QYBE solutions for constant R-Matrices over Grassmann Algebra, 3-generators

Based upon the work of Steven Duplij, Olga Kotulska and Alexander Sadovnikov [1]

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In[158]:=
```
Clear[a, b, c, d, p, q];
```

Assume a, b, c, d, p and q belong to a multiplicative commutative distributive algebra, with both additive and multiplicative inverses.

Based upon the 6-vertex YBE solutions [3], the following matrix forms are created:

In[159]:=
```
R =.;
R = ConstantArray[0, {4, 4}];
R[[1]][[1]] = p;
R[[2]][[2]] = c;
R[[3]][[3]] = b;
R[[4]][[4]] = q;
R[[2]][[3]] = d;
R[[3]][[2]] = a;
R // MatrixForm
```

Out[167]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & c & d & 0 \\ 0 & a & b & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

In[168]:=

```
R12 =.;
R12 = ConstantArray[0, {8, 8}];
R12[[1]][[1]] = p;
R12[[2]][[2]] = p;
R12[[3]][[3]] = c;
R12[[4]][[4]] = c;
R12[[5]][[5]] = b;
R12[[6]][[6]] = b;
R12[[7]][[7]] = q;
R12[[8]][[8]] = q;
R12[[3]][[5]] = d;
R12[[4]][[6]] = d;
R12[[5]][[3]] = a;
R12[[6]][[4]] = a;
R12 // MatrixForm
```

Out[182]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & d & 0 & 0 \\ 0 & 0 & a & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

In[183]:=

```
R13 =.;
R13 = ConstantArray[0, {8, 8}];
R13[[1]][[1]] = p;
R13[[2]][[2]] = c;
R13[[3]][[3]] = p;
R13[[4]][[4]] = c;
R13[[5]][[5]] = b;
R13[[6]][[6]] = q;
R13[[7]][[7]] = b;
R13[[8]][[8]] = q;
R13[[2]][[5]] = d;
R13[[4]][[7]] = d;
R13[[5]][[2]] = a;
R13[[7]][[4]] = a;
R13 // MatrixForm
```

Out[197]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & d & 0 \\ 0 & a & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

In[198]:=

```
R23 =.;
R23 = ConstantArray[0, {8, 8}];
R23[[1]][[1]] = p;
R23[[2]][[2]] = c;
R23[[3]][[3]] = b;
R23[[4]][[4]] = q;
R23[[5]][[5]] = p;
R23[[6]][[6]] = c;
R23[[7]][[7]] = b;
R23[[8]][[8]] = q;
R23[[2]][[3]] = d;
R23[[6]][[7]] = d;
R23[[3]][[2]] = a;
R23[[7]][[6]] = a;
R23 // MatrixForm
```

Out[212]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

From the braid equation get the algebraic equations that must satisfy 0:

In[213]:=

```
equations = Simplify[R12.R13.R23 - R23.R13.R12];
Style[equations // MatrixForm, FontSize → 9]
```

Out[214]=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a\,c\,d & 0 & d\,(-b\,c + p\,(-d + p)) & 0 & 0 & 0 \\ 0 & a\,c\,d & a\,d\,(-a + d) & 0 & -a\,b\,d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a\,c\,d & d\,(b\,c + (d - q)\,q) & 0 \\ 0 & a\,(b\,c + (a - p)\,p) & a\,b\,d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a\,c\,d & 0 & a\,(a - d)\,d & a\,b\,d & 0 \\ 0 & 0 & 0 & a\,(-b\,c + q\,(-a + q)) & 0 & -a\,b\,d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From [1] check the computations:

$$cda = 0, \qquad (28)$$

$$bda = 0, \qquad (29)$$

$$da(d-a) = 0,\ pd(d-p) + cbd = 0, \qquad (30)$$

$$\left.\begin{array}{l} qd(d-q) + cbd = 0, \\ pa(a-p) + cba = 0, \\ qa(a-q) + cba = 0. \end{array}\right\} \qquad (31)$$

Grassmann Algebra 3 generators

In[215]:=

```
★A; DeclareVectorSymbols[ξ1, ξ2, ξ3]
scalars = DeclareScalarSymbols[a0, b0, c0, d0, p0, q0, a12, b12,
  c12, d12, p12, q12, a13, b13, c13, d13, p13, q13, a23, b23, c23, d23, p23, q23]
X = ★G[(1 + ξ1) ∧ (1 + ξ2) ∧ (1 + ξ3)]
```

Out[215]= $\{\xi_1, \xi_2, \xi_3\}$

Out[216]= $\{a_0, a_{12}, a_{13}, a_{23}, b_0, b_{12}, b_{13}, b_{23}, c_0, c_{12}, c_{13}, c_{23}, d_0, d_{12}, d_{13}, d_{23}, p_0, p_{12}, p_{13}, p_{23}, q_0, q_{12}, q_{13}, q_{23}\}$

Out[217]= $1 + \xi_1 + \xi_2 + \xi_3 + \xi_1 \wedge \xi_2 + \xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_3 + \xi_1 \wedge \xi_2 \wedge \xi_3$

