QYBE solutions for constant R-Matrices over Grassmann Algebra With 2-generators & Finite Fields

Based upon the work of Steven Duplij, Olga Kotulska and Alexander Sadovnikov [1]

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Grassmann Algebra Package by John Browne [2]
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In[1]:=
```
Clear[a, b, c, d, p, q];
```

Assume a, b, c, d, p and q belong to a multiplicative commutative distributive algebra, with both additive and multiplicative inverses.

Based upon the 6-vertex YBE solutions [3], the following matrix forms are created:

In[2]:=

```
R =.;
R = ConstantArray[0, {4, 4}];
R[[1]][[1]] = p;
R[[2]][[2]] = c;
R[[3]][[3]] = b;
R[[4]][[4]] = q;
R[[2]][[3]] = d;
R[[3]][[2]] = a;
R // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & c & d & 0 \\ 0 & a & b & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

In[11]:=

```
R12 =.;
R12 = ConstantArray[0, {8, 8}];
R12[[1]][[1]] = p;
R12[[2]][[2]] = p;
R12[[3]][[3]] = c;
R12[[4]][[4]] = c;
R12[[5]][[5]] = b;
R12[[6]][[6]] = b;
R12[[7]][[7]] = q;
R12[[8]][[8]] = q;
R12[[3]][[5]] = d;
R12[[4]][[6]] = d;
R12[[5]][[3]] = a;
R12[[6]][[4]] = a;
R12 // MatrixForm
```

Out[25]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & d & 0 & 0 \\ 0 & 0 & a & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

In[26]:=

```
R13 =.;
R13 = ConstantArray[0, {8, 8}];
R13[[1]][[1]] = p;
R13[[2]][[2]] = c;
R13[[3]][[3]] = p;
R13[[4]][[4]] = c;
R13[[5]][[5]] = b;
R13[[6]][[6]] = q;
R13[[7]][[7]] = b;
R13[[8]][[8]] = q;
R13[[2]][[5]] = d;
R13[[4]][[7]] = d;
R13[[5]][[2]] = a;
R13[[7]][[4]] = a;
R13 // MatrixForm
```

Out[40]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & d & 0 \\ 0 & a & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

In[41]:=

```
R23 =.;
R23 = ConstantArray[0, {8, 8}];
R23[[1]][[1]] = p;
R23[[2]][[2]] = c;
R23[[3]][[3]] = b;
R23[[4]][[4]] = q;
R23[[5]][[5]] = p;
R23[[6]][[6]] = c;
R23[[7]][[7]] = b;
R23[[8]][[8]] = q;
R23[[2]][[3]] = d;
R23[[6]][[7]] = d;
R23[[3]][[2]] = a;
R23[[7]][[6]] = a;
R23 // MatrixForm
```

Out[55]//MatrixForm=

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

From the braid equation get the algebraic equations that must satisfy 0:

In[56]:=
```
equations = Simplify[R12.R13.R23 - R23.R13.R12];
Style[equations // MatrixForm, FontSize → 9]
```

Out[57]=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a c d & 0 & d(-b c + p(-d + p)) & 0 & 0 & 0 \\ 0 & a c d & a d(-a + d) & 0 & -a b d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a c d & d(b c + (d - q) q) & 0 \\ 0 & a(b c + (a - p) p) & a b d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a c d & 0 & a(a - d) d & a b d & 0 \\ 0 & 0 & 0 & a(-b c + q(-a + q)) & 0 & -a b d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From [1] check the computations:

$$cda = 0, \qquad (28)$$

$$bda = 0, \qquad (29)$$

$$da(d-a) = 0,\ pd(d-p) + cbd = 0, \qquad (30)$$

$$\left.\begin{array}{l} qd(d-q) + cbd = 0, \\ pa(a-p) + cba = 0, \\ qa(a-q) + cba = 0. \end{array}\right\} \qquad (31)$$

Load the Grassmann Algebra package [2] :

In[58]:=
```
<< "GrassmannAlgebra`"
```

Declare all the vectors and scalars and make an even algebra, two generators ζ_1, ζ_2 :

In[59]:=
```
★A; DeclareVectorSymbols[ζ1, ζ2]
scalars = DeclareScalarSymbols[a0, b0, c0, d0, p0, q0, a12, b12, c12, d12, p12, q12]
X = ★G[(1 + ζ1) ∧ (1 + ζ2)]
```

Out[59]= $\{\zeta_1, \zeta_2\}$

Out[60]= $\{a_0, a_{12}, b_0, b_{12}, c_0, c_{12}, d_0, d_{12}, p_0, p_{12}, q_0, q_{12}\}$

Out[61]= $1 + \zeta_1 + \zeta_2 + \zeta_1 \wedge \zeta_2$

In[62]:=
```
(* assign the 0s to only get the even terms *)
aa =.;
aa = {a0, 0, 0, a12}.List @@ X

bb =.;
bb = {b0, 0, 0, b12}.List @@ X

cc =.;
cc = {c0, 0, 0, c12}.List @@ X

dd =.;
dd = {d0, 0, 0, d12}.List @@ X

pp =.;
pp = {p0, 0, 0, p12}.List @@ X

qq =.;
qq = {q0, 0, 0, q12}.List @@ X
```

Out[63]= $a_0 + a_{12}\, \zeta_1 \wedge \zeta_2$

Out[65]= $b_0 + b_{12}\, \zeta_1 \wedge \zeta_2$

Out[67]= $c_0 + c_{12}\, \zeta_1 \wedge \zeta_2$

Out[69]= $d_0 + d_{12}\, \zeta_1 \wedge \zeta_2$

Out[71]= $p_0 + p_{12}\, \zeta_1 \wedge \zeta_2$

Out[73]= $q_0 + q_{12}\, \zeta_1 \wedge \zeta_2$

Replace the vars with the Grassmann generators and constants, EQ (28-31) in [1]:

In[74]:=
```
expr1 = GrassmannExpandAndSimplify[cc ∧ dd ∧ aa]
```

Out[74]= $a_0\, c_0\, d_0 + (a_{12}\, c_0\, d_0 + a_0\, c_{12}\, d_0 + a_0\, c_0\, d_{12})\, \zeta_1 \wedge \zeta_2$

Collect the coefficients:

In[75]:=
```
coeff1 = CoefficientList[expr1, {ζ1 ∧ ζ2}]
```

Out[75]= $\{a_0\, c_0\, d_0,\ a_{12}\, c_0\, d_0 + a_0\, c_{12}\, d_0 + a_0\, c_0\, d_{12}\}$

In[76]:=
```
expr2 = GrassmannExpandAndSimplify[bb ∧ dd ∧ aa]
```

Out[76]= $a_0\, b_0\, d_0 + (a_{12}\, b_0\, d_0 + a_0\, b_{12}\, d_0 + a_0\, b_0\, d_{12})\, \zeta_1 \wedge \zeta_2$

In[77]:=
```
coeff2 = CoefficientList[expr2, {ζ1 ∧ ζ2}]
```

Out[77]= $\{a_0\, b_0\, d_0,\ a_{12}\, b_0\, d_0 + a_0\, b_{12}\, d_0 + a_0\, b_0\, d_{12}\}$

In[78]:=

