These are the *Mathematica* solutions for the paper:

“Permutation-type solutions to the Yang-Baxter and other n-simplex equations” by J. Hietarinta.

computations by Dara O Shayda

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In[1]:=

```
<< Notation`
```

This formulation is incomprehensible and I guessed how it works:

In order to write the n-simplex equations in terms of A and B let us further define (in analogue with (2))

$$(A_{K_1\ldots K_n})_i^j = \begin{cases} A_\alpha^\beta, & \text{if } i = K_\alpha,\ j = K_\beta \text{ for some } \alpha,\ \beta, \\ \delta_i^j, & \text{otherwise,} \end{cases} \tag{11}$$

$$(B_{K_1\ldots K_n})_i = \begin{cases} B_i, & \text{if } i = K_\alpha, \text{ for some } \alpha, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

so that

$$(R_{K_1\ldots K_n})^{j_1\ldots j_N}_{i_1\ldots i_N} = \prod_{\mu=1}^{N} \delta^{j_n}_{(A_{K_1\ldots K_n})^\nu_\mu i_\nu + (B_{K_1\ldots K_n})_\mu}, \tag{13}$$

where now the ν summation runs from 1 to N.

In[2]:=
```
Notation[ R_{i_,j_} ⟹ makeA[A, {i_, j_}, 3] ]
Notation[ R_{i_,j_,k_} ⟹ makeA[A, {i_, j_, k_}, 6] ]
Notation[ R_{i_,j_,k_,h_} ⟹ makeA[A, {i_, j_, k_, h_}, 10] ]

makeMatrix[dim_, a_] := Array[Subscript[a, #1, #2] &, dim]

(* makeA generates the A matrices in Hietarinta paper *)

makeA[A_, list_, n_] := Module[{res, i, j, k},

  res = ConstantArray[-1, {n, n}];

  Table[If[MemberQ[list, i] != True, res [[i]][[j]] = KroneckerDelta[i, j];
    res [[j]][[i]] = KroneckerDelta[j, i] ], {i, 1, n}, {j, 1, n}];

  Table[k = 1; Table[
    If[res[[list[[i]]]][[j]] == -1, res[[list[[i]]]][[j]] = A[[i]][[k]]; k = k + 1];,
    {j, 1, n}], {i , 1, Length[list]}];
  res

 ]
```

3-Simplex

In[7]:=
```
A = makeMatrix[{3, 3}, a];
A // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

Rename the entries to match the paper:

In[9]:=
```
A = A /.
   {a_{1,1} → a, a_{1,2} → b, a_{1,3} → c, a_{2,1} → x, a_{2,2} → y, a_{2,3} → z, a_{3,1} → u, a_{3,2} → v, a_{3,3} → w};
A // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} a & b & c \\ x & y & z \\ u & v & w \end{pmatrix}$$

Make the R matrices for 3-implex:

In[11]:= $R_{1,2,3}$ // MatrixForm

Out[11]//MatrixForm=

$$\begin{pmatrix} a & b & c & 0 & 0 & 0 \\ x & y & z & 0 & 0 & 0 \\ u & v & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[12]:=

$R_{1,4,5}$ // MatrixForm

Out[12]//MatrixForm=

$$\begin{pmatrix} a & 0 & 0 & b & c & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ x & 0 & 0 & y & z & 0 \\ u & 0 & 0 & v & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[13]:=

$R_{2,4,6}$ // MatrixForm

Out[13]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & c \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & z \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & u & 0 & v & 0 & w \end{pmatrix}$$

In[14]:=

$R_{3,5,6}$ // MatrixForm

Out[14]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & b & c \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 & y & z \\ 0 & 0 & u & 0 & v & w \end{pmatrix}$$

In order to solve the tetrahedron equation under the present ansatz we first consider the homogeneous part. The equation to solve is just like (4) with $\mathcal{R}$ replaced with A. When the matrix

