

# QYBE solutions for constant R-Matrices over Grassmann Algebra, 4-generators

Based upon the work of Steven Duplij, Olga Kotulska and Alexander Sadovnikov [1]

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```
In[1]:= Clear[a, b, c, d, p, q];
```

Assume a, b, c, d, p and q belong to a multiplicative commutative distributive algebra, with both additive and multiplicative inverses.

Based upon the 6-vertex YBE solutions [3], the following matrix forms are created:

```
In[2]:= R = .;
R = ConstantArray[0, {4, 4}];
R[[1]][[1]] = p;
R[[2]][[2]] = c;
R[[3]][[3]] = b;
R[[4]][[4]] = q;
R[[2]][[3]] = d;
R[[3]][[2]] = a;
R // MatrixForm
```

```
Out[10]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & c & d & 0 \\ 0 & a & b & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

```

```
In[11]:= R12 =. ;
R12 = ConstantArray[0, {8, 8}];
R12[[1]][[1]] = p;
R12[[2]][[2]] = p;
R12[[3]][[3]] = c;
R12[[4]][[4]] = c;
R12[[5]][[5]] = b;
R12[[6]][[6]] = b;
R12[[7]][[7]] = q;
R12[[8]][[8]] = q;
R12[[3]][[5]] = d;
R12[[4]][[6]] = d;
R12[[5]][[3]] = a;
R12[[6]][[4]] = a;
R12 // MatrixForm
```

Out[25]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & d & 0 & 0 \\ 0 & 0 & a & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

In[26]:=

```
R13 =. ;
R13 = ConstantArray[0, {8, 8}];
R13[[1]][[1]] = p;
R13[[2]][[2]] = c;
R13[[3]][[3]] = p;
R13[[4]][[4]] = c;
R13[[5]][[5]] = b;
R13[[6]][[6]] = q;
R13[[7]][[7]] = b;
R13[[8]][[8]] = q;
R13[[2]][[5]] = d;
R13[[4]][[7]] = d;
R13[[5]][[2]] = a;
R13[[7]][[4]] = a;
R13 // MatrixForm
```

Out[40]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & d & 0 \\ 0 & a & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

In[41]:=

```
R23 = .;
R23 = ConstantArray[0, {8, 8}];
R23[[1]][[1]] = p;
R23[[2]][[2]] = c;
R23[[3]][[3]] = b;
R23[[4]][[4]] = q;
R23[[5]][[5]] = p;
R23[[6]][[6]] = c;
R23[[7]][[7]] = b;
R23[[8]][[8]] = q;
R23[[2]][[3]] = d;
R23[[6]][[7]] = d;
R23[[3]][[2]] = a;
R23[[7]][[6]] = a;
R23 // MatrixForm
```

Out[55]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

From the braid equation get the algebraic equations that must satisfy 0:

```
In[56]:= equations = Simplify[R12.R13.R23 - R23.R13.R12];
Style[equations // MatrixForm, FontSize -> 9]
```

$$\text{Out[57]}= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a c d & 0 & d (-b c + p (-d + p)) & 0 & 0 & 0 \\ 0 & a c d & a d (-a + d) & 0 & -a b d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a c d & d (b c + (d - q) q) & 0 \\ 0 & a (b c + (a - p) p) & a b d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a c d & 0 & a (a - d) d & a b d & 0 \\ 0 & 0 & 0 & a (-b c + q (-a + q)) & 0 & -a b d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From [1] check the computations:

$$cda = 0, \quad (28)$$

$$bda = 0, \quad (29)$$

$$da(d - a) = 0, \quad pd(d - p) + cbd = 0, \quad (30)$$

$$\left. \begin{aligned} &qd(d - q) + cbd = 0, \\ &pa(a - p) + cba = 0, \\ &qa(a - q) + cba = 0. \end{aligned} \right\} \quad (31)$$

```
In[58]:= << "GrassmannAlgebra`"
★A; DeclareVectorSymbols[ξ1, ξ2, ξ3, ξ4]
(* Dimension has to be set or the default is 3 *)
DeclareBasis[4];
scalars = DeclareScalarSymbols[a0, b0, c0, d0, p0, q0, a12, b12, c12, d12, p12, q12, a13,
a14, b13, c13, c14, d13, d14, p13, p14, q13, q14, a23, b23, c23, d23, p23, q23, a24, a34,
a1234, b24, b34, b1234, c24, c34, c1234, d24, d34, d1234, p24, p34, p1234, q24, q34, q1234]
X = ★G[(1 + ξ1) ∧ (1 + ξ2) ∧ (1 + ξ3) ∧ (1 + ξ4)]
```

Out[59]= {ξ1, ξ2, ξ3, ξ4}

```
Out[61]= {a0, a12, a13, a14, a23, a24, a1234, b0, b12, b13, b14, b23, b24, b34, b1234,
c0, c12, c13, c14, c23, c24, c34, c1234, d0, d12, d13, d14, d23, d24, d34, d1234,
p0, p12, p13, p14, p23, p24, p34, p1234, q0, q12, q13, q14, q23, q24, q34, q1234}
```

```
Out[62]= 1 + ξ1 + ξ2 + ξ3 + ξ4 + ξ1 ∧ ξ2 + ξ1 ∧ ξ3 + ξ1 ∧ ξ4 + ξ2 ∧ ξ3 + ξ2 ∧ ξ4 +
ξ3 ∧ ξ4 + ξ1 ∧ ξ2 ∧ ξ3 + ξ1 ∧ ξ2 ∧ ξ4 + ξ1 ∧ ξ3 ∧ ξ4 + ξ2 ∧ ξ3 ∧ ξ4 + ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4
```

In[63]:= (\* assign the 0s to only get the even terms \*)
aa = .;
aa = {a0, 0, 0, 0, 0, a12, a13, a14, a23, a24, a34, 0, 0, 0, 0, a1234}.List @@ X

```
bb = .;
bb = {b0, 0, 0, 0, 0, b12, b13, b14, b23, b24, b34, 0, 0, 0, 0, b1234}.List @@ X
```

```
cc = .;
cc = {c0, 0, 0, 0, 0, c12, c13, c14, c23, c24, c34, 0, 0, 0, 0, c1234}.List @@ X
```

```
dd = .;
dd = {d0, 0, 0, 0, 0, d12, d13, d14, d23, d24, d34, 0, 0, 0, 0, d1234}.List @@ X
```

```
pp = .;
pp = {p0, 0, 0, 0, 0, p12, p13, p14, p23, p24, p34, 0, 0, 0, 0, p1234}.List @@ X
```

```
qq = .;
qq = {q0, 0, 0, 0, 0, q12, q13, q14, q23, q24, q34, 0, 0, 0, 0, q1234}.List @@ X
```

Out[64]= a0 + a12 ξ1 ∧ ξ2 + a13 ξ1 ∧ ξ3 + a14 ξ1 ∧ ξ4 + a23 ξ2 ∧ ξ3 + a24 ξ2 ∧ ξ4 + a34 ξ3 ∧ ξ4 + a1234 ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4

Out[66]= b0 + b12 ξ1 ∧ ξ2 + b13 ξ1 ∧ ξ3 + b14 ξ1 ∧ ξ4 + b23 ξ2 ∧ ξ3 + b24 ξ2 ∧ ξ4 + b34 ξ3 ∧ ξ4 + b1234 ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4

Out[68]= c0 + c12 ξ1 ∧ ξ2 + c13 ξ1 ∧ ξ3 + c14 ξ1 ∧ ξ4 + c23 ξ2 ∧ ξ3 + c24 ξ2 ∧ ξ4 + c34 ξ3 ∧ ξ4 + c1234 ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4

Out[70]= d0 + d12 ξ1 ∧ ξ2 + d13 ξ1 ∧ ξ3 + d14 ξ1 ∧ ξ4 + d23 ξ2 ∧ ξ3 + d24 ξ2 ∧ ξ4 + d34 ξ3 ∧ ξ4 + d1234 ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4

Out[72]= p0 + p12 ξ1 ∧ ξ2 + p13 ξ1 ∧ ξ3 + p14 ξ1 ∧ ξ4 + p23 ξ2 ∧ ξ3 + p24 ξ2 ∧ ξ4 + p34 ξ3 ∧ ξ4 + p1234 ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4

Out[74]= q0 + q12 ξ1 ∧ ξ2 + q13 ξ1 ∧ ξ3 + q14 ξ1 ∧ ξ4 + q23 ξ2 ∧ ξ3 + q24 ξ2 ∧ ξ4 + q34 ξ3 ∧ ξ4 + q1234 ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4

```
In[75]:= expr1 = GrassmannExpandAndSimplify[cc ∧ dd ∧ aa]

Out[75]= a0 c0 d0 + (a12 c0 d0 + a0 c12 d0 + a0 c0 d12) ξ1 ∧ ξ2 + (a13 c0 d0 + a0 c13 d0 + a0 c0 d13) ξ1 ∧ ξ3 +
(a14 c0 d0 + a0 c14 d0 + a0 c0 d14) ξ1 ∧ ξ4 + (a23 c0 d0 + a0 c23 d0 + a0 c0 d23) ξ2 ∧ ξ3 +
(a24 c0 d0 + a0 c24 d0 + a0 c0 d24) ξ2 ∧ ξ4 + (a34 c0 d0 + a0 c34 d0 + a0 c0 d34) ξ3 ∧ ξ4 +
(a1234 c0 d0 + a34 c12 d0 - a24 c13 d0 + a23 c14 d0 + a14 c23 d0 - a13 c24 d0 + a12 c34 d0 + a0 c1234 d0 +
a34 c0 d12 + a0 c34 d12 - a24 c0 d13 - a0 c24 d13 + a23 c0 d14 + a0 c23 d14 + a14 c0 d23 +
a0 c14 d23 - a13 c0 d24 - a0 c13 d24 + a12 c0 d34 + a0 c12 d34 + a0 c0 d1234) ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4
```

Collect the coefficients:

```
In[76]:= coeff1 = Flatten[{a0 c0 d0, Coefficient[expr1, {ξ1 ∧ ξ2}],
Coefficient[expr1, {ξ1 ∧ ξ3}], Coefficient[expr1, {ξ1 ∧ ξ4}],
Coefficient[expr1, {ξ2 ∧ ξ3}], Coefficient[expr1, {ξ2 ∧ ξ4}],
Coefficient[expr1, {ξ3 ∧ ξ4}], Coefficient[expr1, {ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4}]}]

Out[76]= {a0 c0 d0, a12 c0 d0 + a0 c12 d0 + a0 c0 d12, a13 c0 d0 + a0 c13 d0 + a0 c0 d13,
a14 c0 d0 + a0 c14 d0 + a0 c0 d14, a23 c0 d0 + a0 c23 d0 + a0 c0 d23,
a24 c0 d0 + a0 c24 d0 + a0 c0 d24, a34 c0 d0 + a0 c34 d0 + a0 c0 d34,
a1234 c0 d0 + a34 c12 d0 - a24 c13 d0 + a23 c14 d0 + a14 c23 d0 - a13 c24 d0 + a12 c34 d0 +
a0 c1234 d0 + a34 c0 d12 + a0 c34 d12 - a24 c0 d13 - a0 c24 d13 + a23 c0 d14 + a0 c23 d14 +
a14 c0 d23 + a0 c14 d23 - a13 c0 d24 - a0 c13 d24 + a12 c0 d34 + a0 c12 d34 + a0 c0 d1234}
```

```
In[77]:= expr2 = GrassmannExpandAndSimplify[bb ∧ dd ∧ aa]

Out[77]= a0 b0 d0 + (a12 b0 d0 + a0 b12 d0 + a0 b0 d12) ξ1 ∧ ξ2 + (a13 b0 d0 + a0 b13 d0 + a0 b0 d13) ξ1 ∧ ξ3 +
(a14 b0 d0 + a0 b14 d0 + a0 b0 d14) ξ1 ∧ ξ4 + (a23 b0 d0 + a0 b23 d0 + a0 b0 d23) ξ2 ∧ ξ3 +
(a24 b0 d0 + a0 b24 d0 + a0 b0 d24) ξ2 ∧ ξ4 + (a34 b0 d0 + a0 b34 d0 + a0 b0 d34) ξ3 ∧ ξ4 +
(a1234 b0 d0 + a34 b12 d0 - a24 b13 d0 + a23 b14 d0 + a14 b23 d0 - a13 b24 d0 + a12 b34 d0 + a0 b1234 d0 +
a34 b0 d12 + a0 b34 d12 - a24 b0 d13 - a0 b24 d13 + a23 b0 d14 + a0 b23 d14 + a14 b0 d23 +
a0 b14 d23 - a13 b0 d24 - a0 b13 d24 + a12 b0 d34 + a0 b12 d34 + a0 b0 d1234) ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4
```

```
In[78]:= coeff2 = Flatten[{a0 b0 d0, Coefficient[expr2, {ξ1 ∧ ξ2}],
Coefficient[expr2, {ξ1 ∧ ξ3}], Coefficient[expr2, {ξ1 ∧ ξ4}],
Coefficient[expr2, {ξ2 ∧ ξ3}], Coefficient[expr2, {ξ2 ∧ ξ4}],
Coefficient[expr2, {ξ3 ∧ ξ4}], Coefficient[expr2, {ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4}]}]

Out[78]= {a0 b0 d0, a12 b0 d0 + a0 b12 d0 + a0 b0 d12, a13 b0 d0 + a0 b13 d0 + a0 b0 d13,
a14 b0 d0 + a0 b14 d0 + a0 b0 d14, a23 b0 d0 + a0 b23 d0 + a0 b0 d23,
a24 b0 d0 + a0 b24 d0 + a0 b0 d24, a34 b0 d0 + a0 b34 d0 + a0 b0 d34,
a1234 b0 d0 + a34 b12 d0 - a24 b13 d0 + a23 b14 d0 + a14 b23 d0 - a13 b24 d0 + a12 b34 d0 +
a0 b1234 d0 + a34 b0 d12 + a0 b34 d12 - a24 b0 d13 - a0 b24 d13 + a23 b0 d14 + a0 b23 d14 +
a14 b0 d23 + a0 b14 d23 - a13 b0 d24 - a0 b13 d24 + a12 b0 d34 + a0 b12 d34 + a0 b0 d1234}
```

In[79]:=

```
expr3 = GrassmannExpandAndSimplify[dd ∧ aa ∧ (dd - aa)]
```

$$\begin{aligned} \text{Out}[79]= & -a_0^2 d_0 + a_0 d_0^2 + (-2 a_0 a_{12} d_0 + a_{12} d_0^2 - a_0^2 d_{12} + 2 a_0 d_0 d_{12}) \zeta_1 \wedge \zeta_2 + \\ & (-2 a_0 a_{13} d_0 + a_{13} d_0^2 - a_0^2 d_{13} + 2 a_0 d_0 d_{13}) \zeta_1 \wedge \zeta_3 + \\ & (-2 a_0 a_{14} d_0 + a_{14} d_0^2 - a_0^2 d_{14} + 2 a_0 d_0 d_{14}) \zeta_1 \wedge \zeta_4 + \\ & (-2 a_0 a_{23} d_0 + a_{23} d_0^2 - a_0^2 d_{23} + 2 a_0 d_0 d_{23}) \zeta_2 \wedge \zeta_3 + \\ & (-2 a_0 a_{24} d_0 + a_{24} d_0^2 - a_0^2 d_{24} + 2 a_0 d_0 d_{24}) \zeta_2 \wedge \zeta_4 + \\ & (-2 a_0 a_{34} d_0 + a_{34} d_0^2 - a_0^2 d_{34} + 2 a_0 d_0 d_{34}) \zeta_3 \wedge \zeta_4 + \\ & (-2 a_{14} a_{23} d_0 + 2 a_{13} a_{24} d_0 - 2 a_{12} a_{34} d_0 - 2 a_0 a_{1234} d_0 + a_{1234} d_0^2 - \\ & 2 a_0 a_{34} d_{12} + 2 a_{34} d_0 d_{12} + 2 a_0 a_{24} d_{13} - 2 a_{24} d_0 d_{13} - 2 a_0 a_{23} d_{14} + 2 a_{23} d_0 d_{14} - \\ & 2 a_0 a_{14} d_{23} + 2 a_{14} d_0 d_{23} + 2 a_0 a_{13} d_{24} - 2 a_{13} d_0 d_{24} - 2 a_0 d_{13} d_{24} - \\ & 2 a_0 a_{12} d_{34} + 2 a_{12} d_0 d_{34} + 2 a_0 d_{12} d_{34} - a_0^2 d_{1234} + 2 a_0 d_0 d_{1234}) \zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \zeta_4 \end{aligned}$$

In[80]:=

```
coeff3 = Flatten[{-a_0^2 d_0 + a_0 d_0^2, Coefficient[expr3, {ξ1 ∧ ξ2}],  

Coefficient[expr3, {ξ1 ∧ ξ3}], Coefficient[expr3, {ξ1 ∧ ξ4}],  

Coefficient[expr3, {ξ2 ∧ ξ3}], Coefficient[expr3, {ξ2 ∧ ξ4}],  

Coefficient[expr3, {ξ3 ∧ ξ4}], Coefficient[expr3, {ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4}]}]
```

$$\begin{aligned} \text{Out}[80]= & \left\{ -a_0^2 d_0 + a_0 d_0^2, -2 a_0 a_{12} d_0 + a_{12} d_0^2 - a_0^2 d_{12} + 2 a_0 d_0 d_{12}, \right. \\ & -2 a_0 a_{13} d_0 + a_{13} d_0^2 - a_0^2 d_{13} + 2 a_0 d_0 d_{13}, -2 a_0 a_{14} d_0 + a_{14} d_0^2 - a_0^2 d_{14} + 2 a_0 d_0 d_{14}, \\ & -2 a_0 a_{23} d_0 + a_{23} d_0^2 - a_0^2 d_{23} + 2 a_0 d_0 d_{23}, -2 a_0 a_{24} d_0 + a_{24} d_0^2 - a_0^2 d_{24} + 2 a_0 d_0 d_{24}, \\ & -2 a_0 a_{34} d_0 + a_{34} d_0^2 - a_0^2 d_{34} + 2 a_0 d_0 d_{34}, -2 a_{14} a_{23} d_0 + 2 a_{13} a_{24} d_0 - 2 a_{12} a_{34} d_0 - \\ & 2 a_0 a_{1234} d_0 + a_{1234} d_0^2 - 2 a_0 a_{34} d_{12} + 2 a_{34} d_0 d_{12} + 2 a_0 a_{24} d_{13} - 2 a_{24} d_0 d_{13} - \\ & 2 a_0 a_{23} d_{14} + 2 a_{23} d_0 d_{14} - 2 a_0 a_{14} d_{23} + 2 a_{14} d_0 d_{23} + 2 a_0 d_{14} d_{23} + 2 a_0 a_{13} d_{24} - \\ & \left. 2 a_{13} d_0 d_{24} - 2 a_0 d_{13} d_{24} - 2 a_0 a_{12} d_{34} + 2 a_{12} d_0 d_{34} + 2 a_0 d_{12} d_{34} - a_0^2 d_{1234} + 2 a_0 d_0 d_{1234} \right\} \end{aligned}$$

In[81]:=

