These are the *Mathematica* solutions for the paper:

“Permutation-type solutions to the Yang-Baxter and other n-simplex equations” by J. Hietarinta.

computations by Dara O Shayda

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```
<< Notation`
```

This formulation is incomprehensible and I guessed how it works:

In order to write the n-simplex equations in terms of A and B let us further define (in analogue with (2))

$$(A_{K_1 \dots K_n})_i^j = \begin{cases} A_\alpha^\beta, & \text{if } i = K_\alpha,\ j = K_\beta \text{ for some } \alpha,\ \beta, \\ \delta_i^j, & \text{otherwise,} \end{cases} \tag{11}$$

$$(B_{K_1 \dots K_n})_i = \begin{cases} B_i, & \text{if } i = K_\alpha, \text{ for some } \alpha, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

so that

$$(R_{K_1 \dots K_n})_{i_1 \dots i_N}^{j_1 \dots j_N} = \prod_{\mu=1}^{N} \delta^{j_n}_{(A_{K_1 \dots K_n})_\mu^\nu i_\nu + (B_{K_1 \dots K_n})_\mu}, \tag{13}$$

where now the ν summation runs from 1 to N.

```
Notation[ R_{i_,j_}  ⟹  makeA[A, {i_, j_}, 3] ]
Notation[ R_{i_,j_,k_}  ⟹  makeA[A, {i_, j_, k_}, 6] ]
Notation[ R_{i_,j_,k_,h_}  ⟹  makeA[A, {i_, j_, k_, h_}, 10] ]

makeMatrix[dim_, a_] := Array[Subscript[a, #1, #2] &, dim]

(* makeA generates the A matrices in Hietarinta paper *)

makeA[A_, list_, n_] := Module[{res, i, j, k},

  res = ConstantArray[-1, {n, n}];

  Table[If[MemberQ[list, i] != True, res [[i]] [[j]] = KroneckerDelta[i, j];
    res [[j]] [[i]] = KroneckerDelta[j, i] ], {i, 1, n}, {j, 1, n}];

  Table[k = 1; Table[
    If[res[[list[[i]]]] [[j]] == -1, res[[list[[i]]]] [[j]] = A[[i]] [[k]]; k = k + 1];,
    {j, 1, n}], {i , 1, Length[list]}];
  res

 ]
```

3-Simplex

In[17]:=

```
A = makeMatrix[{3, 3}, a];
A // MatrixForm
```

Out[18]//MatrixForm=

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

Rename the entries to match the paper:

In[19]:=