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ u & v & w \end{bmatrix}$$

is inserted into the 6×6 matrix $\tilde{A}_{K_1 K_2 K_3}$ the six different ways indicated in (4) and we compute the corresponding matrix product we find 29 equations:

$$\begin{gathered}
abx = 0,\ bxy = 0,\ vyz = 0,\ vwz = 0, \\
bx(b-x) = 0,\ vz(v-z) = 0,\ y(bu-cv) = 0,\ y(-cx+uz) = 0, \\
b(ay+b-1) = 0,\ x(ay+x-1) = 0,\ z(wy+z-1) = 0,\ v(wy+v-1) = 0, \\
abuz + acx + bcu = 0,\ bvxz + cvy + cxy = 0,\ buwz + cuz + cvw = 0, \\
abu + acvx + cux = 0,\ buy + bvxz + uyz = 0,\ cuv + cvwx + uwz = 0, \\
abwz + acz + bcw + c^2 = 0,\ auv + avwx + u^2 + uwx = 0, \\
buvz + cuy + cv^2 - cvz = 0,\ buxz - bcx + cuy + cx^2 = 0, \\
-b^2u - bcvx + bux - cuy = 0,\ -cuy - cvxz + uvz - uz^2 = 0, \\
bwxz + cwy + cxz + cz - c = 0,\ abvz + acy + bcv + bc - c = 0, \\
-auy - avxz - uxz - ux + u = 0,\ -buv - bvwx - uv - uwy + u = 0, \\
-bcu + bu^2z - c^2vx + cuv + cux - cuz = 0.
\end{gathered}$$

By just considering the first four equations the problem can be split into 9 different cases, and each one of them can then be solved rather easily. After eliminating those solutions that reduce to 2-simplex solutions and those with noninvertible A we find 3 basic solutions from which others are obtained by the allowed transformations. These solutions and their nonhomogeneous additions will be discussed below.

10

In[16]:=

$$\text{eqs} = (\mathbf{R}_{1,2,3}.\mathbf{R}_{1,4,5}.\mathbf{R}_{2,4,6}.\mathbf{R}_{3,5,6} - \mathbf{R}_{3,5,6}.\mathbf{R}_{2,4,6}.\mathbf{R}_{1,4,5}.\mathbf{R}_{1,2,3});$$
$$(\mathbf{R}_{1,2,3}.\mathbf{R}_{1,4,5}.\mathbf{R}_{2,4,6}.\mathbf{R}_{3,5,6} - \mathbf{R}_{3,5,6}.\mathbf{R}_{2,4,6}.\mathbf{R}_{1,4,5}.\mathbf{R}_{1,2,3})\ \text{// MatrixForm}$$

Out[17]//MatrixForm=

0	$a\,b\,x$	$a\,c\,x + u\,(b\,c + a\,b\,z)$
$-a\,b\,x$	$-b^2\,x + b\,x^2$	$-b\,c\,x + c\,x^2 + u\,(c\,y + b\,x\,z)$
$-c\,u\,x - a\,(b\,u + c\,v\,x)$	$b\,u\,x - b\,(b\,u + c\,v\,x) - c\,u\,y$	$c\,u\,x - c\,(b\,u + c\,v\,x) - c\,u\,z + u\,(c\,v + l$
$x - x^2 - a\,x\,y$	$-b\,x\,y$	$-c\,x\,y + u\,y\,z$
$u - u\,x - u\,x\,z - a\,(u\,y + v\,x\,z)$	$-u\,y\,z - b\,(u\,y + v\,x\,z)$	$u\,v\,z - u\,z^2 - c\,(u\,y + v\,x\,z)$
$-u^2 - u\,w\,x - a\,(u\,v + v\,w\,x)$	$u - u\,v - b\,(u\,v + v\,w\,x) - u\,w\,y$	$-c\,(u\,v + v\,w\,x) - u\,w\,z$

We get exactly the same equations.

Solve these simultaneous system of equations, '→' means substitute for solution:

Make a list of the polynomials from the R equations:

In[21]:=
```
eqs = DeleteDuplicates[DeleteCases[Flatten[eqs], _Integer]]
```

Out[21]=
$\{a b x,\ a c x + u (b c + a b z),\ -b + b^2 + a b y,\ -c + b c + a c y + v (b c + a b z),$
$c^2 + a c z + w (b c + a b z),\ -a b x,\ -b^2 x + b x^2,\ -b c x + c x^2 + u (c y + b x z),\ b x y,$
$c x y + v (c y + b x z),\ -c + c z + c x z + w (c y + b x z),\ -c u x - a (b u + c v x),$
$b u x - b (b u + c v x) - c u y,\ c u x - c (b u + c v x) - c u z + u (c v + b u z),\ b u y - c v y,$
$c u y - c v z + v (c v + b u z),\ c u z + w (c v + b u z),\ x - x^2 - a x y,\ -b x y,\ -c x y + u y z,$
$v y z,\ -z + w y z + z^2,\ u - u x - u x z - a (u y + v x z),\ -u y z - b (u y + v x z),$
$u v z - u z^2 - c (u y + v x z),\ -v y z,\ v^2 z - v z^2,\ v w z,\ -u^2 - u w x - a (u v + v w x),$
$u - u v - b (u v + v w x) - u w y,\ -c (u v + v w x) - u w z,\ v - v^2 - v w y,\ -v w z\}$