```
expr4 = GrassmannExpandAndSimplify[(pp ∧ dd ∧ (dd - pp)) + cc ∧ bb ∧ dd]
```

$$\begin{aligned} \text{Out}[81]= & b_0 c_0 d_0 + d_0^2 p_0 - d_0 p_0^2 + \\ & (b_{12} c_0 d_0 + b_0 c_{12} d_0 + b_0 c_0 d_{12} + 2 d_0 d_{12} p_0 - d_{12} p_0^2 + d_0^2 p_{12} - 2 d_0 p_0 p_{12}) \zeta_1 \wedge \zeta_2 + \\ & (b_{13} c_0 d_0 + b_0 c_{13} d_0 + b_0 c_0 d_{13} + 2 d_0 d_{13} p_0 - d_{13} p_0^2 + d_0^2 p_{13} - 2 d_0 p_0 p_{13}) \zeta_1 \wedge \zeta_3 + \\ & (b_{14} c_0 d_0 + b_0 c_{14} d_0 + b_0 c_0 d_{14} + 2 d_0 d_{14} p_0 - d_{14} p_0^2 + d_0^2 p_{14} - 2 d_0 p_0 p_{14}) \zeta_1 \wedge \zeta_4 + \\ & (b_{23} c_0 d_0 + b_0 c_{23} d_0 + b_0 c_0 d_{23} + 2 d_0 d_{23} p_0 - d_{23} p_0^2 + d_0^2 p_{23} - 2 d_0 p_0 p_{23}) \zeta_2 \wedge \zeta_3 + \\ & (b_{24} c_0 d_0 + b_0 c_{24} d_0 + b_0 c_0 d_{24} + 2 d_0 d_{24} p_0 - d_{24} p_0^2 + d_0^2 p_{24} - 2 d_0 p_0 p_{24}) \zeta_2 \wedge \zeta_4 + \\ & (b_{34} c_0 d_0 + b_0 c_{34} d_0 + b_0 c_0 d_{34} + 2 d_0 d_{34} p_0 - d_{34} p_0^2 + d_0^2 p_{34} - 2 d_0 p_0 p_{34}) \zeta_3 \wedge \zeta_4 + \\ & (b_{1234} c_0 d_0 + b_{34} c_{12} d_0 - b_{24} c_{13} d_0 + b_{23} c_{14} d_0 + b_{14} c_{23} d_0 - b_{13} c_{24} d_0 + b_{12} c_{34} d_0 + b_0 c_{1234} d_0 + \\ & b_{34} c_0 d_{12} + b_0 c_{34} d_{12} - b_{24} c_0 d_{13} - b_0 c_{24} d_{13} + b_{23} c_0 d_{14} + b_0 c_{23} d_{14} + b_{14} c_0 d_{23} + b_0 c_{14} d_{23} - \\ & b_{13} c_0 d_{24} - b_0 c_{13} d_{24} + b_{12} c_0 d_{34} + b_0 c_{12} d_{34} + b_0 c_0 d_{1234} + 2 d_{14} d_{23} p_0 - 2 d_{13} d_{24} p_0 + \\ & 2 d_{12} d_{34} p_0 + 2 d_0 d_{1234} p_0 - d_{1234} p_0^2 + 2 d_0 d_{34} p_{12} - 2 d_{34} p_0 p_{12} - 2 d_0 d_{24} p_{13} + 2 d_{24} p_0 p_{13} + \\ & 2 d_0 d_{23} p_{14} - 2 d_{23} p_0 p_{14} + 2 d_0 d_{14} p_{23} - 2 d_{14} p_0 p_{23} - 2 d_0 p_{14} p_{23} - 2 d_0 d_{13} p_{24} + 2 d_{13} p_0 p_{24} + \\ & 2 d_0 p_{13} p_{24} + 2 d_0 d_{12} p_{34} - 2 d_{12} p_0 p_{34} - 2 d_0 p_{12} p_{34} + d_0^2 p_{1234} - 2 d_0 p_0 p_{1234}) \zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \zeta_4 \end{aligned}$$

In[82]:=

```
coeff4 = Flatten[ {b0 c0 d0 + d0^2 p0 - d0 p0^2, Coefficient[expr4, {ξ1 ^ ξ2}],
Coefficient[expr4, {ξ1 ^ ξ3}], Coefficient[expr4, {ξ1 ^ ξ4}],
Coefficient[expr4, {ξ2 ^ ξ3}], Coefficient[expr4, {ξ2 ^ ξ4}],
Coefficient[expr4, {ξ3 ^ ξ4}], Coefficient[expr4, {ξ1 ^ ξ2 ^ ξ3 ^ ξ4}] } ]
```

```
Out[82]= {b0 c0 d0 + d0^2 p0 - d0 p0^2, b12 c0 d0 + b0 c12 d0 + b0 c0 d12 + 2 d0 d12 p0 - d12 p0^2 + d0^2 p12 - 2 d0 p0 p12,
b13 c0 d0 + b0 c13 d0 + b0 c0 d13 + 2 d0 d13 p0 - d13 p0^2 + d0^2 p13 - 2 d0 p0 p13,
b14 c0 d0 + b0 c14 d0 + b0 c0 d14 + 2 d0 d14 p0 - d14 p0^2 + d0^2 p14 - 2 d0 p0 p14,
b23 c0 d0 + b0 c23 d0 + b0 c0 d23 + 2 d0 d23 p0 - d23 p0^2 + d0^2 p23 - 2 d0 p0 p23,
b24 c0 d0 + b0 c24 d0 + b0 c0 d24 + 2 d0 d24 p0 - d24 p0^2 + d0^2 p24 - 2 d0 p0 p24,
b34 c0 d0 + b0 c34 d0 + b0 c0 d34 + 2 d0 d34 p0 - d34 p0^2 + d0^2 p34 - 2 d0 p0 p34,
b1234 c0 d0 + b34 c12 d0 - b24 c13 d0 + b23 c14 d0 + b14 c23 d0 - b13 c24 d0 + b12 c34 d0 + b0 c1234 d0 +
b34 c0 d12 + b0 c34 d12 - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 + b0 c23 d14 + b14 c0 d23 + b0 c14 d23 -
b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234 + 2 d14 d23 p0 - 2 d13 d24 p0 +
2 d12 d34 p0 + 2 d0 d1234 p0 - d1234 p0^2 + 2 d0 d34 p12 - 2 d34 p0 p12 - 2 d0 d24 p13 + 2 d24 p0 p13 +
2 d0 d23 p14 - 2 d23 p0 p14 + 2 d0 d14 p23 - 2 d14 p0 p23 - 2 d0 p14 p23 - 2 d0 d13 p24 +
2 d13 p0 p24 + 2 d0 p13 p24 + 2 d0 d12 p34 - 2 d12 p0 p34 - 2 d0 p12 p34 + d0^2 p1234 - 2 d0 p0 p1234}
```

In[83]:=

```
expr5 = GrassmannExpandAndSimplify[(qq ^ dd ^ (dd - qq)) + cc ^ bb ^ dd]
```

```
Out[83]= b0 c0 d0 + d0^2 q0 - d0 q0^2 +
(b12 c0 d0 + b0 c12 d0 + b0 c0 d12 + 2 d0 d12 q0 - d12 q0^2 + d0^2 q12 - 2 d0 q0 q12) ξ1 ^ ξ2 +
(b13 c0 d0 + b0 c13 d0 + b0 c0 d13 + 2 d0 d13 q0 - d13 q0^2 + d0^2 q13 - 2 d0 q0 q13) ξ1 ^ ξ3 +
(b14 c0 d0 + b0 c14 d0 + b0 c0 d14 + 2 d0 d14 q0 - d14 q0^2 + d0^2 q14 - 2 d0 q0 q14) ξ1 ^ ξ4 +
(b23 c0 d0 + b0 c23 d0 + b0 c0 d23 + 2 d0 d23 q0 - d23 q0^2 + d0^2 q23 - 2 d0 q0 q23) ξ2 ^ ξ3 +
(b24 c0 d0 + b0 c24 d0 + b0 c0 d24 + 2 d0 d24 q0 - d24 q0^2 + d0^2 q24 - 2 d0 q0 q24) ξ2 ^ ξ4 +
(b34 c0 d0 + b0 c34 d0 + b0 c0 d34 + 2 d0 d34 q0 - d34 q0^2 + d0^2 q34 - 2 d0 q0 q34) ξ3 ^ ξ4 +
(b1234 c0 d0 + b34 c12 d0 - b24 c13 d0 + b23 c14 d0 + b14 c23 d0 - b13 c24 d0 + b12 c34 d0 + b0 c1234 d0 +
b34 c0 d12 + b0 c34 d12 - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 + b0 c23 d14 + b14 c0 d23 + b0 c14 d23 -
b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234 + 2 d14 d23 q0 - 2 d13 d24 q0 +
2 d12 d34 q0 + 2 d0 d1234 q0 - d1234 q0^2 + 2 d0 d34 q12 - 2 d34 q0 q12 - 2 d0 d24 q13 + 2 d24 q0 q13 +
2 d0 d23 q14 - 2 d23 q0 q14 + 2 d0 d14 q23 - 2 d14 q0 q23 - 2 d0 q14 q23 - 2 d0 d13 q24 + 2 d13 q0 q24 +
2 d0 q13 q24 + 2 d0 d12 q34 - 2 d12 q0 q34 - 2 d0 q12 q34 + d0^2 q1234 - 2 d0 q0 q1234) ξ1 ^ ξ2 ^ ξ3 ^ ξ4
```

In[84]:=

```
coeff5 = Flatten[ {b0 c0 d0 + d0^2 q0 - d0 q0^2, Coefficient[expr5, {ξ1 ^ ξ2}],
Coefficient[expr5, {ξ1 ^ ξ3}], Coefficient[expr5, {ξ1 ^ ξ4}],
Coefficient[expr5, {ξ2 ^ ξ3}], Coefficient[expr5, {ξ2 ^ ξ4}],
Coefficient[expr5, {ξ3 ^ ξ4}], Coefficient[expr5, {ξ1 ^ ξ2 ^ ξ3 ^ ξ4}]} ]
```