```
A = A /.
   {a_{1,1} → a, a_{1,2} → b, a_{1,3} → c, a_{2,1} → x, a_{2,2} → y, a_{2,3} → z, a_{3,1} → u, a_{3,2} → v, a_{3,3} → w};
A // MatrixForm
```

Out[20]//MatrixForm=

$$\begin{pmatrix} a & b & c \\ x & y & z \\ u & v & w \end{pmatrix}$$

Make the R matrices for 3-implex:

In[21]:= $\mathbf{R_{1,2,3}}$ // MatrixForm

Out[21]//MatrixForm=

$$\begin{pmatrix} a & b & c & 0 & 0 & 0 \\ x & y & z & 0 & 0 & 0 \\ u & v & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[22]:=

$\mathbf{R_{1,4,5}}$ // MatrixForm

Out[22]//MatrixForm=

$$\begin{pmatrix} a & 0 & 0 & b & c & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ x & 0 & 0 & y & z & 0 \\ u & 0 & 0 & v & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[23]:=

$\mathbf{R_{2,4,6}}$ // MatrixForm

Out[23]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & c \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & z \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & u & 0 & v & 0 & w \end{pmatrix}$$

In[24]:=

$\mathbf{R_{3,5,6}}$ // MatrixForm

Out[24]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & b & c \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 & y & z \\ 0 & 0 & u & 0 & v & w \end{pmatrix}$$

In order to solve the tetrahedron equation under the present ansatz we first consider the homogeneous part. The equation to solve is just like (4) with $\mathcal{R}$ replaced with A. When the matrix

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ u & v & w \end{bmatrix}$$

is inserted into the 6×6 matrix $\tilde{A}_{K_1 K_2 K_3}$ the six different ways indicated in (4) and we compute the corresponding matrix product we find 29 equations:

$$abx = 0,\ bxy = 0,\ vyz = 0,\ vwz = 0,$$
$$bx(b-x) = 0,\ vz(v-z) = 0,\ y(bu - cv) = 0,\ y(-cx + uz) = 0,$$
$$b(ay + b - 1) = 0,\ x(ay + x - 1) = 0,\ z(wy + z - 1) = 0,\ v(wy + v - 1) = 0,$$
$$abuz + acx + bcu = 0,\ bvxz + cvy + cxy = 0,\ buwz + cuz + cvw = 0,$$
$$abu + acvx + cux = 0,\ buy + bvxz + uyz = 0,\ cuv + cvwx + uwz = 0,$$
$$abwz + acz + bcw + c^2 = 0,\ auv + avwx + u^2 + uwx = 0,$$
$$buvz + cuy + cv^2 - cvz = 0,\ buxz - bcx + cuy + cx^2 = 0,$$
$$-b^2u - bcvx + bux - cuy = 0,\ -cuy - cvxz + uvz - uz^2 = 0,$$
$$bwxz + cwy + cxz + cz - c = 0,\ abvz + acy + bcv + bc - c = 0,$$
$$-auy - avxz - uxz - ux + u = 0,\ -buv - bvwx - uv - uwy + u = 0,$$
$$-bcu + bu^2z - c^2vx + cuv + cux - cuz = 0.$$

By just considering the first four equations the problem can be split into 9 different cases, and each one of them can then be solved rather easily. After eliminating those solutions that reduce to 2-simplex solutions and those with noninvertible A we find 3 basic solutions from which others are obtained by the allowed transformations. These solutions and their nonhomogeneous additions will be discussed below.

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In[25]:=