```
expr3 = GrassmannExpandAndSimplify[dd ∧ aa ∧ (dd - aa)]
```

Out[78]= $-a_0^2\, d_0 + a_0\, d_0^2 + \left(-2\, a_0\, a_{12}\, d_0 + a_{12}\, d_0^2 - a_0^2\, d_{12} + 2\, a_0\, d_0\, d_{12}\right)\, \zeta_1 \wedge \zeta_2$

In[79]:=

```
coeff3 = CoefficientList[expr3, {ζ1 ∧ ζ2}]
```

Out[79]= $\left\{-a_0^2\, d_0 + a_0\, d_0^2,\ -2\, a_0\, a_{12}\, d_0 + a_{12}\, d_0^2 - a_0^2\, d_{12} + 2\, a_0\, d_0\, d_{12}\right\}$

In[80]:=

```
expr4 = GrassmannExpandAndSimplify[(pp ∧ dd ∧ (dd - pp)) + cc ∧ bb ∧ dd]
```

Out[80]= $b_0\, c_0\, d_0 + d_0^2\, p_0 - d_0\, p_0^2 + \left(b_{12}\, c_0\, d_0 + b_0\, c_{12}\, d_0 + b_0\, c_0\, d_{12} + 2\, d_0\, d_{12}\, p_0 - d_{12}\, p_0^2 + d_0^2\, p_{12} - 2\, d_0\, p_0\, p_{12}\right)\, \zeta_1 \wedge \zeta_2$

In[81]:=

```
coeff4 = CoefficientList[expr4, {ζ1 ∧ ζ2}]
```

Out[81]= $\left\{b_0\, c_0\, d_0 + d_0^2\, p_0 - d_0\, p_0^2,\ b_{12}\, c_0\, d_0 + b_0\, c_{12}\, d_0 + b_0\, c_0\, d_{12} + 2\, d_0\, d_{12}\, p_0 - d_{12}\, p_0^2 + d_0^2\, p_{12} - 2\, d_0\, p_0\, p_{12}\right\}$

In[82]:=

```
expr5 = GrassmannExpandAndSimplify[(qq ∧ dd ∧ (dd - qq)) + cc ∧ bb ∧ dd]
```

Out[82]= $b_0\, c_0\, d_0 + d_0^2\, q_0 - d_0\, q_0^2 + \left(b_{12}\, c_0\, d_0 + b_0\, c_{12}\, d_0 + b_0\, c_0\, d_{12} + 2\, d_0\, d_{12}\, q_0 - d_{12}\, q_0^2 + d_0^2\, q_{12} - 2\, d_0\, q_0\, q_{12}\right)\, \zeta_1 \wedge \zeta_2$

In[83]:=

```
coeff5 = CoefficientList[expr5, {ζ1 ∧ ζ2}]
```

Out[83]= $\left\{b_0\, c_0\, d_0 + d_0^2\, q_0 - d_0\, q_0^2,\ b_{12}\, c_0\, d_0 + b_0\, c_{12}\, d_0 + b_0\, c_0\, d_{12} + 2\, d_0\, d_{12}\, q_0 - d_{12}\, q_0^2 + d_0^2\, q_{12} - 2\, d_0\, q_0\, q_{12}\right\}$

In[84]:=

```
expr6 = GrassmannExpandAndSimplify[(pp ∧ aa ∧ (aa - pp)) + cc ∧ bb ∧ aa]
```

Out[84]= $a_0\, b_0\, c_0 + a_0^2\, p_0 - a_0\, p_0^2 + \left(a_{12}\, b_0\, c_0 + a_0\, b_{12}\, c_0 + a_0\, b_0\, c_{12} + 2\, a_0\, a_{12}\, p_0 - a_{12}\, p_0^2 + a_0^2\, p_{12} - 2\, a_0\, p_0\, p_{12}\right)\, \zeta_1 \wedge \zeta_2$

In[85]:=

```
coeff6 = CoefficientList[expr6, {ζ1 ∧ ζ2}]
```

Out[85]= $\left\{a_0\, b_0\, c_0 + a_0^2\, p_0 - a_0\, p_0^2,\ a_{12}\, b_0\, c_0 + a_0\, b_{12}\, c_0 + a_0\, b_0\, c_{12} + 2\, a_0\, a_{12}\, p_0 - a_{12}\, p_0^2 + a_0^2\, p_{12} - 2\, a_0\, p_0\, p_{12}\right\}$

In[86]:=

```
expr7 = GrassmannExpandAndSimplify[(qq ∧ aa ∧ (aa - qq)) + cc ∧ bb ∧ aa]
```

Out[86]= $a_0\, b_0\, c_0 + a_0^2\, q_0 - a_0\, q_0^2 + \left(a_{12}\, b_0\, c_0 + a_0\, b_{12}\, c_0 + a_0\, b_0\, c_{12} + 2\, a_0\, a_{12}\, q_0 - a_{12}\, q_0^2 + a_0^2\, q_{12} - 2\, a_0\, q_0\, q_{12}\right)\, \zeta_1 \wedge \zeta_2$

In[87]:=