Find the Groebner basis, this set has the same gb roots as eqs:

In[32]:=
```
gb = GroebnerBasis[eqs, {a, b, c, x, y, z, u, v, w}]
```

Out[32]=
$\{v w z,\ -v z + v^2 z,\ u w z,\ -u z + u v z,\ -v z + v z^2,\ -u z + u z^2,\ -v + v^2 + v w y,\ -u + u v + u w y,$
$-z + w y z + z^2,\ v y z,\ u y z,\ -u z + u x z,\ -u v - u x + 2 u v x - v w x + v w x^2 - u^2 y,$
$v x - v^2 x - v x^2 + v^2 x^2 + u v y + u x y - 2 u v x y + u^2 y^2,\ -v x z + v x^2 z,\ c v w,$
$-c v + c v^2,\ c u,\ -c v + c v z,\ -c + c w y + c z,\ c v y,\ c v x,\ -c x + c x z,\ c x y,$
$-c x + c x^2,\ c^2 x + c w x,\ -b u + b u v,\ b u^2 + b u w,\ b u z,\ b u y,\ b u + b v w x,$
$-b v x + b v^2 x,\ -b u + b u x,\ c x + b w x z,\ b v x z,\ -b x z + b x z^2,\ b x y,\ -b x + b x^2,$
$-c v + b c v,\ -c x + b c x,\ -b u + b^2 u,\ -b c - c^2 y - c z + 2 b c z - b w z + b^2 w z,$
$-b v z + b^2 v z,\ b c y + c^2 y^2 + b z - b^2 z + c y z - 2 b c y z - b z^2 + b^2 z^2,\ -b x + b^2 x,$
$-a u + a u v + u w - u w x,\ a u z + u^2 z,\ -u + u x + a u y,\ a u + u^2 - u w + 2 u w x + a v w x,$
$-u v - u x - a v x + 2 u v x + a v^2 x - u^2 y,\ a u^2 + u^3 - u^2 w + a u w x + 2 u^2 w x - u w^2 x + u w^2 x^2,$
$u z + a v x z,\ -a x z + w x z - w x^2 z + a x z^2,\ -x + x^2 + a x y,\ a c v + c^2 v,$
$-a c + c w - b c w + a c z,\ -c + b c + a c y,\ a c x,\ -a b v + a b v^2 + b v w - b^2 v w,\ a b u,$
$a c + c^2 - c w + 2 b c w + a b w z,\ c v + a b v z,\ -b c - c^2 y - a b z - c z + 2 b c z + a b z^2,$
$-b + b^2 + a b y,\ a b x,\ a c^2 + c^3 + a b c w - c^2 w + 2 b c^2 w - b c w^2 + b^2 c w^2\}$

Seems the same i.e. same difficulties to solve.

Use Groebner basis to eliminate some vars:

In[34]:=
```
(* did not work *)
gbasis = GroebnerBasis[eqs, {a, b, c}, {x, y, z, u, v, w}]
```

Out[34]= { }

Works this time and we have much fewer equations by elimination:

In[37]:=
```
gbasis = GroebnerBasis[eqs, {a, b, c, x}, {y, z, u, v, w}]
```

Out[37]=
$\{-c x + c x^2,\ -b x + b x^2,\ -c x + b c x,\ -b x + b^2 x,\ a c x,\ a b x\}$

Solve for 0s:

In[39]:=
```
Solve[gbasis == 0, {a, b, c, x}]
```
Out[39]=
```
{{x → 0}, {b → 0, c → 0}, {a → 0, b → 1, x → 1}}
```

Now we can classify:

Case : {*x* → 0}

In[53]:=
```
sol = Solve[DeleteDuplicates[DeleteCases[Simplify[eqs /. {x → 0}], _Integer]] == 0,
   {a, b, c, y, z, u, v, w} ];
```