```
(R1,2,3.R1,4,5.R2,4,6.R3,5,6 - R3,5,6.R2,4,6.R1,4,5.R1,2,3) // MatrixForm
```

$$(\mathbf{R}_{1,2,3}.\mathbf{R}_{1,4,5}.\mathbf{R}_{2,4,6}.\mathbf{R}_{3,5,6} - \mathbf{R}_{3,5,6}.\mathbf{R}_{2,4,6}.\mathbf{R}_{1,4,5}.\mathbf{R}_{1,2,3})\ //\ \mathbf{MatrixForm}$$

Out[25]//MatrixForm=

$$\begin{pmatrix} 0 & a\,b\,x & a\,c\,x + u\,(b\,c + a\,b\,z) \\ -a\,b\,x & -b^2\,x + b\,x^2 & -b\,c\,x + c\,x^2 + u\,(c\,y + b\,x\,z) \\ -c\,u\,x - a\,(b\,u + c\,v\,x) & b\,u\,x - b\,(b\,u + c\,v\,x) - c\,u\,y & c\,u\,x - c\,(b\,u + c\,v\,x) - c\,u\,z + u\,(c\,v + \text{l} \\ x - x^2 - a\,x\,y & -b\,x\,y & -c\,x\,y + u\,y\,z \\ u - u\,x - u\,x\,z - a\,(u\,y + v\,x\,z) & -u\,y\,z - b\,(u\,y + v\,x\,z) & u\,v\,z - u\,z^2 - c\,(u\,y + v\,x\,z) \\ -u^2 - u\,w\,x - a\,(u\,v + v\,w\,x) & u - u\,v - b\,(u\,v + v\,w\,x) - u\,w\,y & -c\,(u\,v + v\,w\,x) - u\,w\,z \end{pmatrix}$$

We get exactly the same equations.

Solve these simultaneous system of equations, '→' means substitute for solution:

3-Simplex Solutions

In[26]:=
```
sol = Solve[R1,2,3.R1,4,5.R2,4,6.R3,5,6 - R3,5,6.R2,4,6.R1,4,5.R1,2,3 ==
    ConstantArray[0, {6, 6}], {a, b, c, x, y, z, u, v, w}];
```

Substitute them back into the original A matrix:

In[27]:=
```
sol = DeleteDuplicates[sol];
Table[ A /. sol[[i]] // MatrixForm , {i, 1, Length[sol]}]
```

Out[28]=

$$\left\{\begin{pmatrix} -\frac{u}{v} & 0 & 0 \\ x & \frac{v(-1+x)}{u} & 0 \\ u & v & \frac{u-uv}{v(-1+x)} \end{pmatrix}, \begin{pmatrix} \frac{u-ux}{(-1+v)x} & 0 & 0 \\ x & \frac{(-1+v)x}{u} & 0 \\ u & v & -\frac{u}{x} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix}, \begin{pmatrix} \frac{1-x}{y} & 0 & 0 \\ x & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix},\right.$$

$$\begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} \frac{1-x}{y} & 0 & 0 \\ x & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix}, \begin{pmatrix} \frac{c-bc}{b(-1+z)} & b & c \\ 0 & \frac{b(-1+z)}{c} & z \\ 0 & 0 & -\frac{c}{b} \end{pmatrix},$$

$$\begin{pmatrix} -\frac{c}{z} & b & c \\ 0 & \frac{(-1+b)z}{c} & z \\ 0 & 0 & \frac{c-cz}{(-1+b)z} \end{pmatrix}, \begin{pmatrix} -u & 0 & 0 \\ 1 & 0 & 0 \\ u & 1 & w \end{pmatrix}, \begin{pmatrix} -\frac{u}{v} & 0 & 0 \\ 0 & -\frac{v}{u} & 0 \\ u & v & \frac{u(-1+v)}{v} \end{pmatrix}, \begin{pmatrix} \frac{u(-1+x)}{x} & 0 & 0 \\ x & -\frac{x}{u} & 0 \\ u & 0 & -\frac{u}{x} \end{pmatrix}, \begin{pmatrix} \frac{1-x}{y} & 0 & 0 \\ x & y & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1-x}{y} & 0 & 0 \\ x & y & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} \frac{1-x}{y} & 0 & 0 \\ x & y & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix}, \begin{pmatrix} \frac{1-x}{y} & 0 & 0 \\ x & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & z \\ 0 & 0 & \frac{1-z}{y} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1-b}{y} & b & 0 \\ 0 & y & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 0 \\ 0 & v & \frac{1-v}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ u & 1 & -u \end{pmatrix}, \begin{pmatrix} a & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & -c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} -\frac{c}{z} & 0 & c \\ 0 & -\frac{z}{c} & z \\ 0 & 0 & \frac{c(-1+z)}{z} \end{pmatrix},$$

$$\begin{pmatrix} \frac{(-1+b)\,c}{b} & b & c \\ 0 & -\frac{b}{c} & 0 \\ 0 & 0 & -\frac{c}{b} \end{pmatrix}, \begin{pmatrix} -c & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & w \end{pmatrix}, \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & y & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -u & 0 & 0 \\ 1 & 0 & 1 \\ u & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ u & 1 & -u \end{pmatrix}, \begin{pmatrix} 0 & 1 & c \\ 1 & 0 & 1 \\ 0 & 0 & -c \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -c & 1 & c \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\Big\}$$

3-Simplex Solutions: In a, b, c

In[29]:= **solabc = Solve[**$R_{1,2,3}.R_{1,4,5}.R_{2,4,6}.R_{3,5,6} - R_{3,5,6}.R_{2,4,6}.R_{1,4,5}.R_{1,2,3}$ **==**
ConstantArray[0, {6, 6}], {x, y, z, u, v, w}];

```
solabc = DeleteDuplicates[solabc];
Table[ A /. solabc[[i]] // MatrixForm , {i, 1, Length[solabc]}]
```

$$\left\{\begin{pmatrix} a & b & c \\ 0 & \frac{1-b}{a} & \frac{a\,b+c-b\,c}{a\,b} \\ 0 & 0 & -\frac{c}{b} \end{pmatrix}, \begin{pmatrix} a & b & c \\ 0 & \frac{1-b}{a} & -\frac{c}{a} \\ 0 & 0 & \frac{-a-c}{-1+b} \end{pmatrix}\right\}$$