```
coeff7 = CoefficientList[expr7, {ζ1 ∧ ζ2}]
```

Out[87]= $\left\{a_0\, b_0\, c_0 + a_0^2\, q_0 - a_0\, q_0^2,\ a_{12}\, b_0\, c_0 + a_0\, b_{12}\, c_0 + a_0\, b_0\, c_{12} + 2\, a_0\, a_{12}\, q_0 - a_{12}\, q_0^2 + a_0^2\, q_{12} - 2\, a_0\, q_0\, q_{12}\right\}$

Simplify and remove the duplicates and make a list of the equations, compare against the [1], 14 system of equations:

Substitution of these equations into the system of Eqs.(28)~(31) yields the system of equations below

$$c_0 d_0 a_0 = 0, \quad b_0 d_0 a_0 = 0, \quad d_0 a_0 (d_0 - a_0) = 0, \qquad (50)$$

$$\begin{aligned}
&p_0 d_0 (d_0 - p_0) + c_0 b_0 d_0 = 0, \quad q_0 d_0 (d_0 - q_0) + c_0 b_0 d_0 = 0,\\
&p_0 a_0 (a_0 - p_0) + c_0 b_0 a_0 = 0, \quad q_0 a_0 (a_0 - q_0) + c_0 b_0 a_0 = 0,\\
&c_0 d_0 a_{12} + c_0 a_0 d_{12} + d_0 a_0 c_{12} = 0,\\
&b_0 d_0 a_{12} + b_0 a_0 d_{12} + d_0 a_0 b_{12} = 0,\\
&d_0 a_0 (d_{12} - a_{12}) + d_0 (d_0 - a_0) a_{12} + a_0 (d_0 - a_0) d_{12} = 0,\\
&p_0 d_0 (d_{12} - p_{12}) + p_0 (d_0 - p_0) d_{12} +\\
&d_0 (d_0 - p_0) p_{12} + c_0 b_0 d_{12} + c_0 d_0 b_{12} + b_0 d_0 c_{12} = 0,\\
&q_0 d_0 (d_{12} - q_{12}) + q_0 (d_0 - q_0) d_{12} + d_0 (d_0 - q_0) q_{12} +\\
&c_0 b_0 d_{12} + c_0 d_0 b_{12} + b_0 d_0 c_{12} = 0,\\
&p_0 a_0 (a_{12} - p_{12}) + p_0 (a_0 - p_0) a_{12} + a_0 (a_0 - p_0) p_{12} +\\
&c_0 b_0 a_{12} + c_0 a_0 b_{12} + b_0 a_0 c_{12} = 0,\\
&q_0 a_0 (a_{12} - q_{12}) + q_0 (a_0 - q_0) a_{12} + a_0 (a_0 - q_0) q_{12} +\\
&c_0 b_0 a_{12} + c_0 a_0 b_{12} + b_0 a_0 c_{12} = 0.
\end{aligned} \qquad (51)$$

In[88]:=
```
(* Exactly 14 equations *)
coeffSystem = DeleteDuplicates[
  Simplify[Join[coeff1, coeff2, coeff3, coeff4, coeff5, coeff6, coeff7]]]
Length[coeffSystem]
```

Out[88]=

$$\{a_0 c_0 d_0,\ a_{12} c_0 d_0 + a_0 (c_{12} d_0 + c_0 d_{12}),\ a_0 b_0 d_0,\ a_{12} b_0 d_0 + a_0 (b_{12} d_0 + b_0 d_{12}),$$
$$a_0 d_0 (-a_0 + d_0),\ a_{12} d_0^2 - a_0^2 d_{12} + 2 a_0 d_0 (-a_{12} + d_{12}),\ d_0 (b_0 c_0 + (d_0 - p_0) p_0),$$
$$b_{12} c_0 d_0 + b_0 (c_{12} d_0 + c_0 d_{12}) + 2 d_0 d_{12} p_0 - d_{12} p_0^2 + d_0^2 p_{12} - 2 d_0 p_0 p_{12},$$
$$d_0 (b_0 c_0 + (d_0 - q_0) q_0),\ b_{12} c_0 d_0 + b_0 (c_{12} d_0 + c_0 d_{12}) + 2 d_0 d_{12} q_0 - d_{12} q_0^2 + d_0^2 q_{12} - 2 d_0 q_0 q_{12},$$
$$a_0 (b_0 c_0 + (a_0 - p_0) p_0),\ a_{12} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{12} c_0 + b_0 c_{12} + (a_0 - 2 p_0) p_{12}),$$
$$a_0 (b_0 c_0 + (a_0 - q_0) q_0),\ a_{12} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{12} c_0 + b_0 c_{12} + (a_0 - 2 q_0) q_{12})\}$$

Out[89]= 14

Case I: Compare with [1]:

Case I: d_0=0. Eqs.(50)~(51) have the following form:

$$\left.\begin{array}{l} d_0 = 0,\ \ p_0 a_0 (a_0 - p_0) + c_0 b_0 a_0 = 0, \\ q_0 a_0 (a_0 - q_0) + c_0 b_0 a_0 = 0, \end{array}\right\} \qquad (52.1)$$

$$a_0^2 d_{12} = 0, \qquad (52.2)$$

$$c_0 a_0 d_{12} = 0,\ \ b_0 a_0 d_{12} = 0, \qquad (52.3)$$

$$p_0^2 d_{12} - c_0 b_0 d_{12} = 0,\ \ q_0^2 d_{12} - c_0 b_0 d_{12} = 0, \qquad (52.4)$$

$$p_0 a_0 (a_{12} - p_{12}) + p_0 (a_0 - p_0) a_{12} + a_0 (a_0 - p_0) p_{12} + c_0 b_0 a_{12} + c_0 a_0 b_{12} + b_0 a_0 c_{12} = 0, \qquad (52.5)$$

$$q_0 a_0 (a_{12} - q_{12}) + q_0 (a_0 - q_0) a_{12} + a_0 (a_0 - q_0) q_{12} + c_0 b_0 a_{12} + c_0 a_0 b_{12} + b_0 a_0 c_{12} = 0. \qquad (52.6)$$

In[90]:=
```
(* exactly 9 equations *)
caseI = DeleteCases[DeleteDuplicates[coeffSystem /. {d0 → 0}], _Integer]
Length[caseI]
```

Out[90]=
$$\{a_0 c_0 d_{12},\ a_0 b_0 d_{12},\ -a_0^2 d_{12},\ b_0 c_0 d_{12} - d_{12} p_0^2,\ b_0 c_0 d_{12} - d_{12} q_0^2,\ a_0 (b_0 c_0 + (a_0 - p_0) p_0),\ a_{12} (b_0 c_0 + (2 a_0 - p_0) p_0) + a_0 (b_{12} c_0 + b_0 c_{12} + (a_0 - 2 p_0) p_{12}),\ a_0 (b_0 c_0 + (a_0 - q_0) q_0),\ a_{12} (b_0 c_0 + (2 a_0 - q_0) q_0) + a_0 (b_{12} c_0 + b_0 c_{12} + (a_0 - 2 q_0) q_{12})\}$$

Out[91]= 9

Solve this system of equations:

In[92]:=