```
Out[84]= {b0 c0 d0 + d0^2 q0 - d0 q0^2, b12 c0 d0 + b0 c12 d0 + b0 c0 d12 + 2 d0 d12 q0 - d12 q0^2 + d0^2 q12 - 2 d0 q0 q12,
b13 c0 d0 + b0 c13 d0 + b0 c0 d13 + 2 d0 d13 q0 - d13 q0^2 + d0^2 q13 - 2 d0 q0 q13,
b14 c0 d0 + b0 c14 d0 + b0 c0 d14 + 2 d0 d14 q0 - d14 q0^2 + d0^2 q14 - 2 d0 q0 q14,
b23 c0 d0 + b0 c23 d0 + b0 c0 d23 + 2 d0 d23 q0 - d23 q0^2 + d0^2 q23 - 2 d0 q0 q23,
b24 c0 d0 + b0 c24 d0 + b0 c0 d24 + 2 d0 d24 q0 - d24 q0^2 + d0^2 q24 - 2 d0 q0 q24,
b34 c0 d0 + b0 c34 d0 + b0 c0 d34 + 2 d0 d34 q0 - d34 q0^2 + d0^2 q34 - 2 d0 q0 q34,
b1234 c0 d0 + b34 c12 d0 - b24 c13 d0 + b23 c14 d0 + b14 c23 d0 - b13 c24 d0 + b12 c34 d0 + b0 c1234 d0 +
b34 c0 d12 + b0 c34 d12 - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 + b0 c23 d14 + b14 c0 d23 + b0 c14 d23 -
b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234 + 2 d14 d23 q0 - 2 d13 d24 q0 +
2 d12 d34 q0 + 2 d0 d1234 q0 - d1234 q0^2 + 2 d0 d34 q12 - 2 d14 q0 q12 - 2 d0 d24 q13 + 2 d24 q0 q13 +
2 d0 d23 q14 - 2 d23 q0 q14 + 2 d0 d14 q23 - 2 d14 q0 q23 - 2 d0 q14 q23 - 2 d0 d13 q24 +
2 d13 q0 q24 + 2 d0 q13 q24 + 2 d0 d12 q34 - 2 d12 q0 q34 - 2 d0 q12 q34 + d0^2 q1234 - 2 d0 q0 q1234}
```

In[85]:=

```
expr6 = GrassmannExpandAndSimplify[(pp ^ aa ^ (aa - pp)) + cc ^ bb ^ aa]
```

```
Out[85]= a0 b0 c0 + a0^2 p0 - a0 p0^2 +
(a12 b0 c0 + a0 b12 c0 + a0 b0 c12 + 2 a0 a12 p0 - a12 p0^2 + a0^2 p12 - 2 a0 p0 p12) ξ1 ^ ξ2 +
(a13 b0 c0 + a0 b13 c0 + a0 b0 c13 + 2 a0 a13 p0 - a13 p0^2 + a0^2 p13 - 2 a0 p0 p13) ξ1 ^ ξ3 +
(a14 b0 c0 + a0 b14 c0 + a0 b0 c14 + 2 a0 a14 p0 - a14 p0^2 + a0^2 p14 - 2 a0 p0 p14) ξ1 ^ ξ4 +
(a23 b0 c0 + a0 b23 c0 + a0 b0 c23 + 2 a0 a23 p0 - a23 p0^2 + a0^2 p23 - 2 a0 p0 p23) ξ2 ^ ξ3 +
(a24 b0 c0 + a0 b24 c0 + a0 b0 c24 + 2 a0 a24 p0 - a24 p0^2 + a0^2 p24 - 2 a0 p0 p24) ξ2 ^ ξ4 +
(a34 b0 c0 + a0 b34 c0 + a0 b0 c34 + 2 a0 a34 p0 - a34 p0^2 + a0^2 p34 - 2 a0 p0 p34) ξ3 ^ ξ4 +
(a1234 b0 c0 + a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 + a12 b34 c0 + a0 b1234 c0 +
a34 b0 c12 + a0 b34 c12 - a24 b0 c13 - a0 b24 c13 + a23 b0 c14 + a0 b23 c14 + a14 b0 c23 + a0 b14 c23 -
a13 b0 c24 - a0 b13 c24 + a12 b0 c34 + a0 b12 c34 + a0 b0 c1234 + 2 a14 a23 p0 - 2 a13 a24 p0 +
2 a12 a34 p0 + 2 a0 a1234 p0 - a1234 p0^2 + 2 a0 a34 p12 - 2 a34 p0 p12 - 2 a0 a24 p13 + 2 a24 p0 p13 +
2 a0 a23 p14 - 2 a23 p0 p14 + 2 a0 a14 p23 - 2 a14 p0 p23 - 2 a0 p14 p23 - 2 a0 a13 p24 + 2 a13 p0 p24 +
2 a0 p13 p24 + 2 a0 a12 p34 - 2 a12 p0 p34 - 2 a0 p12 p34 + a0^2 p1234 - 2 a0 p0 p1234) ξ1 ^ ξ2 ^ ξ3 ^ ξ4
```

In[86]:=

```

coeff6 = Flatten[{a0 b0 c0 + a0^2 p0 - a0 p0^2, Coefficient[expr6, {ξ1 ∧ ξ2}],
Coefficient[expr6, {ξ1 ∧ ξ3}], Coefficient[expr6, {ξ1 ∧ ξ4}],
Coefficient[expr6, {ξ2 ∧ ξ3}], Coefficient[expr6, {ξ2 ∧ ξ4}],
Coefficient[expr6, {ξ3 ∧ ξ4}], Coefficient[expr6, {ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4}]}]

Out[86]= {a0 b0 c0 + a0^2 p0 - a0 p0^2, a12 b0 c0 + a0 b12 c0 + a0 b0 c12 + 2 a0 a12 p0 - a12 p0^2 + a0^2 p12 - 2 a0 p0 p12,
a13 b0 c0 + a0 b13 c0 + a0 b0 c13 + 2 a0 a13 p0 - a13 p0^2 + a0^2 p13 - 2 a0 p0 p13,
a14 b0 c0 + a0 b14 c0 + a0 b0 c14 + 2 a0 a14 p0 - a14 p0^2 + a0^2 p14 - 2 a0 p0 p14,
a23 b0 c0 + a0 b23 c0 + a0 b0 c23 + 2 a0 a23 p0 - a23 p0^2 + a0^2 p23 - 2 a0 p0 p23,
a24 b0 c0 + a0 b24 c0 + a0 b0 c24 + 2 a0 a24 p0 - a24 p0^2 + a0^2 p24 - 2 a0 p0 p24,
a34 b0 c0 + a0 b34 c0 + a0 b0 c34 + 2 a0 a34 p0 - a34 p0^2 + a0^2 p34 - 2 a0 p0 p34,
a1234 b0 c0 + a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 + a12 b34 c0 + a0 b1234 c0 +
a34 b0 c12 + a0 b34 c12 - a24 b0 c13 - a0 b24 c13 + a23 b0 c14 + a0 b23 c14 + a14 b0 c23 + a0 b14 c23 -
a13 b0 c24 - a0 b13 c24 + a12 b0 c34 + a0 b12 c34 + a0 b0 c1234 + 2 a14 a23 p0 - 2 a13 a24 p0 +
2 a12 a34 p0 + 2 a0 a1234 p0 - a1234 p0^2 + 2 a0 a34 p12 - 2 a34 p0 p12 - 2 a0 a24 p13 + 2 a24 p0 p13 +
2 a0 a23 p14 - 2 a23 p0 p14 + 2 a0 a14 p23 - 2 a14 p0 p23 - 2 a0 p14 p23 - 2 a0 a13 p24 +
2 a13 p0 p24 + 2 a0 p13 p24 + 2 a0 a12 p34 - 2 a12 p0 p34 - 2 a0 p12 p34 + a0^2 p1234 - 2 a0 p0 p1234}

```

In[87]:=

```

expr7 = GrassmannExpandAndSimplify[(qq ∧ aa ∧ (aa - qq)) + cc ∧ bb ∧ aa]

Out[87]= a0 b0 c0 + a0^2 q0 - a0 q0^2 +
(a12 b0 c0 + a0 b12 c0 + a0 b0 c12 + 2 a0 a12 q0 - a12 q0^2 + a0^2 q12 - 2 a0 q0 q12) ξ1 ∧ ξ2 +
(a13 b0 c0 + a0 b13 c0 + a0 b0 c13 + 2 a0 a13 q0 - a13 q0^2 + a0^2 q13 - 2 a0 q0 q13) ξ1 ∧ ξ3 +
(a14 b0 c0 + a0 b14 c0 + a0 b0 c14 + 2 a0 a14 q0 - a14 q0^2 + a0^2 q14 - 2 a0 q0 q14) ξ1 ∧ ξ4 +
(a23 b0 c0 + a0 b23 c0 + a0 b0 c23 + 2 a0 a23 q0 - a23 q0^2 + a0^2 q23 - 2 a0 q0 q23) ξ2 ∧ ξ3 +
(a24 b0 c0 + a0 b24 c0 + a0 b0 c24 + 2 a0 a24 q0 - a24 q0^2 + a0^2 q24 - 2 a0 q0 q24) ξ2 ∧ ξ4 +
(a34 b0 c0 + a0 b34 c0 + a0 b0 c34 + 2 a0 a34 q0 - a34 q0^2 + a0^2 q34 - 2 a0 q0 q34) ξ3 ∧ ξ4 +
(a1234 b0 c0 + a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 + a12 b34 c0 + a0 b1234 c0 +
a34 b0 c12 + a0 b34 c12 - a24 b0 c13 - a0 b24 c13 + a23 b0 c14 + a0 b23 c14 + a14 b0 c23 + a0 b14 c23 -
a13 b0 c24 - a0 b13 c24 + a12 b0 c34 + a0 b12 c34 + a0 b0 c1234 + 2 a14 a23 q0 - 2 a13 a24 q0 +
2 a12 a34 q0 + 2 a0 a1234 q0 - a1234 q0^2 + 2 a0 a34 q12 - 2 a34 q0 q12 - 2 a0 a24 q13 + 2 a24 q0 q13 +
2 a0 a23 q14 - 2 a23 q0 q14 + 2 a0 a14 q23 - 2 a14 q0 q23 - 2 a0 q14 q23 - 2 a0 a13 q24 + 2 a13 q0 q24 +
2 a0 q13 q24 + 2 a0 a12 q34 - 2 a12 q0 q34 - 2 a0 q12 q34 + a0^2 q1234 - 2 a0 q0 q1234) ξ1 ∧ ξ2 ∧ ξ3 ∧ ξ4

```

In[88]:=