In[218]:=

```
(* assign the 0s to only get the even terms *)
aa =.;
aa = {a0, 0, 0, 0, a12, a13, a23 , 0}.List @@ X

bb =.;
bb = {b0, 0, 0, 0, b12, b13, b23 , 0}.List @@ X

cc =.;
cc = {c0, 0, 0, 0, c12, c13, c23 , 0}.List @@ X

dd =.;
dd = {d0, 0, 0, 0, d12, d13, d23 , 0}.List @@ X

pp =.;
pp = {p0, 0, 0, 0, p12, p13, p23 , 0}.List @@ X

qq =.;
qq = {q0, 0, 0, 0, q12, q13, q23 , 0}.List @@ X
```

Out[219]= $a_0 + a_{12}\, \xi_1 \wedge \xi_2 + a_{13}\, \xi_1 \wedge \xi_3 + a_{23}\, \xi_2 \wedge \xi_3$

Out[221]= $b_0 + b_{12}\, \xi_1 \wedge \xi_2 + b_{13}\, \xi_1 \wedge \xi_3 + b_{23}\, \xi_2 \wedge \xi_3$

Out[223]= $c_0 + c_{12}\, \xi_1 \wedge \xi_2 + c_{13}\, \xi_1 \wedge \xi_3 + c_{23}\, \xi_2 \wedge \xi_3$

Out[225]= $d_0 + d_{12}\, \xi_1 \wedge \xi_2 + d_{13}\, \xi_1 \wedge \xi_3 + d_{23}\, \xi_2 \wedge \xi_3$

Out[227]= $p_0 + p_{12}\, \xi_1 \wedge \xi_2 + p_{13}\, \xi_1 \wedge \xi_3 + p_{23}\, \xi_2 \wedge \xi_3$

Out[229]= $q_0 + q_{12}\, \xi_1 \wedge \xi_2 + q_{13}\, \xi_1 \wedge \xi_3 + q_{23}\, \xi_2 \wedge \xi_3$

In[230]:=

```
expr1 = GrassmannExpandAndSimplify[cc ∧ dd ∧ aa]
```

Out[230]= $a_0\, c_0\, d_0 + (a_{12}\, c_0\, d_0 + a_0\, c_{12}\, d_0 + a_0\, c_0\, d_{12})\, \zeta_1 \wedge \zeta_2 + (a_{13}\, c_0\, d_0 + a_0\, c_{13}\, d_0 + a_0\, c_0\, d_{13})\, \zeta_1 \wedge \zeta_3 + (a_{23}\, c_0\, d_0 + a_0\, c_{23}\, d_0 + a_0\, c_0\, d_{23})\, \zeta_2 \wedge \zeta_3$

Collect the coefficients:

In[231]:=

```
coeff1 = Flatten[{a0 c0 d0, Coefficient[expr1, {ζ1 ∧ ζ2}],
   Coefficient[expr1, {ζ1 ∧ ζ3}], Coefficient[expr1, {ζ2 ∧ ζ3}]}]
```

Out[231]= $\{a_0\, c_0\, d_0,\ a_{12}\, c_0\, d_0 + a_0\, c_{12}\, d_0 + a_0\, c_0\, d_{12},\ a_{13}\, c_0\, d_0 + a_0\, c_{13}\, d_0 + a_0\, c_0\, d_{13},\ a_{23}\, c_0\, d_0 + a_0\, c_{23}\, d_0 + a_0\, c_0\, d_{23}\}$

In[232]:=

```
expr2 = GrassmannExpandAndSimplify[bb ∧ dd ∧ aa]
```

Out[232]= $a_0\, b_0\, d_0 + (a_{12}\, b_0\, d_0 + a_0\, b_{12}\, d_0 + a_0\, b_0\, d_{12})\, \zeta_1 \wedge \zeta_2 + (a_{13}\, b_0\, d_0 + a_0\, b_{13}\, d_0 + a_0\, b_0\, d_{13})\, \zeta_1 \wedge \zeta_3 + (a_{23}\, b_0\, d_0 + a_0\, b_{23}\, d_0 + a_0\, b_0\, d_{23})\, \zeta_2 \wedge \zeta_3$

In[233]:=

```
coeff2 = Flatten[{a0 b0 d0, Coefficient[expr2, {ζ1 ∧ ζ2}],
   Coefficient[expr2, {ζ1 ∧ ζ3}], Coefficient[expr2, {ζ2 ∧ ζ3}]}]
```

Out[233]= $\{a_0\, b_0\, d_0,\ a_{12}\, b_0\, d_0 + a_0\, b_{12}\, d_0 + a_0\, b_0\, d_{12},\ a_{13}\, b_0\, d_0 + a_0\, b_{13}\, d_0 + a_0\, b_0\, d_{13},\ a_{23}\, b_0\, d_0 + a_0\, b_{23}\, d_0 + a_0\, b_0\, d_{23}\}$