```
solCaseI = Solve[caseI == 0, scalars]
```

Out[92]=

$$\Big\{\{a_0 \to 0,\ a_{12} \to 0,\ d_{12} \to 0\},\ \Big\{a_0 \to 0,\ c_0 \to \frac{q_0^2}{b_0},\ p_0 \to -q_0\Big\},$$

$$\Big\{a_0 \to 0,\ c_0 \to \frac{q_0^2}{b_0},\ p_0 \to q_0\Big\},\ \{a_0 \to 0,\ b_0 \to 0,\ p_0 \to 0,\ q_0 \to 0\},$$

$$\Big\{a_0 \to 0,\ c_0 \to \frac{q_0^2}{b_0},\ d_{12} \to 0,\ p_0 \to -q_0\Big\},\ \Big\{a_0 \to 0,\ c_0 \to \frac{q_0^2}{b_0},\ d_{12} \to 0,\ p_0 \to q_0\Big\},$$

$$\{a_0 \to 0,\ b_0 \to 0,\ d_{12} \to 0,\ p_0 \to 0,\ q_0 \to 0\},\ \{a_0 \to 0,\ b_0 \to 0,\ d_{12} \to 0,\ p_0 \to 0,\ q_0 \to 0\},$$

$$\Big\{c_0 \to \frac{q_0\ (-a_0 + q_0)}{b_0},\ d_{12} \to 0,\ p_0 \to a_0 - q_0,\ p_{12} \to \frac{-a_0\ a_{12}\ b_0 - b_0^2\ c_{12} + a_{12}\ b_0\ q_0 + a_0\ b_{12}\ q_0 - b_{12}\ q_0^2}{b_0\ (-a_0 + 2\ q_0)},$$

$$q_{12} \to \frac{b_0^2\ c_{12} + a_{12}\ b_0\ q_0 - a_0\ b_{12}\ q_0 + b_{12}\ q_0^2}{b_0\ (-a_0 + 2\ q_0)}\Big\},\ \Big\{c_0 \to \frac{q_0\ (-a_0 + q_0)}{b_0},\ d_{12} \to 0,\ p_0 \to q_0,$$

$$p_{12} \to \frac{b_0^2\ c_{12} + a_{12}\ b_0\ q_0 - a_0\ b_{12}\ q_0 + b_{12}\ q_0^2}{b_0\ (-a_0 + 2\ q_0)},\ q_{12} \to \frac{b_0^2\ c_{12} + a_{12}\ b_0\ q_0 - a_0\ b_{12}\ q_0 + b_{12}\ q_0^2}{b_0\ (-a_0 + 2\ q_0)}\Big\},$$

$$\Big\{c_0 \to -\frac{a_0^2}{4\ b_0},\ c_{12} \to -\frac{a_0\ (2\ a_{12}\ b_0 - a_0\ b_{12})}{4\ b_0^2},\ d_{12} \to 0,\ p_0 \to \frac{a_0}{2},\ q_0 \to \frac{a_0}{2}\Big\},$$

$$\Big\{b_0 \to 0,\ d_{12} \to 0,\ p_0 \to 0,\ p_{12} \to -\frac{b_{12}\ c_0}{a_0},\ q_0 \to 0,\ q_{12} \to -\frac{b_{12}\ c_0}{a_0}\Big\},$$

$$\Big\{b_0 \to 0,\ d_{12} \to 0,\ p_0 \to a_0,\ p_{12} \to \frac{a_0\ a_{12} + b_{12}\ c_0}{a_0},\ q_0 \to 0,\ q_{12} \to -\frac{b_{12}\ c_0}{a_0}\Big\},$$

$$\Big\{b_0 \to 0,\ d_{12} \to 0,\ p_0 \to 0,\ p_{12} \to -\frac{b_{12}\ c_0}{a_0},\ q_0 \to a_0,\ q_{12} \to \frac{a_0\ a_{12} + b_{12}\ c_0}{a_0}\Big\},$$

$$\Big\{b_0 \to 0,\ d_{12} \to 0,\ p_0 \to a_0,\ p_{12} \to \frac{a_0\ a_{12} + b_{12}\ c_0}{a_0},\ q_0 \to a_0,\ q_{12} \to \frac{a_0\ a_{12} + b_{12}\ c_0}{a_0}\Big\}\Big\}$$

Let’s compute the original R matrices, but plug in the Grassmann numbers:

In[93]:=

```
R12Grassmann = R12 /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
Style[R12Grassmann // MatrixForm, FontSize → 9]
```

Out[94]=

$$\begin{pmatrix}
p_0 + p_{12}\,\zeta_1\wedge\zeta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_0 + p_{12}\,\zeta_1\wedge\zeta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_0 + c_{12}\,\zeta_1\wedge\zeta_2 & 0 & d_0 + d_{12}\,\zeta_1\wedge\zeta_2 & 0 & 0 & 0 \\
0 & 0 & 0 & c_0 + c_{12}\,\zeta_1\wedge\zeta_2 & 0 & d_0 + d_{12}\,\zeta_1\wedge\zeta_2 & 0 & 0 \\
0 & 0 & a_0 + a_{12}\,\zeta_1\wedge\zeta_2 & 0 & b_0 + b_{12}\,\zeta_1\wedge\zeta_2 & 0 & 0 & 0 \\
0 & 0 & 0 & a_0 + a_{12}\,\zeta_1\wedge\zeta_2 & 0 & b_0 + b_{12}\,\zeta_1\wedge\zeta_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_0 + q_{12}\,\zeta_1\wedge\zeta_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_0 + q_{12}\,\zeta_1\wedge\zeta_2
\end{pmatrix}$$

In [1] they only reported one of the solutions:

In[95]:=

```
R12Grassmann3 = Flatten[R12Grassmann /. {solCaseI[[3]]}, 1];
R12Grassmann3 /. {a0 → 0, d0 → 0, b0 → (t^2) / r, p0 → t, q0 → t, c0 → r, c12 → v,
   b12 → w, d12 → y, p12 → l, q12 → m, a12 → n, ξ1 ∧ ξ2 -> ξ1 ξ2} // MatrixForm
```

Out[96]//MatrixForm=

$$\begin{pmatrix} t+l\,\zeta_1\zeta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t+l\,\zeta_1\zeta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r+v\,\zeta_1\zeta_2 & 0 & y\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & r+v\,\zeta_1\zeta_2 & 0 & y\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & n\,\zeta_1\zeta_2 & 0 & \frac{t^2}{r}+w\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & n\,\zeta_1\zeta_2 & 0 & \frac{t^2}{r}+w\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t+m\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t+m\,\zeta_1\zeta_2 \end{pmatrix}$$

Remark: *We shorthand* $\zeta_1 \wedge \zeta_2$ *to* $\zeta_1 \zeta_2$ *.*

Also let's use the paper's [1] substitution for parameters, check and see if we got the same matrix:

$$\left.\begin{array}{l} c = r + v\xi_1\xi_2,\ \ b = t^2/r + w\xi_1\xi_2,\ \ d = y\xi_1\xi_2, \\ p = t + l\xi_1\xi_2,\ \ q = t + m\xi_1\xi_2,\ \ a = n\xi_1\xi_2. \end{array}\right\} \quad (54)$$

Let us write the resulting 6-vertex $\boldsymbol{R}$-matrix

$$\boldsymbol{R} = \begin{pmatrix} t+l\xi_1\xi_2 & 0 & 0 & 0 \\ 0 & r+v\xi_1\xi_2 & y\xi_1\xi_2 & 0 \\ 0 & n\xi_1\xi_2 & t^2/r+w\xi_1\xi_2 & 0 \\ 0 & 0 & 0 & t+m\xi_1\xi_2 \end{pmatrix}$$

One of our solutions matches the paper [1] EQ (55) for Case I:

In[97]:=