```
coeff7 = Flatten[{{a0 b0 c0 + a0^2 q0 - a0 q0^2, Coefficient[expr7, {ξ1 ^ ξ2}],
Coefficient[expr7, {ξ1 ^ ξ3}], Coefficient[expr7, {ξ1 ^ ξ4}],
Coefficient[expr7, {ξ2 ^ ξ3}], Coefficient[expr7, {ξ2 ^ ξ4}],
Coefficient[expr7, {ξ3 ^ ξ4}], Coefficient[expr7, {ξ1 ^ ξ2 ^ ξ3 ^ ξ4}]}]
```

```
Out[88]= {a0 b0 c0 + a0^2 q0 - a0 q0^2, a12 b0 c0 + a0 b12 c0 + a0 b0 c12 + 2 a0 a12 q0 - a12 q0^2 + a0^2 q12 - 2 a0 q0 q12,
a13 b0 c0 + a0 b13 c0 + a0 b0 c13 + 2 a0 a13 q0 - a13 q0^2 + a0^2 q13 - 2 a0 q0 q13,
a14 b0 c0 + a0 b14 c0 + a0 b0 c14 + 2 a0 a14 q0 - a14 q0^2 + a0^2 q14 - 2 a0 q0 q14,
a23 b0 c0 + a0 b23 c0 + a0 b0 c23 + 2 a0 a23 q0 - a23 q0^2 + a0^2 q23 - 2 a0 q0 q23,
a24 b0 c0 + a0 b24 c0 + a0 b0 c24 + 2 a0 a24 q0 - a24 q0^2 + a0^2 q24 - 2 a0 q0 q24,
a34 b0 c0 + a0 b34 c0 + a0 b0 c34 + 2 a0 a34 q0 - a34 q0^2 + a0^2 q34 - 2 a0 q0 q34,
a1234 b0 c0 + a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 + a12 b34 c0 + a0 b1234 c0 +
a34 b0 c12 + a0 b34 c12 - a24 b0 c13 - a0 b24 c13 + a23 b0 c14 + a0 b23 c14 + a14 b0 c23 + a0 b14 c23 -
a13 b0 c24 - a0 b13 c24 + a12 b0 c34 + a0 b12 c34 + a0 b0 c1234 + 2 a14 a23 q0 - 2 a13 a24 q0 +
2 a12 a34 q0 + 2 a0 a1234 q0 - a1234 q0^2 + 2 a0 a34 q12 - 2 a34 q0 q12 - 2 a0 a24 q13 + 2 a24 q0 q13 +
2 a0 a23 q14 - 2 a23 q0 q14 + 2 a0 a14 q23 - 2 a14 q0 q23 - 2 a0 q14 q23 - 2 a0 a13 q24 +
2 a13 q0 q24 + 2 a0 q13 q24 + 2 a0 a12 q34 - 2 a12 q0 q34 - 2 a0 q12 q34 + a0^2 q1234 - 2 a0 q0 q1234}
```

In[89]:= (\* Exactly 56 equations \*)

```
coeffSystem = DeleteDuplicates[
Simplify[Join[coeff1, coeff2, coeff3, coeff4, coeff5, coeff6, coeff7]]]
Length[coeffSystem]
```