In[234]:=

```
expr3  = GrassmannExpandAndSimplify[dd ∧ aa ∧ (dd - aa)]
```

Out[234]= $-a_0^2\, d_0 + a_0\, d_0^2 + \left(-2\, a_0\, a_{12}\, d_0 + a_{12}\, d_0^2 - a_0^2\, d_{12} + 2\, a_0\, d_0\, d_{12}\right)\, \zeta_1 \wedge \zeta_2 + \left(-2\, a_0\, a_{13}\, d_0 + a_{13}\, d_0^2 - a_0^2\, d_{13} + 2\, a_0\, d_0\, d_{13}\right)\, \zeta_1 \wedge \zeta_3 + \left(-2\, a_0\, a_{23}\, d_0 + a_{23}\, d_0^2 - a_0^2\, d_{23} + 2\, a_0\, d_0\, d_{23}\right)\, \zeta_2 \wedge \zeta_3$

In[235]:=

```
coeff3 = Flatten[{-a0^2 d0 + a0 d0^2, Coefficient[expr3, {ζ1 ∧ ζ2}],
   Coefficient[expr3, {ζ1 ∧ ζ3}], Coefficient[expr3, {ζ2 ∧ ζ3}]}]
```

Out[235]= $\left\{-a_0^2\, d_0 + a_0\, d_0^2,\ -2\, a_0\, a_{12}\, d_0 + a_{12}\, d_0^2 - a_0^2\, d_{12} + 2\, a_0\, d_0\, d_{12},\ -2\, a_0\, a_{13}\, d_0 + a_{13}\, d_0^2 - a_0^2\, d_{13} + 2\, a_0\, d_0\, d_{13},\ -2\, a_0\, a_{23}\, d_0 + a_{23}\, d_0^2 - a_0^2\, d_{23} + 2\, a_0\, d_0\, d_{23}\right\}$

In[236]:=

```
expr4 = GrassmannExpandAndSimplify[(pp ∧ dd ∧ (dd - pp)) + cc ∧ bb ∧ dd]
```

Out[236]= $b_0\, c_0\, d_0 + d_0^2\, p_0 - d_0\, p_0^2 + \left(b_{12}\, c_0\, d_0 + b_0\, c_{12}\, d_0 + b_0\, c_0\, d_{12} + 2\, d_0\, d_{12}\, p_0 - d_{12}\, p_0^2 + d_0^2\, p_{12} - 2\, d_0\, p_0\, p_{12}\right)\, \zeta_1 \wedge \zeta_2 + \left(b_{13}\, c_0\, d_0 + b_0\, c_{13}\, d_0 + b_0\, c_0\, d_{13} + 2\, d_0\, d_{13}\, p_0 - d_{13}\, p_0^2 + d_0^2\, p_{13} - 2\, d_0\, p_0\, p_{13}\right)\, \zeta_1 \wedge \zeta_3 + \left(b_{23}\, c_0\, d_0 + b_0\, c_{23}\, d_0 + b_0\, c_0\, d_{23} + 2\, d_0\, d_{23}\, p_0 - d_{23}\, p_0^2 + d_0^2\, p_{23} - 2\, d_0\, p_0\, p_{23}\right)\, \zeta_2 \wedge \zeta_3$

In[237]:=

```
coeff4 = Flatten[{b0 c0 d0 + d0^2 p0 - d0 p0^2, Coefficient[expr4, {ξ1 ∧ ξ2}],
   Coefficient[expr4, {ξ1 ∧ ξ3}], Coefficient[expr4, {ξ2 ∧ ξ3}]}]
```

Out[237]=

$$\{b_0 c_0 d_0 + d_0^2 p_0 - d_0 p_0^2,\ b_{12} c_0 d_0 + b_0 c_{12} d_0 + b_0 c_0 d_{12} + 2 d_0 d_{12} p_0 - d_{12} p_0^2 + d_0^2 p_{12} - 2 d_0 p_0 p_{12},\ b_{13} c_0 d_0 + b_0 c_{13} d_0 + b_0 c_0 d_{13} + 2 d_0 d_{13} p_0 - d_{13} p_0^2 + d_0^2 p_{13} - 2 d_0 p_0 p_{13},\ b_{23} c_0 d_0 + b_0 c_{23} d_0 + b_0 c_0 d_{23} + 2 d_0 d_{23} p_0 - d_{23} p_0^2 + d_0^2 p_{23} - 2 d_0 p_0 p_{23}\}$$

In[238]:=

```
expr5 = GrassmannExpandAndSimplify[(qq ∧ dd ∧ (dd - qq)) + cc ∧ bb ∧ dd]
```