```
RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
RGrassmann3 = Flatten[RGrassmann /. {solCaseI[[3]]}, 1];
RGrassmann3 /. {a0 → 0, d0 → 0, b0 → (t^2) / r, p0 → t, q0 → t, c0 → r, c12 → v,
   b12 → w, d12 → y, p12 → l, q12 → m, a12 → n, ξ1 ∧ ξ2 -> ξ1 ξ2} // MatrixForm
```

Out[99]//MatrixForm=

$$\begin{pmatrix} t+l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r+v\,\zeta_1\zeta_2 & y\,\zeta_1\zeta_2 & 0 \\ 0 & n\,\zeta_1\zeta_2 & \frac{t^2}{r}+w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & t+m\,\zeta_1\zeta_2 \end{pmatrix}$$

Let’s tabulate all the solutions for Case I:

In[100]:=
```
Column[Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
  RGrassmannFinal = Flatten[RGrassmann /. {solCaseI[[i]]}, 1];
  RGrassmannFinal /. {a₀ → 0, d₀ → 0, b₀ → (t^2) / r, p₀ → t, q₀ → t, c₀ → r, c₁₂ → v,
     b₁₂ → w, d₁₂ → y, p₁₂ → l, q₁₂ → m, a₁₂ → n, ζ₁ ∧ ζ₂ -> ζ₁ ζ₂} // MatrixForm,
  {i, 1, Length[solCaseI]}]]
```

$$\begin{pmatrix} t + l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & t + m\,\zeta_1\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} -t + l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & y\,\zeta_1\zeta_2 & 0 \\ 0 & n\,\zeta_1\zeta_2 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & t + m\,\zeta_1\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} t + l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & y\,\zeta_1\zeta_2 & 0 \\ 0 & n\,\zeta_1\zeta_2 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & t + m\,\zeta_1\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & y\,\zeta_1\zeta_2 & 0 \\ 0 & n\,\zeta_1\zeta_2 & w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & m\,\zeta_1\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} -t + l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\zeta_2 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & t + m\,\zeta_1\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} t + l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\zeta_2 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & t + m\,\zeta_1\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\zeta_2 & w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & m\,\zeta_1\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} l\,\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\zeta_2 & w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & m\,\zeta_1\zeta_2 \end{pmatrix}$$

Out[100]=

$$\begin{pmatrix} -t + \frac{r\left(\frac{n\,t^3}{r} - \frac{t^4\,v}{r^2} - t^2\,w\right)\zeta_1\zeta_2}{2\,t^3} & 0 & 0 & 0 \\ 0 & r + v\,\zeta_1\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\zeta_2 & \frac{t^2}{r} + w\,\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & t + \frac{r\left(\frac{n\,t^3}{r} + \frac{t^4\,v}{r^2} + t^2\,w\right)\zeta_1\zeta_2}{2\,t^3} \end{pmatrix}$$

$$\begin{pmatrix} t+\frac{r\left(\frac{n\,t^3}{r}+\frac{t^4\,v}{r^2}+t^2\,w\right)\zeta_1\,\zeta_2}{2\,t^3} & 0 & 0 & 0 \\ 0 & r+v\,\zeta_1\,\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\,\zeta_2 & \frac{t^2}{r}+w\,\zeta_1\,\zeta_2 & 0 \\ 0 & 0 & 0 & t+\frac{r\left(\frac{n\,t^3}{r}+\frac{t^4\,v}{r^2}+t^2\,w\right)\zeta_1\,\zeta_2}{2\,t^3} \end{pmatrix}$$

$$\begin{pmatrix} l\,\zeta_1\,\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & n\,\zeta_1\,\zeta_2 & \frac{t^2}{r}+w\,\zeta_1\,\zeta_2 & 0 \\ 0 & 0 & 0 & m\,\zeta_1\,\zeta_2 \end{pmatrix}$$

$$\begin{pmatrix} \text{ComplexInfinity} & 0 & 0 & 0 \\ 0 & r+v\,\zeta_1\,\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\,\zeta_2 & w\,\zeta_1\,\zeta_2 & 0 \\ 0 & 0 & 0 & \text{ComplexInfinity} \end{pmatrix}$$

$$\begin{pmatrix} \text{ComplexInfinity} & 0 & 0 & 0 \\ 0 & r+v\,\zeta_1\,\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\,\zeta_2 & w\,\zeta_1\,\zeta_2 & 0 \\ 0 & 0 & 0 & \text{ComplexInfinity} \end{pmatrix}$$

$$\begin{pmatrix} \text{ComplexInfinity} & 0 & 0 & 0 \\ 0 & r+v\,\zeta_1\,\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\,\zeta_2 & w\,\zeta_1\,\zeta_2 & 0 \\ 0 & 0 & 0 & \text{ComplexInfinity} \end{pmatrix}$$

$$\begin{pmatrix} \text{ComplexInfinity} & 0 & 0 & 0 \\ 0 & r+v\,\zeta_1\,\zeta_2 & 0 & 0 \\ 0 & n\,\zeta_1\,\zeta_2 & w\,\zeta_1\,\zeta_2 & 0 \\ 0 & 0 & 0 & \text{ComplexInfinity} \end{pmatrix}$$

Case 2: Compare with [1]:

Case II: a_0=0. Eqs.(50)~(51) lead to the following:

$$a_0 = 0, \tag{56.1}$$

$$\left.\begin{aligned} p_0 d_0(d_0 - p_0) + c_0 b_0 d_0 &= 0, \\ q_0 d_0(d_0 - q_0) + c_0 b_0 d_0 &= 0, \end{aligned}\right\} \tag{56.2}$$

$$c_0 d_0 a_{12} = 0, \quad b_0 d_0 a_{12} = 0, \tag{56.3}$$

$$d_0^2 a_{12} = 0, \tag{56.4}$$

$$p_0 d_0(d_{12} - p_{12}) + p_0(d_0 - p_0)d_{12} + d_0(d_0 - p_0)$$

$$\times p_{12} + c_0 b_0 d_{12} + c_0 d_0 b_{12} + b_0 d_0 c_{12} = 0, \qquad (56.5)$$

$$q_0 d_0 (d_{12} - q_{12}) + q_0 (d_0 - q_0) d_{12} + d_0 (d_0 - q_0) q_{12} +$$
$$c_0 b_0 d_{12} + c_0 d_0 b_{12} + b_0 d_0 c_{12} = 0, \qquad (56.6)$$

$$p_0^2 a_{12} - c_0 b_0 a_{12} = 0, \quad q_0^2 a_{12} - c_0 b_0 a_{12} = 0. \qquad (56.7)$$

In[101]:=
```
(* exactly 9 equations *)
caseII = DeleteCases[DeleteDuplicates[coeffSystem /. {a0 → 0}], _Integer]
Length[caseII]
```

Out[101]= $\{a_{12} c_0 d_0, a_{12} b_0 d_0, a_{12} d_0^2, d_0 (b_0 c_0 + (d_0 - p_0) p_0),$
$b_{12} c_0 d_0 + b_0 (c_{12} d_0 + c_0 d_{12}) + 2 d_0 d_{12} p_0 - d_{12} p_0^2 + d_0^2 p_{12} - 2 d_0 p_0 p_{12},$
$d_0 (b_0 c_0 + (d_0 - q_0) q_0), b_{12} c_0 d_0 + b_0 (c_{12} d_0 + c_0 d_{12}) + 2 d_0 d_{12} q_0 - d_{12} q_0^2 + d_0^2 q_{12} - 2 d_0 q_0 q_{12},$
$a_{12} (b_0 c_0 - p_0^2), a_{12} (b_0 c_0 - q_0^2)\}$

Out[102]= 9

Since $a_{12} d_0^2 = 0$ then $d_0 = 0$ this case coincides with the case I.

Case III:

Case III: c_0=b_0=0, d_0=a_0. Then Eqs.(50)~51) are as follows:

$$c_0 = 0, \; b_0 = 0, \; d_0 = a_0, \qquad (57.1)$$

$$p_0 d_0 (d_0 - p_0) = 0, \; q_0 d_0 (d_0 - q_0) = 0, \qquad (57.2)$$

$$d_0 a_0 c_{12} = 0, \; d_0 a_0 b_{12} = 0, \qquad (57.3)$$

$$d_0 a_0 (d_{12} - a_{12}) = 0, \qquad (57.4)$$

$$p_0 d_0 (d_{12} - p_{12}) + p_0 (d_0 - p_0) d_{12} + d_0 (d_0 - p_0) p_{12} = 0, \qquad (57.5)$$

$$q_0 d_0 (d_{12} - q_{12}) + q_0 (d_0 - q_0) d_{12} + d_0 (d_0 - q_0) q_{12} = 0. \qquad (57.6)$$

Apparently [1] is missing two equations:

In[103]:=