```
Out[89]= {a0 c0 d0, a12 c0 d0 + a0 (c12 d0 + c0 d12), a13 c0 d0 + a0 (c13 d0 + c0 d13),
a14 c0 d0 + a0 (c14 d0 + c0 d14), a23 c0 d0 + a0 (c23 d0 + c0 d23),
a24 c0 d0 + a0 (c24 d0 + c0 d24), a34 c0 d0 + a0 (c34 d0 + c0 d34),
a1234 c0 d0 - a24 c13 d0 + a23 c14 d0 + a14 c23 d0 - a13 c24 d0 + a12 c34 d0 + a0 c1234 d0 +
a0 c34 d12 + a34 (c12 d0 + c0 d12) - a24 c0 d13 - a0 c24 d13 + a23 c0 d14 + a0 c23 d14 +
a14 c0 d23 + a0 c14 d23 - a13 c0 d24 - a0 c13 d24 + a12 c0 d34 + a0 c12 d34 + a0 c0 d1234, a0 b0 d0,
a12 b0 d0 + a0 (b12 d0 + b0 d12), a13 b0 d0 + a0 (b13 d0 + b0 d13), a14 b0 d0 + a0 (b14 d0 + b0 d14),
a23 b0 d0 + a0 (b23 d0 + b0 d23), a24 b0 d0 + a0 (b24 d0 + b0 d24), a34 b0 d0 + a0 (b34 d0 + b0 d34),
a1234 b0 d0 - a24 b13 d0 + a23 b14 d0 + a14 b23 d0 - a13 b24 d0 + a12 b34 d0 + a0 b1234 d0 +
a0 b34 d12 + a34 (b12 d0 + b0 d12) - a24 b0 d13 - a0 b24 d13 + a23 b0 d14 + a0 b23 d14 +
a14 b0 d23 + a0 b14 d23 - a13 b0 d24 - a0 b13 d24 + a12 b0 d34 + a0 b12 d34 + a0 b0 d1234,
a0 d0 (-a0 + d0), a12 d0^2 - a0^2 d12 + 2 a0 d0 (-a12 + d12), a13 d0^2 - a0^2 d13 + 2 a0 d0 (-a13 + d13),
a14 d0^2 - a0^2 d14 + 2 a0 d0 (-a14 + d14), a23 d0^2 - a0^2 d23 + 2 a0 d0 (-a23 + d23),
a24 d0^2 - a0^2 d24 + 2 a0 d0 (-a24 + d24), a34 d0^2 - a0^2 d34 + 2 a0 d0 (-a34 + d34),
-2 a12 a34 d0 - 2 a0 a1234 d0 + a1234 d0^2 - 2 a0 a34 d12 + 2 a34 d0 d12 + 2 a0 a24 d13 - 2 a24 d0 d13 -
2 a0 a23 d14 + 2 a23 d0 d14 + 2 a0 d14 d23 - 2 a14 (a23 d0 + (a0 - d0) d23) - 2 a0 d13 d24 +
2 a13 (a24 d0 + (a0 - d0) d24) - 2 a0 a12 d34 + 2 a12 d0 d34 + 2 a0 d12 d34 - a0^2 d1234 + 2 a0 d0 d1234,
d0 (b0 c0 + (d0 - p0) p0), b12 c0 d0 + b0 (c12 d0 + c0 d12) + 2 d0 d12 p0 - d12 p0^2 + d0^2 p12 - 2 d0 p0 p12,
b13 c0 d0 + b0 (c13 d0 + c0 d13) + 2 d0 d13 p0 - d13 p0^2 + d0^2 p13 - 2 d0 p0 p13,
b14 c0 d0 + b0 (c14 d0 + c0 d14) + 2 d0 d14 p0 - d14 p0^2 + d0^2 p14 - 2 d0 p0 p14,
b23 c0 d0 + b0 (c23 d0 + c0 d23) + 2 d0 d23 p0 - d23 p0^2 + d0^2 p23 - 2 d0 p0 p23,
b24 c0 d0 + b0 (c24 d0 + c0 d24) + 2 d0 d24 p0 - d24 p0^2 + d0^2 p24 - 2 d0 p0 p24,
b34 c0 d0 + b0 (c34 d0 + c0 d34) + 2 d0 d34 p0 - d34 p0^2 + d0^2 p34 - 2 d0 p0 p34,
b1234 c0 d0 - b24 c13 d0 + b23 c14 d0 + b14 c23 d0 - b13 c24 d0 + b12 c34 d0 + b0 c1234 d0 + b0 c34 d12 +
b34 (c12 d0 + c0 d12) - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 + b0 c23 d14 + b14 c0 d23 + b0 c14 d23 -
b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234 + 2 d14 d23 p0 - 2 d13 d24 p0 +
2 d12 d34 p0 + 2 d0 d1234 p0 - d1234 p0^2 + 2 d0 d34 p12 - 2 d34 p0 p12 - 2 d0 d24 p13 + 2 d24 p0 p13 +
```

$$\begin{aligned}
& 2 d_0 d_{23} p_{14} - 2 d_{23} p_0 p_{14} + 2 d_0 d_{14} p_{23} - 2 d_{14} p_0 p_{23} - 2 d_0 p_{14} p_{23} - 2 d_0 d_{13} p_{24} + \\
& 2 d_{13} p_0 p_{24} + 2 d_0 p_{13} p_{24} + 2 d_0 d_{12} p_{34} - 2 d_{12} p_0 p_{34} - 2 d_0 p_{12} p_{34} + d_0^2 p_{1234} - 2 d_0 p_0 p_{1234}, \\
& d_0 (b_0 c_0 + (d_0 - q_0) q_0), b_{12} c_0 d_0 + b_0 (c_{12} d_0 + c_0 d_{12}) + 2 d_0 d_{12} q_0 - d_{12} q_0^2 + d_0^2 q_{12} - 2 d_0 q_0 q_{12}, \\
& b_{13} c_0 d_0 + b_0 (c_{13} d_0 + c_0 d_{13}) + 2 d_0 d_{13} q_0 - d_{13} q_0^2 + d_0^2 q_{13} - 2 d_0 q_0 q_{13}, \\
& b_{14} c_0 d_0 + b_0 (c_{14} d_0 + c_0 d_{14}) + 2 d_0 d_{14} q_0 - d_{14} q_0^2 + d_0^2 q_{14} - 2 d_0 q_0 q_{14}, \\
& b_{23} c_0 d_0 + b_0 (c_{23} d_0 + c_0 d_{23}) + 2 d_0 d_{23} q_0 - d_{23} q_0^2 + d_0^2 q_{23} - 2 d_0 q_0 q_{23}, \\
& b_{24} c_0 d_0 + b_0 (c_{24} d_0 + c_0 d_{24}) + 2 d_0 d_{24} q_0 - d_{24} q_0^2 + d_0^2 q_{24} - 2 d_0 q_0 q_{24}, \\
& b_{34} c_0 d_0 + b_0 (c_{34} d_0 + c_0 d_{34}) + 2 d_0 d_{34} q_0 - d_{34} q_0^2 + d_0^2 q_{34} - 2 d_0 q_0 q_{34}, \\
& b_{1234} c_0 d_0 - b_{24} c_{13} d_0 + b_{23} c_{14} d_0 + b_{14} c_{23} d_0 - b_{13} c_{24} d_0 + b_{12} c_{34} d_0 + b_0 c_{1234} d_0 + b_0 c_{34} d_{12} + \\
& b_{34} (c_{12} d_0 + c_0 d_{12}) - b_{24} c_0 d_{13} - b_0 c_{24} d_{13} + b_{23} c_0 d_{14} + b_0 c_{23} d_{14} + b_{14} c_0 d_{23} + b_0 c_{14} d_{23} - \\
& b_{13} c_0 d_{24} - b_0 c_{13} d_{24} + b_{12} c_0 d_{34} + b_0 c_{12} d_{34} + b_0 c_0 d_{1234} + 2 d_{14} d_{23} q_0 - 2 d_{13} d_{24} q_0 + \\
& 2 d_{12} d_{34} q_0 + 2 d_0 d_{1234} q_0 - d_{1234} q_0^2 + 2 d_0 d_{34} q_{12} - 2 d_{34} q_0 q_{12} - 2 d_0 d_{24} q_{13} + 2 d_{24} q_0 q_{13} + \\
& 2 d_0 d_{23} q_{14} - 2 d_{23} q_0 q_{14} + 2 d_0 d_{14} q_{23} - 2 d_{14} q_0 q_{23} - 2 d_0 q_{14} q_{23} - 2 d_0 d_{13} q_{24} + \\
& 2 d_{13} q_0 q_{24} + 2 d_0 q_{13} q_{24} + 2 d_0 d_{12} q_{34} - 2 d_{12} q_0 q_{34} - 2 d_0 q_{12} q_{34} + d_0^2 q_{1234} - 2 d_0 q_0 q_{1234}, \\
& a_0 (b_0 c_0 + (a_0 - p_0) p_0), a_{12} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{12} c_0 + b_0 c_{12} + (a_0 - 2 p_0) p_{12}), \\
& a_{13} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{13} c_0 + b_0 c_{13} + (a_0 - 2 p_0) p_{13}), \\
& a_{14} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{14} c_0 + b_0 c_{14} + (a_0 - 2 p_0) p_{14}), \\
& a_{23} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{23} c_0 + b_0 c_{23} + (a_0 - 2 p_0) p_{23}), \\
& a_{24} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{24} c_0 + b_0 c_{24} + (a_0 - 2 p_0) p_{24}), \\
& a_{34} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{34} c_0 + b_0 c_{34} + (a_0 - 2 p_0) p_{34}), \\
& - a_{24} b_{13} c_0 + a_{23} b_{14} c_0 + a_{14} b_{23} c_0 - a_{13} b_{24} c_0 + a_{12} b_{34} c_0 + a_0 b_{1234} c_0 + a_0 b_{34} c_{12} - a_{24} b_0 c_{13} - \\
& a_0 b_{24} c_{13} + a_{23} b_0 c_{14} + a_0 b_{23} c_{14} + a_{14} b_0 c_{23} + a_0 b_{14} c_{23} - a_{13} b_0 c_{24} - a_0 b_{13} c_{24} + a_{12} b_0 c_{34} + \\
& a_0 b_{12} c_{34} + a_0 b_0 c_{1234} + 2 a_{14} a_{23} p_0 - 2 a_{13} a_{24} p_0 + a_{1234} (b_0 c_0 + (2 a_0 - p_0) p_0) + \\
& a_{34} (b_{12} c_0 + b_0 c_{12} + 2 a_{12} p_0 + 2 a_0 p_{12} - 2 p_0 p_{12}) - 2 a_0 a_{24} p_{13} + 2 a_{24} p_0 p_{13} + \\
& 2 a_0 a_{23} p_{14} - 2 a_{23} p_0 p_{14} + 2 a_0 a_{14} p_{23} - 2 a_{14} p_0 p_{23} - 2 a_0 p_{14} p_{23} - 2 a_0 a_{13} p_{24} + \\
& 2 a_{13} p_0 p_{24} + 2 a_0 p_{13} p_{24} + 2 a_0 a_{12} p_{34} - 2 a_{12} p_0 p_{34} - 2 a_0 p_{12} p_{34} + a_0^2 p_{1234} - 2 a_0 p_0 p_{1234}, \\
& a_0 (b_0 c_0 + (a_0 - q_0) q_0), a_{12} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{12} c_0 + b_0 c_{12} + (a_0 - 2 q_0) q_{12}), \\
& a_{13} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{13} c_0 + b_0 c_{13} + (a_0 - 2 q_0) q_{13}), \\
& a_{14} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{14} c_0 + b_0 c_{14} + (a_0 - 2 q_0) q_{14}), \\
& a_{23} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{23} c_0 + b_0 c_{23} + (a_0 - 2 q_0) q_{23}), \\
& a_{24} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{24} c_0 + b_0 c_{24} + (a_0 - 2 q_0) q_{24}), \\
& a_{34} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{34} c_0 + b_0 c_{34} + (a_0 - 2 q_0) q_{34}), \\
& - a_{24} b_{13} c_0 + a_{23} b_{14} c_0 + a_{14} b_{23} c_0 - a_{13} b_{24} c_0 + a_{12} b_{34} c_0 + a_0 b_{1234} c_0 + a_0 b_{34} c_{12} - a_{24} b_0 c_{13} - \\
& a_0 b_{24} c_{13} + a_{23} b_0 c_{14} + a_0 b_{23} c_{14} + a_{14} b_0 c_{23} + a_0 b_{14} c_{23} - a_{13} b_0 c_{24} - a_0 b_{13} c_{24} + a_{12} b_0 c_{34} + \\
& a_0 b_{12} c_{34} + a_0 b_0 c_{1234} + 2 a_{14} a_{23} q_0 - 2 a_{13} a_{24} q_0 + a_{1234} (b_0 c_0 + (2 a_0 - q_0) q_0) + \\
& a_{34} (b_{12} c_0 + b_0 c_{12} + 2 a_{12} q_0 + 2 a_0 q_{12} - 2 q_0 q_{12}) - 2 a_0 a_{24} q_{13} + 2 a_{24} q_0 q_{13} + \\
& 2 a_0 a_{23} q_{14} - 2 a_{23} q_0 q_{14} + 2 a_0 a_{14} q_{23} - 2 a_{14} q_0 q_{23} - 2 a_0 q_{14} q_{23} - 2 a_0 a_{13} q_{24} + \\
& 2 a_{13} q_0 q_{24} + 2 a_0 q_{13} q_{24} + 2 a_0 a_{12} q_{34} - 2 a_{12} q_0 q_{34} - 2 a_0 q_{12} q_{34} + a_0^2 q_{1234} - 2 a_0 q_0 q_{1234} \}
\end{aligned}$$

Out[90]= 56

```
In[91]:= (* exactly 49 equations *)
caseI = DeleteCases[DeleteDuplicates[
  coeffSystem /. {c0 → 0, b0 → 0, d0 → a0, q0 → a0, p0 → a0}], _Integer]
Length[
caseI]