Out[238]=

$$b_0 c_0 d_0 + d_0^2 q_0 - d_0 q_0^2 + (b_{12} c_0 d_0 + b_0 c_{12} d_0 + b_0 c_0 d_{12} + 2 d_0 d_{12} q_0 - d_{12} q_0^2 + d_0^2 q_{12} - 2 d_0 q_0 q_{12})\ \zeta_1 \wedge \zeta_2 + (b_{13} c_0 d_0 + b_0 c_{13} d_0 + b_0 c_0 d_{13} + 2 d_0 d_{13} q_0 - d_{13} q_0^2 + d_0^2 q_{13} - 2 d_0 q_0 q_{13})\ \zeta_1 \wedge \zeta_3 + (b_{23} c_0 d_0 + b_0 c_{23} d_0 + b_0 c_0 d_{23} + 2 d_0 d_{23} q_0 - d_{23} q_0^2 + d_0^2 q_{23} - 2 d_0 q_0 q_{23})\ \zeta_2 \wedge \zeta_3$$

In[239]:=

```
coeff5 = Flatten[{b0 c0 d0 + d0^2 q0 - d0 q0^2, Coefficient[expr5, {ξ1 ∧ ξ2}],
   Coefficient[expr5, {ξ1 ∧ ξ3}], Coefficient[expr5, {ξ2 ∧ ξ3}]}]
```

Out[239]=

$$\{b_0 c_0 d_0 + d_0^2 q_0 - d_0 q_0^2,\ b_{12} c_0 d_0 + b_0 c_{12} d_0 + b_0 c_0 d_{12} + 2 d_0 d_{12} q_0 - d_{12} q_0^2 + d_0^2 q_{12} - 2 d_0 q_0 q_{12},\ b_{13} c_0 d_0 + b_0 c_{13} d_0 + b_0 c_0 d_{13} + 2 d_0 d_{13} q_0 - d_{13} q_0^2 + d_0^2 q_{13} - 2 d_0 q_0 q_{13},\ b_{23} c_0 d_0 + b_0 c_{23} d_0 + b_0 c_0 d_{23} + 2 d_0 d_{23} q_0 - d_{23} q_0^2 + d_0^2 q_{23} - 2 d_0 q_0 q_{23}\}$$

In[240]:=

```
expr6 = GrassmannExpandAndSimplify[(pp ∧ aa ∧ (aa - pp)) + cc ∧ bb ∧ aa]
```

Out[240]=

$$a_0 b_0 c_0 + a_0^2 p_0 - a_0 p_0^2 + (a_{12} b_0 c_0 + a_0 b_{12} c_0 + a_0 b_0 c_{12} + 2 a_0 a_{12} p_0 - a_{12} p_0^2 + a_0^2 p_{12} - 2 a_0 p_0 p_{12})\ \zeta_1 \wedge \zeta_2 + (a_{13} b_0 c_0 + a_0 b_{13} c_0 + a_0 b_0 c_{13} + 2 a_0 a_{13} p_0 - a_{13} p_0^2 + a_0^2 p_{13} - 2 a_0 p_0 p_{13})\ \zeta_1 \wedge \zeta_3 + (a_{23} b_0 c_0 + a_0 b_{23} c_0 + a_0 b_0 c_{23} + 2 a_0 a_{23} p_0 - a_{23} p_0^2 + a_0^2 p_{23} - 2 a_0 p_0 p_{23})\ \zeta_2 \wedge \zeta_3$$

In[242]:=

```
coeff6 = Flatten[{a0 b0 c0 + a0^2 p0 - a0 p0^2, Coefficient[expr6, {ξ1 ∧ ξ2}],
   Coefficient[expr6, {ξ1 ∧ ξ3}], Coefficient[expr6, {ξ2 ∧ ξ3}]}]
```

Out[242]=

$$\{a_0 b_0 c_0 + a_0^2 p_0 - a_0 p_0^2,\ a_{12} b_0 c_0 + a_0 b_{12} c_0 + a_0 b_0 c_{12} + 2 a_0 a_{12} p_0 - a_{12} p_0^2 + a_0^2 p_{12} - 2 a_0 p_0 p_{12},\ a_{13} b_0 c_0 + a_0 b_{13} c_0 + a_0 b_0 c_{13} + 2 a_0 a_{13} p_0 - a_{13} p_0^2 + a_0^2 p_{13} - 2 a_0 p_0 p_{13},\ a_{23} b_0 c_0 + a_0 b_{23} c_0 + a_0 b_0 c_{23} + 2 a_0 a_{23} p_0 - a_{23} p_0^2 + a_0^2 p_{23} - 2 a_0 p_0 p_{23}\}$$

In[243]:=

```
expr7 = GrassmannExpandAndSimplify[(qq ∧ aa ∧ (aa - qq)) + cc ∧ bb ∧ aa]
```

Out[243]=

$$a_0 b_0 c_0 + a_0^2 q_0 - a_0 q_0^2 + (a_{12} b_0 c_0 + a_0 b_{12} c_0 + a_0 b_0 c_{12} + 2 a_0 a_{12} q_0 - a_{12} q_0^2 + a_0^2 q_{12} - 2 a_0 q_0 q_{12})\ \zeta_1 \wedge \zeta_2 + (a_{13} b_0 c_0 + a_0 b_{13} c_0 + a_0 b_0 c_{13} + 2 a_0 a_{13} q_0 - a_{13} q_0^2 + a_0^2 q_{13} - 2 a_0 q_0 q_{13})\ \zeta_1 \wedge \zeta_3 + (a_{23} b_0 c_0 + a_0 b_{23} c_0 + a_0 b_0 c_{23} + 2 a_0 a_{23} q_0 - a_{23} q_0^2 + a_0^2 q_{23} - 2 a_0 q_0 q_{23})\ \zeta_2 \wedge \zeta_3$$