```
(* exactly 9 equations *)
caseIII =
 DeleteCases[DeleteDuplicates[coeffSystem /. {d0 → a0, c0 → 0, b0 → 0}], _Integer]
Length[caseIII]
```

Out[103]= $\{a_0^2\, c_{12},\ a_0^2\, b_{12},\ a_0^2\, a_{12} - a_0^2\, d_{12} + 2\, a_0^2\, (-a_{12} + d_{12}),\ a_0\, (a_0 - p_0)\, p_0,$
$2\, a_0\, d_{12}\, p_0 - d_{12}\, p_0^2 + a_0^2\, p_{12} - 2\, a_0\, p_0\, p_{12},\ a_0\, (a_0 - q_0)\, q_0,\ 2\, a_0\, d_{12}\, q_0 - d_{12}\, q_0^2 + a_0^2\, q_{12} - 2\, a_0\, q_0\, q_{12},$
$a_{12}\, (2\, a_0 - p_0)\, p_0 + a_0\, (a_0 - 2\, p_0)\, p_{12},\ a_{12}\, (2\, a_0 - q_0)\, q_0 + a_0\, (a_0 - 2\, q_0)\, q_{12}\}$

Out[104]= 9

Let's solve the system of equations:

In[105]:=

```
solCaseIII = Solve[caseIII == 0, scalars]
```

Out[105]= $\{\{a_0 \to 0,\ a_{12} \to 0,\ d_{12} \to 0\},\ \{a_0 \to 0,\ p_0 \to 0,\ q_0 \to 0\},$
$\{a_0 \to 0,\ a_{12} \to 0,\ d_{12} \to 0,\ q_0 \to 0\},\ \{a_0 \to 0,\ b_{12} \to 0,\ c_{12} \to 0,\ d_{12} \to a_{12},\ p_0 \to 0,\ q_0 \to 0\},$
$\{b_{12} \to 0,\ c_{12} \to 0,\ d_{12} \to a_{12},\ p_0 \to 0,\ p_{12} \to 0,\ q_0 \to 0,\ q_{12} \to 0\},$
$\{b_{12} \to 0,\ c_{12} \to 0,\ d_{12} \to a_{12},\ p_0 \to a_0,\ p_{12} \to a_{12},\ q_0 \to 0,\ q_{12} \to 0\},$
$\{b_{12} \to 0,\ c_{12} \to 0,\ d_{12} \to a_{12},\ p_0 \to 0,\ p_{12} \to 0,\ q_0 \to a_0,\ q_{12} \to a_{12}\},$
$\{b_{12} \to 0,\ c_{12} \to 0,\ d_{12} \to a_{12},\ p_0 \to a_0,\ p_{12} \to a_{12},\ q_0 \to a_0,\ q_{12} \to a_{12}\}\}$

Compute the R matrices, see they match [1]:

Eq.(57.3) shows that $a_0=b_0=c_0=d_0=0$, then the coefficients p_0, q_0, a_{12}, b_{12}, c_{12}, d_{12}, p_{12}, q_{12} remain undefined. The solution will also be 8-parametrical $p_0=t$, $q_0=r$, $a_{12}=l$, $b_{12}=m$, $c_{12}=n$, $d_{12}=k$, $p_{12}=w$, $q_{12}=s$ and can be represented in the following form

$$\boldsymbol{R} = \begin{pmatrix} t + w\xi_1\xi_2 & 0 & 0 & 0 \\ 0 & n\xi_1\xi_2 & k\xi_1\xi_2 & 0 \\ 0 & l\xi_1\xi_2 & m\xi_1\xi_2 & 0 \\ 0 & 0 & 0 & r + s\xi_1\xi_2 \end{pmatrix}$$

Case III is not matching!

In[106]:=

```
Column[Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
  RGrassmannFinal = Flatten[RGrassmann /. {solCaseIII[[i]]}, 1];
  RGrassmannFinal /. {b0 → 0, a0 → 0, d0 → 0, p0 → t, q0 → r, c0 → 0, a12 → l,
     q12 → s, p12 → w, b12 → m, c12 → n, d12 → k, ξ1 ∧ ξ2 -> ξ1 ξ2} // MatrixForm,
  {i, 1, Length[solCaseIII]}]]
```

Out[106]=

$$\begin{pmatrix} t + w\,\xi_1\,\xi_2 & 0 & 0 & 0 \\ 0 & n\,\xi_1\,\xi_2 & 0 & 0 \\ 0 & 0 & m\,\xi_1\,\xi_2 & 0 \\ 0 & 0 & 0 & r + s\,\xi_1\,\xi_2 \end{pmatrix}$$

$$\begin{pmatrix} w\,\xi_1\,\xi_2 & 0 & 0 & 0 \\ 0 & n\,\xi_1\,\xi_2 & k\,\xi_1\,\xi_2 & 0 \\ 0 & l\,\xi_1\,\xi_2 & m\,\xi_1\,\xi_2 & 0 \\ 0 & 0 & 0 & s\,\xi_1\,\xi_2 \end{pmatrix}$$

$$\begin{pmatrix} t + w\,\xi_1\,\xi_2 & 0 & 0 & 0 \\ 0 & n\,\xi_1\,\xi_2 & 0 & 0 \\ 0 & 0 & m\,\xi_1\,\xi_2 & 0 \\ 0 & 0 & 0 & s\,\xi_1\,\xi_2 \end{pmatrix}$$

$$\begin{pmatrix} w\,\xi_1\,\xi_2 & 0 & 0 & 0 \\ 0 & 0 & l\,\xi_1\,\xi_2 & 0 \\ 0 & l\,\xi_1\,\xi_2 & 0 & 0 \\ 0 & 0 & 0 & s\,\xi_1\,\xi_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & l\,\xi_1\,\xi_2 & 0 \\ 0 & l\,\xi_1\,\xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l\,\xi_1\,\xi_2 & 0 & 0 & 0 \\ 0 & 0 & l\,\xi_1\,\xi_2 & 0 \\ 0 & l\,\xi_1\,\xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & l\,\xi_1\,\xi_2 & 0 \\ 0 & l\,\xi_1\,\xi_2 & 0 & 0 \\ 0 & 0 & 0 & l\,\xi_1\,\xi_2 \end{pmatrix}$$

$$\begin{pmatrix} l\,\xi_1\,\xi_2 & 0 & 0 & 0 \\ 0 & 0 & l\,\xi_1\,\xi_2 & 0 \\ 0 & l\,\xi_1\,\xi_2 & 0 & 0 \\ 0 & 0 & 0 & l\,\xi_1\,\xi_2 \end{pmatrix}$$

Finite Fields: Modulus Arithmetic

$\mathbb{Z}_2$

System Parameters need to be increased for Solver:

```
In[107]:= SetSystemOptions["ReduceOptions" → "MaxModularPoints" → 100 000 000];

In[146]:= solCaseI = DeleteDuplicates[Solve[caseI == 0, scalars, Modulus → 2]];

In[147]:= Length[solCaseI]

Out[147]= 544

In[148]:= randIndex = RandomInteger[{1, Length[solCaseI]}, 100 ];
          DeleteDuplicates[
           Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
            RGrassmannFinal = Flatten[RGrassmann /. {solCaseI[[randIndex[[i]]]]}, 1];
            Style[RGrassmannFinal /. {a0 → 0, d0 → 0, b0 → 0, p0 → t, q0 → t, c0 → 0,
                c12 → v, b12 → w, d12 → y, p12 → l, q12 → m, a12 → n, ξ1 ∧ ξ2 -> ξ1 ξ2,
                ξ1 ∧ ξ3 -> ξ1 ξ3, ξ2 ∧ ξ3 -> ξ2 ξ3} // MatrixForm, FontSize → 7],
            {i,
             1,
             100}]]
```

Out[149]=

$$\left\{\begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\right.$$

$$\begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & \zeta_1 \zeta_2 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\zeta_1 \zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1 \zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 0 & 0 \\ 0 & \zeta_1 \zeta_2 & 1 & 0 \\ 0 & 0 & 0 & \zeta_1 \zeta_2 \end{pmatrix}\right\}$$

$\mathbb{Z}_3$

In[117]:=
```
SetSystemOptions["ReduceOptions" → "MaxModularPoints" → 100 000 000];
```

In[138]:=
```
solCaseI = DeleteDuplicates[Solve[caseI == 0, scalars, Modulus → 3]];
```

In[139]:=
```
Length[solCaseI]
```

Out[139]= 16 389

Output some random 100 solutions:

In[140]:=
```
randIndex = RandomInteger[{1, Length[solCaseI]}, 100 ];
DeleteDuplicates[
 Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
  RGrassmannFinal = Flatten[RGrassmann /. {solCaseI[[randIndex[[i]]]]}, 1];
  Style[RGrassmannFinal /. {a0 → 0, d0 → 0, b0 → 0, p0 → t, q0 → t, c0 → 0,
      c12 → v, b12 → w, d12 → y, p12 → l, q12 → m, a12 → n, ξ1 ∧ ξ2 -> ξ1 ξ2,
      ξ1 ∧ ξ3 -> ξ1 ξ3, ξ2 ∧ ξ3 -> ξ2 ξ3} // MatrixForm, FontSize → 7],
  {i,
   1,
   100}]]
```

Out[141]=

$$\left\{\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&1+\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&0&0\\0&0&0&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&1+2\zeta_1\zeta_2&2&0\\0&0&0&1\end{pmatrix},\right.$$

$$\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2&0&0\\0&2\zeta_1\zeta_2&2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&2\end{pmatrix},$$

$$\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2&0&0\\0&0&\zeta_1\zeta_2&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&2&0&0\\0&0&2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&0&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&1+2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&\zeta_1\zeta_2&2&0\\0&0&0&2\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&1+\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&0&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&2+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&2\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0\\0&2+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&1+\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&0&2\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}2&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&1+\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&1&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&1&0&0\\0&\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&2\end{pmatrix},$$

$$\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&0&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&2+\zeta_1\zeta_2&2&0\\0&0&0&2\end{pmatrix},\begin{pmatrix}2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&2+\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&2+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}2&0&0&0\\0&1+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&1&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&2+\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&\zeta_1\zeta_2&0&0\\0&0&\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0\\0&1&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2+2\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}2&0&0&0\\0&1&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&0&2+\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1+\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&1&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0\\0&1+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&1+\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2+2\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2&0&0\\0&\zeta_1\zeta_2&2&0\\0&0&0&2\end{pmatrix},$$

$$\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&1&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&1&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&0&0\\0&0&0&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&\zeta_1\zeta_2&0&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&0&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&1&0&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&0&\zeta_1\zeta_2&0\\0&0&1+2\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2+2\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2+2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2+\zeta_1\zeta_2&0&0\\0&0&2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2&0&0&0\\0&2+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&2+2\zeta_1\zeta_2&0\\0&0&0&2\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0\\0&2+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&0&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&1&0&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&1&0&0\\0&0&\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&0&0\\0&0&1+\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&0&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&2&0&0\\0&2+2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&2\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix}\Big\}$$

$\mathbb{Z}_4$

```
In[107]:= SetSystemOptions["ReduceOptions" → "MaxModularPoints" → 100 000 000];