In[54]:=

```
Table[(A /. {x → 0}) /. sol[[i]] // MatrixForm, {i, 1, Length[sol]}]
```

Out[54]=

$$\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1-v}{w} & 0 \\ 0 & v & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1-z}{w} & z \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ 0 & \frac{1-b}{a} & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1-v}{w} & 0 \\ 0 & v & w \end{pmatrix},\right.$$

$$\begin{pmatrix} a & b & 0 \\ 0 & \frac{1-b}{a} & 0 \\ 0 & v & \frac{a(-1+v)}{-1+b} \end{pmatrix}, \begin{pmatrix} -\frac{c}{z} & b & c \\ 0 & \frac{(-1+b)z}{c} & z \\ 0 & 0 & \frac{c-cz}{(-1+b)z} \end{pmatrix}, \begin{pmatrix} \frac{c-bc}{b(-1+z)} & b & c \\ 0 & \frac{b(-1+z)}{c} & z \\ 0 & 0 & -\frac{c}{b} \end{pmatrix}, \begin{pmatrix} -\frac{w}{-1+v} & 0 & 0 \\ 0 & \frac{1-v}{w} & 0 \\ 0 & v & w \end{pmatrix},$$

$$\begin{pmatrix} -\frac{u}{v} & 0 & 0 \\ 0 & -\frac{v}{u} & 0 \\ u & v & \frac{u(-1+v)}{v} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1-z}{w} & z \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} -\frac{w}{-1+z} & 0 & 0 \\ 0 & \frac{1-z}{w} & z \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{w} & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ 0 & \frac{1-b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a & b & 0 \\ 0 & \frac{1-b}{a} & 0 \\ 0 & 0 & -\frac{a}{-1+b} \end{pmatrix}, \begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix},$$

$$\begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{u}{v} & 0 & 0 \\ 0 & -\frac{v}{u} & 0 \\ u & v & \frac{u(-1+v)}{v} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} -\frac{c}{z} & 0 & c \\ 0 & -\frac{z}{c} & z \\ 0 & 0 & \frac{c(-1+z)}{z} \end{pmatrix},$$

$$\begin{pmatrix} a & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & -c \end{pmatrix}, \begin{pmatrix} a & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & -c \end{pmatrix}, \begin{pmatrix} -c & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix},$$

$$\left.\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} -c & 1 & c \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\}$$

Case : $\{b \to 0, c \to 0\}$

In[55]:=

```
sol =
  Solve[DeleteDuplicates[DeleteCases[Simplify[eqs /. {b → 0, c → 0}], _Integer]] == 0,
   {a, x, y, z, u, v, w} ];
```

In[56]:=
```
Table[(A /. {b → 0, c → 0}) /. sol[[i]] // MatrixForm, {i, 1, Length[sol]}]
```

Out[56]=

$$\Bigg\{\begin{pmatrix} a & 0 & 0 \\ x & \frac{1-x}{a} & z \\ 0 & 0 & \frac{-a+a z}{-1+x} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ x & \frac{1-x}{a} & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix},$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ x & \frac{1-x}{a} & 0 \\ u & -\frac{u}{a} & \frac{-a-u}{-1+x} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ x & \frac{1-x}{a} & 0 \\ u & \frac{u+a x-u x}{a x} & -\frac{u}{x} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ x & \frac{1-x}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ x & \frac{1-x}{a} & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a} & z \\ 0 & 0 & a-a z \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & v & a-a v \end{pmatrix},$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ u & -\frac{u}{a} & a+u \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 0 \\ u & 1+u y & -u \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ -a & 1 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 1 \\ -a & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\Bigg\}$$

Case : $\{a \to 0, b \to 1, x \to 1\}$

In[57]:=
```
sol = Solve[
   DeleteDuplicates[DeleteCases[Simplify[eqs /. {a → 0, b → 1, x → 1}], _Integer]] == 0,
   {c, y, z, u, v, w}];
```

In[58]:=
```
Table[(A /. {a → 0, b → 1, x → 1}) /. sol[[i]] // MatrixForm, {i, 1, Length[sol]}]
```

Out[58]=

$$\Bigg\{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ u & 1 & -u \end{pmatrix}, \begin{pmatrix} 0 & 1 & c \\ 1 & 0 & 1 \\ 0 & 0 & -c \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\Bigg\}$$