In[244]:=

```
coeff7 = Flatten[{a0 b0 c0 + a0^2 q0 - a0 q0^2, Coefficient[expr7, {ξ1 ∧ ξ2}],
   Coefficient[expr7, {ξ1 ∧ ξ3}], Coefficient[expr7, {ξ2 ∧ ξ3}]}]
```

Out[244]= $\{a_0\, b_0\, c_0 + a_0^2\, q_0 - a_0\, q_0^2,\ a_{12}\, b_0\, c_0 + a_0\, b_{12}\, c_0 + a_0\, b_0\, c_{12} + 2\, a_0\, a_{12}\, q_0 - a_{12}\, q_0^2 + a_0^2\, q_{12} - 2\, a_0\, q_0\, q_{12},$
$a_{13}\, b_0\, c_0 + a_0\, b_{13}\, c_0 + a_0\, b_0\, c_{13} + 2\, a_0\, a_{13}\, q_0 - a_{13}\, q_0^2 + a_0^2\, q_{13} - 2\, a_0\, q_0\, q_{13},$
$a_{23}\, b_0\, c_0 + a_0\, b_{23}\, c_0 + a_0\, b_0\, c_{23} + 2\, a_0\, a_{23}\, q_0 - a_{23}\, q_0^2 + a_0^2\, q_{23} - 2\, a_0\, q_0\, q_{23}\}$

In[245]:=

```
coeffSystem = DeleteDuplicates[
  Simplify[Join[coeff1, coeff2, coeff3, coeff4, coeff5, coeff6, coeff7]]]
Length[coeffSystem]
```

Out[245]= $\{a_0\, c_0\, d_0,\ a_{12}\, c_0\, d_0 + a_0\, (c_{12}\, d_0 + c_0\, d_{12}),\ a_{13}\, c_0\, d_0 + a_0\, (c_{13}\, d_0 + c_0\, d_{13}),$
$a_{23}\, c_0\, d_0 + a_0\, (c_{23}\, d_0 + c_0\, d_{23}),\ a_0\, b_0\, d_0,\ a_{12}\, b_0\, d_0 + a_0\, (b_{12}\, d_0 + b_0\, d_{12}),$
$a_{13}\, b_0\, d_0 + a_0\, (b_{13}\, d_0 + b_0\, d_{13}),\ a_{23}\, b_0\, d_0 + a_0\, (b_{23}\, d_0 + b_0\, d_{23}),\ a_0\, d_0\, (-a_0 + d_0),$
$a_{12}\, d_0^2 - a_0^2\, d_{12} + 2\, a_0\, d_0\, (-a_{12} + d_{12}),\ a_{13}\, d_0^2 - a_0^2\, d_{13} + 2\, a_0\, d_0\, (-a_{13} + d_{13}),$
$a_{23}\, d_0^2 - a_0^2\, d_{23} + 2\, a_0\, d_0\, (-a_{23} + d_{23}),\ d_0\, (b_0\, c_0 + (d_0 - p_0)\, p_0),$
$b_{12}\, c_0\, d_0 + b_0\, (c_{12}\, d_0 + c_0\, d_{12}) + 2\, d_0\, d_{12}\, p_0 - d_{12}\, p_0^2 + d_0^2\, p_{12} - 2\, d_0\, p_0\, p_{12},$
$b_{13}\, c_0\, d_0 + b_0\, (c_{13}\, d_0 + c_0\, d_{13}) + 2\, d_0\, d_{13}\, p_0 - d_{13}\, p_0^2 + d_0^2\, p_{13} - 2\, d_0\, p_0\, p_{13},$
$b_{23}\, c_0\, d_0 + b_0\, (c_{23}\, d_0 + c_0\, d_{23}) + 2\, d_0\, d_{23}\, p_0 - d_{23}\, p_0^2 + d_0^2\, p_{23} - 2\, d_0\, p_0\, p_{23},$
$d_0\, (b_0\, c_0 + (d_0 - q_0)\, q_0),\ b_{12}\, c_0\, d_0 + b_0\, (c_{12}\, d_0 + c_0\, d_{12}) + 2\, d_0\, d_{12}\, q_0 - d_{12}\, q_0^2 + d_0^2\, q_{12} - 2\, d_0\, q_0\, q_{12},$
$b_{13}\, c_0\, d_0 + b_0\, (c_{13}\, d_0 + c_0\, d_{13}) + 2\, d_0\, d_{13}\, q_0 - d_{13}\, q_0^2 + d_0^2\, q_{13} - 2\, d_0\, q_0\, q_{13},$
$b_{23}\, c_0\, d_0 + b_0\, (c_{23}\, d_0 + c_0\, d_{23}) + 2\, d_0\, d_{23}\, q_0 - d_{23}\, q_0^2 + d_0^2\, q_{23} - 2\, d_0\, q_0\, q_{23},$
$a_0\, (b_0\, c_0 + (a_0 - p_0)\, p_0),\ a_{12}\, (b_0\, c_0 + (2\, a_0 - p_0)\, p_0) + a_0\, (b_{12}\, c_0 + b_0\, c_{12} + (a_0 - 2\, p_0)\, p_{12}),$
$a_{13}\, (b_0\, c_0 + (2\, a_0 - p_0)\, p_0) + a_0\, (b_{13}\, c_0 + b_0\, c_{13} + (a_0 - 2\, p_0)\, p_{13}),$
$a_{23}\, (b_0\, c_0 + (2\, a_0 - p_0)\, p_0) + a_0\, (b_{23}\, c_0 + b_0\, c_{23} + (a_0 - 2\, p_0)\, p_{23}),$
$a_0\, (b_0\, c_0 + (a_0 - q_0)\, q_0),\ a_{12}\, (b_0\, c_0 + (2\, a_0 - q_0)\, q_0) + a_0\, (b_{12}\, c_0 + b_0\, c_{12} + (a_0 - 2\, q_0)\, q_{12}),$
$a_{13}\, (b_0\, c_0 + (2\, a_0 - q_0)\, q_0) + a_0\, (b_{13}\, c_0 + b_0\, c_{13} + (a_0 - 2\, q_0)\, q_{13}),$
$a_{23}\, (b_0\, c_0 + (2\, a_0 - q_0)\, q_0) + a_0\, (b_{23}\, c_0 + b_0\, c_{23} + (a_0 - 2\, q_0)\, q_{23})\}$