Out[91]= {a0^2 c12, a0^2 c13, a0^2 c14, a0^2 c23, a0^2 c24, a0^2 c34,
a0 a34 c12 - a0 a24 c13 + a0 a23 c14 + a0 a14 c23 - a0 a13 c24 + a0 a12 c34 + a0^2 c1234 + a0 c34 d12 -
a0 c24 d13 + a0 c23 d14 + a0 c14 d23 - a0 c13 d24 + a0 c12 d34, a0^2 b12, a0^2 b13, a0^2 b14, a0^2 b23,
a0^2 b24, a0^2 b34, a0 a34 b12 - a0 a24 b13 + a0 a23 b14 + a0 a14 b23 - a0 a13 b24 + a0 a12 b34 +
a0^2 b1234 + a0 b34 d12 - a0 b24 d13 + a0 b23 d14 + a0 b14 d23 - a0 b13 d24 + a0 b12 d34,
a0^2 a12 - a0^2 d12 + 2 a0^2 (-a12 + d12), a0^2 a13 - a0^2 d13 + 2 a0^2 (-a13 + d13),
a0^2 a14 - a0^2 d14 + 2 a0^2 (-a14 + d14), a0^2 a23 - a0^2 d23 + 2 a0^2 (-a23 + d23),
a0^2 a24 - a0^2 d24 + 2 a0^2 (-a24 + d24), a0^2 a34 - a0^2 d34 + 2 a0^2 (-a34 + d34),
-2 a0 a14 a23 + 2 a0 a13 a24 - 2 a0 a12 a34 - a0^2 a1234 + 2 a0 d14 d23 - 2 a0 d13 d24 + 2 a0 d12 d34 + a0^2 d1234,
a0^2 d12 - a0^2 p12, a0^2 d13 - a0^2 p13, a0^2 d14 - a0^2 p14, a0^2 d23 - a0^2 p23, a0^2 d24 - a0^2 p24, a0^2 d34 - a0^2 p34,
a0 b34 c12 - a0 b24 c13 + a0 b23 c14 + a0 b14 c23 - a0 b13 c24 + a0 b12 c34 + 2 a0 d14 d23 -
2 a0 d13 d24 + 2 a0 d12 d34 + a0^2 d1234 - 2 a0 p14 p23 + 2 a0 p13 p24 - 2 a0 p12 p34 - a0^2 p1234,
a0^2 d12 - a0^2 q12, a0^2 d13 - a0^2 q13, a0^2 d14 - a0^2 q14, a0^2 d23 - a0^2 q23, a0^2 d24 - a0^2 q24, a0^2 d34 - a0^2 q34,
a0 b34 c12 - a0 b24 c13 + a0 b23 c14 + a0 b14 c23 - a0 b13 c24 + a0 b12 c34 + 2 a0 d14 d23 -
2 a0 d13 d24 + 2 a0 d12 d34 + a0^2 d1234 - 2 a0 q14 q23 + 2 a0 q13 q24 - 2 a0 q12 q34 - a0^2 q1234,
a0^2 a12 - a0^2 p12, a0^2 a13 - a0^2 p13, a0^2 a14 - a0^2 p14, a0^2 a23 - a0^2 p23, a0^2 a24 - a0^2 p24, a0^2 a34 - a0^2 p34,
2 a0 a14 a23 - 2 a0 a13 a24 + 2 a0 a12 a34 + a0^2 a1234 + a0 b34 c12 - a0 b24 c13 + a0 b23 c14 +
a0 b14 c23 - a0 b13 c24 + a0 b12 c34 - 2 a0 p14 p23 + 2 a0 p13 p24 - 2 a0 p12 p34 - a0^2 p1234,
a0^2 a12 - a0^2 q12, a0^2 a13 - a0^2 q13, a0^2 a14 - a0^2 q14, a0^2 a23 - a0^2 q23, a0^2 a24 - a0^2 q24, a0^2 a34 - a0^2 q34,
2 a0 a14 a23 - 2 a0 a13 a24 + 2 a0 a12 a34 + a0^2 a1234 + a0 b34 c12 - a0 b24 c13 + a0 b23 c14 +
a0 b14 c23 - a0 b13 c24 + a0 b12 c34 - 2 a0 q14 q23 + 2 a0 q13 q24 - 2 a0 q12 q34 - a0^2 q1234}
```

Out[92]= 49

**Case I:**  $c_0 \rightarrow 0, b_0 \rightarrow 0, d_0 \rightarrow a_0, q_0 \rightarrow a_0, p_0 \rightarrow a_0$ 

In[93]:= solCaseI = Solve[caseI == 0, scalars]

```
Out[93]= { {a0 → 0}, {a0 → 0, a12 → d12, a13 → d13, a14 → d14, b12 → 0, b13 → 0, b14 → 0,
b23 → 0, b24 → 0, b34 → 0, c12 → 0, c13 → 0, c14 → 0, c23 → 0, c24 → 0, c34 → 0,
d23 → a23, d24 → a24, d34 → a34, p12 → d12, p13 → d13, p14 → d14, p23 → a23, p24 → a24,
p34 → a34, q12 → d12, q13 → d13, q14 → d14, q23 → a23, q24 → a24, q34 → a34},
{a12 → d12, a13 → d13, a14 → d14, b12 → 0, b13 → 0, b14 → 0, b23 → 0, b24 → 0, b34 → 0,
b1234 → 0, c12 → 0, c13 → 0, c14 → 0, c23 → 0, c24 → 0, c34 → 0, c1234 → 0, d23 → a23, d24 → a24,
d34 → a34, d1234 → a1234, p12 → d12, p13 → d13, p14 → d14, p23 → a23, p24 → a24, p34 → a34,
p1234 → a1234, q12 → d12, q13 → d13, q14 → d14, q23 → a23, q24 → a24, q34 → a34, q1234 → a1234} }
```

```
In[94]:= (*  $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 \rightarrow \xi_1 \xi_2 \xi_3 \xi_4$  has to be first in the list of subs *)
Column[Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
RGrassmannFinal = Flatten[RGrassmann /. {solCaseI[[i]]}, 1];
Style[RGrassmannFinal /.
{ $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 \rightarrow \xi_1 \xi_2 \xi_3 \xi_4$ ,  $\xi_1 \wedge \xi_2 \rightarrow \xi_1 \xi_2$ ,  $\xi_1 \wedge \xi_3 \rightarrow \xi_1 \xi_3$ ,  $\xi_2 \wedge \xi_3 \rightarrow \xi_2 \xi_3$ ,
 $\xi_2 \wedge \xi_4 \rightarrow \xi_2 \xi_4$ ,  $\xi_1 \wedge \xi_4 \rightarrow \xi_1 \xi_4$ ,  $\xi_3 \wedge \xi_4 \rightarrow \xi_3 \xi_4}$  // MatrixForm, FontSize → 6],
{i, 1, Length[solCaseI]}]]]

Out[94]= 
$$\left\{ \begin{array}{l} P_0 + P_{12} \xi_1 \xi_2 + P_{13} \xi_1 \xi_3 + P_{23} \xi_2 \xi_3 + P_{14} \xi_1 \xi_4 + P_{24} \xi_2 \xi_4 + P_{34} \xi_3 \xi_4 + P_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ 0 \\ 0 \\ 0 \end{array} \right. \begin{array}{l} 0 \\ c_{01} + c_{12} \xi_1 \xi_2 + c_{13} \xi_1 \xi_3 + c_{23} \xi_2 \xi_3 + c_{14} \xi_1 \xi_4 + c_{24} \xi_2 \xi_4 + c_{34} \xi_3 \xi_4 + c_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ a_{12} \xi_1 \xi_2 + a_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 + a_{14} \xi_1 \xi_4 + a_{24} \xi_2 \xi_4 + a_{34} \xi_3 \xi_4 + a_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ 0 \end{array} \begin{array}{l} d_0 + d_{12} \xi_1 \xi_2 + d_{13} \xi_1 \xi_3 + d_{23} \xi_2 \xi_3 \\ b_0 + b_{12} \xi_1 \xi_2 + b_{13} \xi_1 \xi_3 + b_{23} \xi_2 \xi_3 \end{array}$$


$$\left\{ \begin{array}{l} P_0 + d_{12} \xi_1 \xi_2 + d_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 + a_{14} \xi_1 \xi_4 + a_{24} \xi_2 \xi_4 + a_{34} \xi_3 \xi_4 + P_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ 0 \\ 0 \\ 0 \end{array} \right. \begin{array}{l} 0 \\ c_0 + c_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ d_{12} \xi_1 \xi_2 + d_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 + d_{14} \xi_1 \xi_4 + a_{24} \xi_2 \xi_4 + a_{34} \xi_3 \xi_4 + a_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ 0 \end{array} \begin{array}{l} d_0 + d_{12} \xi_1 \xi_2 + d_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 \\ b_0 + \end{array}$$


$$\left\{ \begin{array}{l} P_0 + d_{12} \xi_1 \xi_2 + d_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 + d_{14} \xi_1 \xi_4 + a_{24} \xi_2 \xi_4 + a_{34} \xi_3 \xi_4 + a_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ 0 \\ 0 \\ 0 \end{array} \right. \begin{array}{l} 0 \\ c_0 \\ a_0 + d_{12} \xi_1 \xi_2 + d_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 + d_{14} \xi_1 \xi_4 + a_{24} \xi_2 \xi_4 + a_{34} \xi_3 \xi_4 + a_{1234} \xi_1 \xi_2 \xi_3 \xi_4 \\ 0 \end{array} \begin{array}{l} d_0 + d_{12} \xi_1 \xi_2 + d_{13} \xi_1 \xi_3 + a_{23} \xi_2 \xi_3 \\ \end{array}$$

```

```
In[95]:= caseII = DeleteCases[
  DeleteDuplicates[coeffSystem //. {a0 → 0, d0 → a0, q0 → a0, p0 → a0}], _Integer]
Length[
  caseII]

Out[95]= {a34 c0 d12 - a24 c0 d13 + a23 c0 d14 + a14 c0 d23 - a13 c0 d24 + a12 c0 d34,
  a34 b0 d12 - a24 b0 d13 + a23 b0 d14 + a14 b0 d23 - a13 b0 d24 + a12 b0 d34,
  b0 c0 d12, b0 c0 d13, b0 c0 d14, b0 c0 d23, b0 c0 d24, b0 c0 d34,
  b34 c0 d12 + b0 c34 d12 - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 + b0 c23 d14 + b14 c0 d23 + b0 c14 d23 -
  b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234, a12 b0 c0, a13 b0 c0, a14 b0 c0,
  a23 b0 c0, a24 b0 c0, a34 b0 c0, a1234 b0 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 +
  a12 b34 c0 + a34 (b12 c0 + b0 c12) - a24 b0 c13 + a23 b0 c14 + a14 b0 c23 - a13 b0 c24 + a12 b0 c34}
```

Let's assume  $b$  and  $c$  are non-zero.

