & 5 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 & 6 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \left. \begin{pmatrix} 0 & 6 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 0 & 3 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 0 & 4 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 0 & 5 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 0 & 6 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}
\end{aligned}$$