Out[246]= 28

In[247]:=

```
(* exactly 12 equations *)
caseI = DeleteCases[DeleteDuplicates[coeffSystem /. {d0 → 0, a0 → 0}], _Integer]
Length[caseI]
```

Out[247]= $\{b_0\, c_0\, d_{12} - d_{12}\, p_0^2,\ b_0\, c_0\, d_{13} - d_{13}\, p_0^2,\ b_0\, c_0\, d_{23} - d_{23}\, p_0^2,\ b_0\, c_0\, d_{12} - d_{12}\, q_0^2,$
$b_0\, c_0\, d_{13} - d_{13}\, q_0^2,\ b_0\, c_0\, d_{23} - d_{23}\, q_0^2,\ a_{12}\, (b_0\, c_0 - p_0^2),\ a_{13}\, (b_0\, c_0 - p_0^2),$
$a_{23}\, (b_0\, c_0 - p_0^2),\ a_{12}\, (b_0\, c_0 - q_0^2),\ a_{13}\, (b_0\, c_0 - q_0^2),\ a_{23}\, (b_0\, c_0 - q_0^2)\}$

Out[248]= 12

In[249]:=

```
solCaseI = Solve[caseI == 0, scalars]
```

Out[249]= $\{\{p_0 \to -\sqrt{b_0}\,\sqrt{c_0},\ q_0 \to -\sqrt{b_0}\,\sqrt{c_0}\},\ \{p_0 \to -\sqrt{b_0}\,\sqrt{c_0},\ q_0 \to \sqrt{b_0}\,\sqrt{c_0}\},$
$\{p_0 \to \sqrt{b_0}\,\sqrt{c_0},\ q_0 \to -\sqrt{b_0}\,\sqrt{c_0}\},\ \{p_0 \to \sqrt{b_0}\,\sqrt{c_0},\ q_0 \to \sqrt{b_0}\,\sqrt{c_0}\},$
$\{a_{12} \to 0,\ a_{13} \to 0,\ a_{23} \to 0,\ d_{12} \to 0,\ d_{13} \to 0,\ d_{23} \to 0\}\}$

In[250]:=

```
Column[Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
  RGrassmannFinal = Flatten[RGrassmann /. {solCaseI[[i]]}, 1];
  Style[RGrassmannFinal /. {a0 → 0, d0 → 0, b0 → (t^2) / r, p0 → t, q0 → t,
      c0 → r, c12 → v, b12 → w, d12 → y, p12 → l, q12 → m, a12 → n, ζ1 ∧ ζ2 -> ζ1 ζ2,
      ζ1 ∧ ζ3 -> ζ1 ζ3, ζ2 ∧ ζ3 -> ζ2 ζ3} // MatrixForm, FontSize → 6],
  {i, 1, Length[solCaseI]}]]
```