In[128]:= solCaseI = DeleteDuplicates[Solve[caseI == 0, scalars, Modulus → 4]];
```

In[129]:=
```
Length[solCaseI]
```

Out[129]= 358 400

Output some random 100 solutions:

In[136]:=
```
randIndex = RandomInteger[{1, Length[solCaseI]}, 100 ];
DeleteDuplicates[
 Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
  RGrassmannFinal = Flatten[RGrassmann /. {solCaseI[[randIndex[[i]]]]}, 1];
  Style[RGrassmannFinal /. {a0 → 0, d0 → 0, b0 → 0, p0 → t, q0 → t, c0 → 0,
      c12 → v, b12 → w, d12 → y, p12 → l, q12 → m, a12 → n, ζ1 ∧ ζ2 -> ζ1 ζ2,
      ζ1 ∧ ζ3 -> ζ1 ζ3, ζ2 ∧ ζ3 -> ζ2 ζ3} // MatrixForm, FontSize → 7],
  {i,
   1,
   100}]]
```

Out[137]=

$$\left\{\begin{pmatrix}0&0&0&0\\0&3+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&3\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&2+3\zeta_1\zeta_2&0&0\\0&2+\zeta_1\zeta_2&2+3\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+3\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2+2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2&2+\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix},\right.$$

$$\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&2+3\zeta_1\zeta_2&3+\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+3\zeta_1\zeta_2&0&0&0\\0&1&3\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2&0&0&0\\0&2+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&3\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&2+2\zeta_1\zeta_2&0&0\\0&0&0&3\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}1+2\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&3\zeta_1\zeta_2&0\\0&0&1&0\\0&0&0&1+3\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}1+3\zeta_1\zeta_2&0&0&0\\0&3+\zeta_1\zeta_2&0&0\\0&0&1&0\\0&0&0&3+2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&3\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&2+3\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2+3\zeta_1\zeta_2&2&0\\0&0&0&2+3\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&2+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&1+\zeta_1\zeta_2&0\\0&0&0&3\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+3\zeta_1\zeta_2&0&0&0\\0&2+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2+3\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&2+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2+\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}3+\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&0&1+\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&0&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2&3+3\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&2&3\zeta_1\zeta_2&0\\0&2+3\zeta_1\zeta_2&2+2\zeta_1\zeta_2&0\\0&0&0&2\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&3&2\zeta_1\zeta_2&0\\0&2+2\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&3\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}2&0&0&0\\0&2\zeta_1\zeta_2&3\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&3+3\zeta_1\zeta_2&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&2\zeta_1\zeta_2&0&0\\0&0&3+2\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&3&0&0\\0&0&3+\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2&2\zeta_1\zeta_2&0\\0&0&0&3\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&3+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2&0&0\\0&0&0&2\end{pmatrix}, \begin{pmatrix}3+2\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&3\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+3\zeta_1\zeta_2&0\\0&0&0&1+\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+3\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2+2\zeta_1\zeta_2&2+2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}2+\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&3+3\zeta_1\zeta_2&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}3+3\zeta_1\zeta_2&0&0&0\\0&1+3\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&3&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}3\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&3\zeta_1\zeta_2&0\\0&2+\zeta_1\zeta_2&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}3&0&0&0\\0&3+\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&3&0\\0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}3&0&0&0\\0&1&3\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&1+\zeta_1\zeta_2&0\\0&0&0&1+2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&3\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&2\end{pmatrix}, \begin{pmatrix}2+3\zeta_1\zeta_2&0&0&0\\0&1&2\zeta_1\zeta_2&0\\0&2+\zeta_1\zeta_2&2+3\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}3+\zeta_1\zeta_2&0&0&0\\0&1+3\zeta_1\zeta_2&0&0\\0&0&3+\zeta_1\zeta_2&0\\0&0&0&3+\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}3&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&2+3\zeta_1\zeta_2&0\\0&0&0&2\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&3\zeta_1\zeta_2&3\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&3&0\\0&0&0&2\end{pmatrix}, \begin{pmatrix}3+2\zeta_1\zeta_2&0&0&0\\0&\zeta_1\zeta_2&0&0\\0&0&2+2\zeta_1\zeta_2&0\\0&0&0&3\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}1+3\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&0&0\\0&2+\zeta_1\zeta_2&3&0\\0&0&0&3+\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix}3\zeta_1\zeta_2&0&0&0\\0&1+\zeta_1\zeta_2&3\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&0&0&2+2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+2\zeta_1\zeta_2&0&0&0\\0&0&0&0\\0&2+2\zeta_1\zeta_2&1+2\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&2+3\zeta_1\zeta_2&0&0\\0&\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&0&0&2+\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}1+\zeta_1\zeta_2&0&0&0\\0&1+2\zeta_1\zeta_2&0&0\\0&0&3+\zeta_1\zeta_2&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}3\zeta_1\zeta_2&0&0&0\\0&2+2\zeta_1\zeta_2&3\zeta_1\zeta_2&0\\0&\zeta_1\zeta_2&2+2\zeta_1\zeta_2&0\\0&0&0&2\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}2+3\zeta_1\zeta_2&0&0&0\\0&2\zeta_1\zeta_2&\zeta_1\zeta_2&0\\0&2+\zeta_1\zeta_2&0&0\\0&0&0&\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}3&0&0&0\\0&1+\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&1&0\\0&0&0&3+3\zeta_1\zeta_2\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&1+3\zeta_1\zeta_2&2\zeta_1\zeta_2&0\\0&2\zeta_1\zeta_2&2&0\\0&0&0&3\zeta_1\zeta_2\end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 2+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 2+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+2\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1\zeta_2 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 1+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 2+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & 3+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 3+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+2\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 1+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 3+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 3+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 1+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1\zeta_2 & 0 \\ 0 & 2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+3\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3 & 3\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 3+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 3+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}\Big\}$$

$\mathbb{Z}_5$

In[107]:=
```
SetSystemOptions["ReduceOptions" → "MaxModularPoints" → 100 000 000];
```

In[142]:=
```
solCaseI = DeleteDuplicates[Solve[caseI == 0, scalars, Modulus → 5]];
```

In[143]:=
```
Length[solCaseI]
```

Out[143]= 1 041 625

Output some random 100 solutions:

In[144]:=
```
randIndex = RandomInteger[{1, Length[solCaseI]}, 100 ];
DeleteDuplicates[
 Table[RGrassmann = R /. {a → aa, b → bb, c → cc, d → dd, p → pp, q → qq};
  RGrassmannFinal = Flatten[RGrassmann /. {solCaseI[[randIndex[[i]]]]}, 1];
  Style[RGrassmannFinal /. {a_0 → 0, d_0 → 0, b_0 → 0, p_0 → t, q_0 → t, c_0 → 0,
      c_12 → v, b_12 → w, d_12 → y, p_12 → l, q_12 → m, a_12 → n, ξ_1 ∧ ξ_2 -> ξ_1 ξ_2,
      ξ_1 ∧ ξ_3 -> ξ_1 ξ_3, ξ_2 ∧ ξ_3 -> ξ_2 ξ_3} // MatrixForm, FontSize → 7],
  {i,
   1,
   100}]]
```

Out[145]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 1+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 4+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix},\right.$$

$$\begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+4\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 3+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4+3\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 4+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+4\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 3+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 4+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1 & 2\zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 4+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 4+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1+\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 4+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+2\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 2 & 0 \\ 0 & 0 & 0 & 4+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 1+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+3\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & 1+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+3\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 1+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 4\zeta_1\zeta_2 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3+4\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 3+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2+4\zeta_1\zeta_2 & 4 & 0 \\ 0 & 0 & 0 & 4+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 1+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+4\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+2\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 2+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4+4\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 4 & 0 \\ 0 & 0 & 0 & 1+4\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 3+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 4+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+3\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 1+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 4+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4+3\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 1+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1+2\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & 4\zeta_1\zeta_2 & 1+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+2\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & 1 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 0 & 0 \\ 0 & 2+4\zeta_1\zeta_2 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 4+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 0 & 0 \\ 0 & 4 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+2\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 3+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 2+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 3+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 4+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 4+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2+4\zeta_1\zeta_2 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4+2\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 3\zeta_1\zeta_2 & 4+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1+2\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 4\zeta_1\zeta_2 & 4+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+4\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 4+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+4\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 4+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+4\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 3+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+4\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & \zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 0 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1+\zeta_1\zeta_2 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 4+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+4\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4 & 3\zeta_1\zeta_2 & 0 \\ 0 & 4\zeta_1\zeta_2 & 4+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3+\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & \zeta_1\zeta_2 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+2\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 4+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 4+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2+2\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 2\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 4\zeta_1\zeta_2 & 3+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 1+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3+2\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 4+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 2+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 1+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4+2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 3+4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 3+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 2+2\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 2+\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 2\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 1+3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 1+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4+2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 1+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1+4\zeta_1\zeta_2 \end{pmatrix},$$

$$\begin{pmatrix} 2+3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2+\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 3\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 4\zeta_1\zeta_2 & 0 & 0 \\ 0 & 2\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 4+4\zeta_1\zeta_2 & 0 & 0 & 0 \\ 0 & 3+3\zeta_1\zeta_2 & 3\zeta_1\zeta_2 & 0 \\ 0 & 2\zeta_1\zeta_2 & 2 & 0 \\ 0 & 0 & 0 & 4+3\zeta_1\zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+\zeta_1\zeta_2 & 4\zeta_1\zeta_2 & 0 \\ 0 & 0 & 3\zeta_1\zeta_2 & 0 \\ 0 & 0 & 0 & 2\zeta_1\zeta_2 \end{pmatrix} \}$$