```
In[97]:= Factor[caseII]

Out[97]= {c0 (a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34),  

b0 (a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34), b0 c0 d12, b0 c0 d13, b0 c0 d14,  

b0 c0 d23, b0 c0 d24, b0 c0 d34, b34 c0 d12 + b0 c34 d12 - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 +  

b0 c23 d14 + b14 c0 d23 + b0 c14 d23 - b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234,  

a12 b0 c0, a13 b0 c0, a14 b0 c0, a23 b0 c0, a24 b0 c0, a34 b0 c0,  

a1234 b0 c0 + a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 +  

a12 b34 c0 + a34 b0 c12 - a24 b0 c13 + a23 b0 c14 + a14 b0 c23 - a13 b0 c24 + a12 b0 c34}
```

```
In[98]:= caseIII1 = DeleteDuplicates[ { (a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34) ,
  (a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34) , b0 c0 d12 , b0 c0 d13 , b0 c0 d14 ,
  b0 c0 d23 , b0 c0 d24 , b0 c0 d34 , b34 c0 d12 + b0 c34 d12 - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 +
  b0 c23 d14 + b14 c0 d23 + b0 c14 d23 - b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234 ,
  a12 b0 c0 , a13 b0 c0 , a14 b0 c0 , a23 b0 c0 , a24 b0 c0 , a34 b0 c0 ,
  a1234 b0 c0 + a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 + a12 b34 c0 +
  a34 b0 c12 - a24 b0 c13 + a23 b0 c14 + a14 b0 c23 - a13 b0 c24 + a12 b0 c34} ]
Out[98]= {a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34 ,
  b0 c0 d12 , b0 c0 d13 , b0 c0 d14 , b0 c0 d23 , b0 c0 d24 , b0 c0 d34 ,
  b34 c0 d12 + b0 c34 d12 - b24 c0 d13 - b0 c24 d13 + b23 c0 d14 + b0 c23 d14 + b14 c0 d23 + b0 c14 d23 -
  b13 c0 d24 - b0 c13 d24 + b12 c0 d34 + b0 c12 d34 + b0 c0 d1234 , a12 b0 c0 , a13 b0 c0 , a14 b0 c0 ,
  a23 b0 c0 , a24 b0 c0 , a34 b0 c0 + a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 -
  a13 b24 c0 + a12 b34 c0 + a34 b0 c12 - a24 b0 c13 + a23 b0 c14 + a14 b0 c23 - a13 b0 c24 + a12 b0 c34}
```

We get a simple solution where  $b_0 \rightarrow 0$ ,  $c_0 \rightarrow 0$  which is against our assumptions, and  $a_{1234} \rightarrow 0$ ,  $d_{1234} \rightarrow 0$  which is not a desirable solution since we need to be of 4th degree , see following:

```
In[99]:= solCaseIII1 =
  Solve[(caseIII1 /. {d12 → 0, d13 → 0, d14 → 0, d23 → 0, d24 → 0, d34 → 0, a12 → 0, a13 → 0,
    a14 → 0, a23 → 0, a24 → 0, a34 → 0}) == 0, scalars]
Out[99]= { {b0 → 0} , {c0 → 0} , {a1234 → 0, d1234 → 0} }
```

Let's go back and assume  $b_0 \rightarrow 0$  and  $c_0$  non-zero:

```
In[100]:= caseII2 = DeleteCases[DeleteDuplicates[
  coeffSystem //.{a0 → 0, d0 → a0, q0 → a0, p0 → a0, b0 → 0}], _Integer]
Out[100]= {a34 c0 d12 - a24 c0 d13 + a23 c0 d14 + a14 c0 d23 - a13 c0 d24 + a12 c0 d34 ,
  b34 c0 d12 - b24 c0 d13 + b23 c0 d14 + b14 c0 d23 - b13 c0 d24 + b12 c0 d34 ,
  a34 b12 c0 - a24 b13 c0 + a23 b14 c0 + a14 b23 c0 - a13 b24 c0 + a12 b34 c0}
In[101]:= Factor[caseII2]
Out[101]= {c0 (a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34) ,
  c0 (b34 d12 - b24 d13 + b23 d14 + b14 d23 - b13 d24 + b12 d34) ,
  (a34 b12 - a24 b13 + a23 b14 + a14 b23 - a13 b24 + a12 b34) c0}
```

Remove  $c_0$ :

```
In[102]:= caseII2 = DeleteDuplicates[{ (a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34) ,
  (b34 d12 - b24 d13 + b23 d14 + b14 d23 - b13 d24 + b12 d34) ,
  (a34 b12 - a24 b13 + a23 b14 + a14 b23 - a13 b24 + a12 b34) }]
Out[102]= {a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34 ,
  b34 d12 - b24 d13 + b23 d14 + b14 d23 - b13 d24 + b12 d34 ,
  a34 b12 - a24 b13 + a23 b14 + a14 b23 - a13 b24 + a12 b34}
```

Too few equations but we can find nice enough relationship between them for parametric solutions:

```
In[103]:= solCaseIII2 = Solve[caseIII2 == 0, {a34, b34, a24}]
Out[103]= 
$$\left\{ \begin{array}{l} a_{34} \rightarrow -\frac{a_{12} d_{13} (-b_{24} d_{13} + b_{23} d_{14} + b_{14} d_{23} - b_{13} d_{24} + b_{12} d_{34})}{d_{12} (b_{13} d_{12} - b_{12} d_{13})} - \\ \quad - \frac{(a_{23} b_{14} + a_{14} b_{23} - a_{13} b_{24}) d_{13} + b_{13} (a_{23} d_{14} + a_{14} d_{23} - a_{13} d_{24} + a_{12} d_{34})}{b_{13} d_{12} - b_{12} d_{13}}, \\ b_{34} \rightarrow -\frac{-b_{24} d_{13} + b_{23} d_{14} + b_{14} d_{23} - b_{13} d_{24} + b_{12} d_{34}}{d_{12}}, \\ a_{24} \rightarrow -\frac{1}{b_{13} d_{12} - b_{12} d_{13}} (-a_{23} b_{14} d_{12} - a_{14} b_{23} d_{12} + a_{13} b_{24} d_{12} - a_{12} b_{24} d_{13} + a_{23} b_{12} d_{14} + \\ \quad a_{12} b_{23} d_{14} + a_{14} b_{12} d_{23} + a_{12} b_{14} d_{23} - a_{13} b_{12} d_{24} - a_{12} b_{13} d_{24} + 2 a_{12} b_{12} d_{34}) \end{array} \right\}$$

```

Let's go back and assume  $c_0 \rightarrow 0$  and  $b_0$  non-zero:

```
In[104]:= caseIII3 = DeleteCases[DeleteDuplicates[
coeffSystem //. { $a_0 \rightarrow 0$ ,  $d_0 \rightarrow a_0$ ,  $q_0 \rightarrow a_0$ ,  $p_0 \rightarrow a_0$ ,  $c_0 \rightarrow 0$ }], _Integer]
Out[104]= { $a_{34} b_0 d_{12} - a_{24} b_0 d_{13} + a_{23} b_0 d_{14} + a_{14} b_0 d_{23} - a_{13} b_0 d_{24} + a_{12} b_0 d_{34}$ ,
 $b_0 c_{34} d_{12} - b_0 c_{24} d_{13} + b_0 c_{23} d_{14} + b_0 c_{14} d_{23} - b_0 c_{13} d_{24} + b_0 c_{12} d_{34}$ ,
 $a_{34} b_0 c_{12} - a_{24} b_0 c_{13} + a_{23} b_0 c_{14} + a_{14} b_0 c_{23} - a_{13} b_0 c_{24} + a_{12} b_0 c_{34}$ }
```

```
In[105]:= Factor[caseIII3]
Out[105]= { $b_0 (a_{34} d_{12} - a_{24} d_{13} + a_{23} d_{14} + a_{14} d_{23} - a_{13} d_{24} + a_{12} d_{34})$ ,
 $b_0 (c_{34} d_{12} - c_{24} d_{13} + c_{23} d_{14} + c_{14} d_{23} - c_{13} d_{24} + c_{12} d_{34})$ ,
 $b_0 (a_{34} c_{12} - a_{24} c_{13} + a_{23} c_{14} + a_{14} c_{23} - a_{13} c_{24} + a_{12} c_{34})$ }
```

Remove  $b_0$ :

```
In[106]:= caseIII3 = DeleteDuplicates[{ (a34 d12 - a24 d13 + a23 d14 + a14 d23 - a13 d24 + a12 d34),
(c34 d12 - c24 d13 + c23 d14 + c14 d23 - c13 d24 + c12 d34),
(a34 c12 - a24 c13 + a23 c14 + a14 c23 - a13 c24 + a12 c34) }]
Out[106]= { $a_{34} d_{12} - a_{24} d_{13} + a_{23} d_{14} + a_{14} d_{23} - a_{13} d_{24} + a_{12} d_{34}$ ,
 $c_{34} d_{12} - c_{24} d_{13} + c_{23} d_{14} + c_{14} d_{23} - c_{13} d_{24} + c_{12} d_{34}$ ,
 $a_{34} c_{12} - a_{24} c_{13} + a_{23} c_{14} + a_{14} c_{23} - a_{13} c_{24} + a_{12} c_{34}$ }
```

Too few equations but we can find nice enough relationship between them for parametric solutions:

```
In[107]:= solCaseIII3 = Solve[caseIII3 == 0, {a34, c34, a24}]
Out[107]= 
$$\left\{ \begin{array}{l} a_{34} \rightarrow -\frac{a_{12} d_{13} (-c_{24} d_{13} + c_{23} d_{14} + c_{14} d_{23} - c_{13} d_{24} + c_{12} d_{34})}{d_{12} (c_{13} d_{12} - c_{12} d_{13})} - \\ \quad (- (a_{23} c_{14} + a_{14} c_{23} - a_{13} c_{24}) d_{13} + c_{13} (a_{23} d_{14} + a_{14} d_{23} - a_{13} d_{24} + a_{12} d_{34})) / \\ \quad (c_{13} d_{12} - c_{12} d_{13}), c_{34} \rightarrow -\frac{-c_{24} d_{13} + c_{23} d_{14} + c_{14} d_{23} - c_{13} d_{24} + c_{12} d_{34}}{d_{12}}, \\ a_{24} \rightarrow -\frac{1}{c_{13} d_{12} - c_{12} d_{13}} (-a_{23} c_{14} d_{12} - a_{14} c_{23} d_{12} + a_{13} c_{24} d_{12} - a_{12} c_{24} d_{13} + a_{23} c_{12} d_{14} + \\ \quad a_{12} c_{23} d_{14} + a_{14} c_{12} d_{23} + a_{12} c_{14} d_{23} - a_{13} c_{12} d_{24} - a_{12} c_{13} d_{24} + 2 a_{12} c_{12} d_{34}) \end{array} \right\}$$

```