Out[250]=

$$\begin{pmatrix} -\sqrt{r}\sqrt{\frac{t^2}{r}} + l\,\zeta_1\zeta_2 + p_{13}\,\zeta_1\zeta_3 + p_{23}\,\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 + c_{13}\,\zeta_1\zeta_3 + c_{23}\,\zeta_2\zeta_3 & y\,\zeta_1\zeta_2 + d_{13}\,\zeta_1\zeta_3 + d_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & n\,\zeta_1\zeta_2 + a_{13}\,\zeta_1\zeta_3 + a_{23}\,\zeta_2\zeta_3 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 + b_{13}\,\zeta_1\zeta_3 + b_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & -\sqrt{r}\sqrt{\frac{t^2}{r}} + m\,\zeta_1\zeta_2 + q_{13}\,\zeta_1\zeta_3 + q_{23}\,\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} -\sqrt{r}\sqrt{\frac{t^2}{r}} + l\,\zeta_1\zeta_2 + p_{13}\,\zeta_1\zeta_3 + p_{23}\,\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 + c_{13}\,\zeta_1\zeta_3 + c_{23}\,\zeta_2\zeta_3 & y\,\zeta_1\zeta_2 + d_{13}\,\zeta_1\zeta_3 + d_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & n\,\zeta_1\zeta_2 + a_{13}\,\zeta_1\zeta_3 + a_{23}\,\zeta_2\zeta_3 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 + b_{13}\,\zeta_1\zeta_3 + b_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & \sqrt{r}\sqrt{\frac{t^2}{r}} + m\,\zeta_1\zeta_2 + q_{13}\,\zeta_1\zeta_3 + q_{23}\,\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{r}\sqrt{\frac{t^2}{r}} + l\,\zeta_1\zeta_2 + p_{13}\,\zeta_1\zeta_3 + p_{23}\,\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 + c_{13}\,\zeta_1\zeta_3 + c_{23}\,\zeta_2\zeta_3 & y\,\zeta_1\zeta_2 + d_{13}\,\zeta_1\zeta_3 + d_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & n\,\zeta_1\zeta_2 + a_{13}\,\zeta_1\zeta_3 + a_{23}\,\zeta_2\zeta_3 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 + b_{13}\,\zeta_1\zeta_3 + b_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & -\sqrt{r}\sqrt{\frac{t^2}{r}} + m\,\zeta_1\zeta_2 + q_{13}\,\zeta_1\zeta_3 + q_{23}\,\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{r}\sqrt{\frac{t^2}{r}} + l\,\zeta_1\zeta_2 + p_{13}\,\zeta_1\zeta_3 + p_{23}\,\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 + c_{13}\,\zeta_1\zeta_3 + c_{23}\,\zeta_2\zeta_3 & y\,\zeta_1\zeta_2 + d_{13}\,\zeta_1\zeta_3 + d_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & n\,\zeta_1\zeta_2 + a_{13}\,\zeta_1\zeta_3 + a_{23}\,\zeta_2\zeta_3 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 + b_{13}\,\zeta_1\zeta_3 + b_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & \sqrt{r}\sqrt{\frac{t^2}{r}} + m\,\zeta_1\zeta_2 + q_{13}\,\zeta_1\zeta_3 + q_{23}\,\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} t + l\,\zeta_1\zeta_2 + p_{13}\,\zeta_1\zeta_3 + p_{23}\,\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 + c_{13}\,\zeta_1\zeta_3 + c_{23}\,\zeta_2\zeta_3 & 0 & 0 \\ 0 & 0 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 + b_{13}\,\zeta_1\zeta_3 + b_{23}\,\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & t + m\,\zeta_1\zeta_2 + q_{13}\,\zeta_1\zeta_3 + q_{23}\,\zeta_2\zeta_3 \end{pmatrix}$$

In[251]:=

```
(* exactly 17 equations *)
caseIII =
 DeleteCases[DeleteDuplicates[coeffSystem /. {d0 → 0, b0 → 0, c0 → 0}], _Integer]
Length[caseIII]
```

Out[251]=

$$\{-a_0^2\, d_{12},\ -a_0^2\, d_{13},\ -a_0^2\, d_{23},\ -d_{12}\, p_0^2,\ -d_{13}\, p_0^2,\ -d_{23}\, p_0^2,\ -d_{12}\, q_0^2,\ -d_{13}\, q_0^2,\ -d_{23}\, q_0^2,\ a_0\,(a_0 - p_0)\, p_0,\ a_{12}\,(2\,a_0 - p_0)\, p_0 + a_0\,(a_0 - 2\,p_0)\, p_{12},\ a_{13}\,(2\,a_0 - p_0)\, p_0 + a_0\,(a_0 - 2\,p_0)\, p_{13},\ a_{23}\,(2\,a_0 - p_0)\, p_0 + a_0\,(a_0 - 2\,p_0)\, p_{23},\ a_0\,(a_0 - q_0)\, q_0,\ a_{12}\,(2\,a_0 - q_0)\, q_0 + a_0\,(a_0 - 2\,q_0)\, q_{12},\ a_{13}\,(2\,a_0 - q_0)\, q_0 + a_0\,(a_0 - 2\,q_0)\, q_{13},\ a_{23}\,(2\,a_0 - q_0)\, q_0 + a_0\,(a_0 - 2\,q_0)\, q_{23}\}$$

Out[252]= 17

In[253]:= **solCaseIII = Solve[caseIII == 0, scalars]**

Out[253]= $\{\{a_0 \to 0, p_0 \to 0, q_0 \to 0\}, \{a_0 \to 0, d_{12} \to 0, d_{13} \to 0, d_{23} \to 0, p_0 \to 0, q_0 \to 0\},$
$\{a_0 \to 0, a_{12} \to 0, a_{13} \to 0, a_{23} \to 0, d_{12} \to 0, d_{13} \to 0, d_{23} \to 0\},$
$\{a_0 \to 0, a_{12} \to 0, a_{13} \to 0, a_{23} \to 0, d_{12} \to 0, d_{13} \to 0, d_{23} \to 0, q_0 \to 0\},$
$\{d_{12} \to 0, d_{13} \to 0, d_{23} \to 0, p_0 \to 0, p_{12} \to 0, p_{13} \to 0, p_{23} \to 0, q_0 \to 0, q_{12} \to 0, q_{13} \to 0, q_{23} \to 0\},$
$\{d_{12} \to 0, d_{13} \to 0, d_{23} \to 0, p_0 \to a_0, p_{12} \to a_{12}, p_{13} \to a_{13}, p_{23} \to a_{23}, q_0 \to 0,$
$q_{12} \to 0, q_{13} \to 0, q_{23} \to 0\}, \{d_{12} \to 0, d_{13} \to 0, d_{23} \to 0, p_0 \to 0, p_{12} \to 0, p_{13} \to 0,$
$p_{23} \to 0, q_0 \to a_0, q_{12} \to a_{12}, q_{13} \to a_{13}, q_{23} \to a_{23}\}, \{d_{12} \to 0, d_{13} \to 0, d_{23} \to 0,$
$p_0 \to a_0, p_{12} \to a_{12}, p_{13} \to a_{13}, p_{23} \to a_{23}, q_0 \to a_0, q_{12} \to a_{12}, q_{13} \to a_{13}, q_{23} \to a_{23}\}\}$

In[254]:=

$$\begin{aligned}&\mathbf{Column[Table[RGrassmann = R\ /.\ \{a \to aa,\ b \to bb,\ c \to cc,\ d \to dd,\ p \to pp,\ q \to qq\};}\\
&\quad\mathbf{RGrassmannFinal = Flatten[RGrassmann\ /.\ \{solCaseIII[[i]]\}, 1];}\\
&\quad\mathbf{Style[RGrassmannFinal\ /.\ \{a_0 \to 0,\ d_0 \to 0,\ b_0 \to 0,\ p_0 \to t,\ q_0 \to t,\ c_0 \to 0,}\\
&\qquad\mathbf{c_{12} \to v,\ b_{12} \to w,\ d_{12} \to y,\ p_{12} \to l,\ q_{12} \to m,\ a_{12} \to n,\ \zeta_1 \wedge \zeta_2 \text{ -> } \zeta_1\,\zeta_2,}\\
&\qquad\mathbf{\zeta_1 \wedge \zeta_3 \text{ -> } \zeta_1\,\zeta_3,\ \zeta_2 \wedge \zeta_3 \text{ -> } \zeta_2\,\zeta_3\}\ //\ MatrixForm,\ FontSize \to 6],}\\
&\quad\mathbf{\{i, 1, Length[solCaseIII]\}]]}\end{aligned}$$

Out[254]=

$$\begin{pmatrix} l\zeta_1\zeta_2 + p_{13}\zeta_1\zeta_3 + p_{23}\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & y\zeta_1\zeta_2 + d_{13}\zeta_1\zeta_3 + d_{23}\zeta_2\zeta_3 & 0 \\ 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & m\zeta_1\zeta_2 + q_{13}\zeta_1\zeta_3 + q_{23}\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} l\zeta_1\zeta_2 + p_{13}\zeta_1\zeta_3 + p_{23}\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & 0 & 0 \\ 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & m\zeta_1\zeta_2 + q_{13}\zeta_1\zeta_3 + q_{23}\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} t + l\zeta_1\zeta_2 + p_{13}\zeta_1\zeta_3 + p_{23}\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & 0 & 0 \\ 0 & 0 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & t + m\zeta_1\zeta_2 + q_{13}\zeta_1\zeta_3 + q_{23}\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} t + l\zeta_1\zeta_2 + p_{13}\zeta_1\zeta_3 + p_{23}\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & 0 & 0 \\ 0 & 0 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & m\zeta_1\zeta_2 + q_{13}\zeta_1\zeta_3 + q_{23}\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & 0 & 0 \\ 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & 0 & 0 \\ 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & 0 & 0 \\ 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & 0 & 0 & 0 \\ 0 & v\zeta_1\zeta_2 + c_{13}\zeta_1\zeta_3 + c_{23}\zeta_2\zeta_3 & 0 & 0 \\ 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 & w\zeta_1\zeta_2 + b_{13}\zeta_1\zeta_3 + b_{23}\zeta_2\zeta_3 & 0 \\ 0 & 0 & 0 & n\zeta_1\zeta_2 + a_{13}\zeta_1\zeta_3 + a_{23}\zeta_2\zeta_3 \end{pmatrix}$$