## Chapter 9

## 9. Counting Permutations/Combinations

## Permutations

A permutation of a number of objects is a re-arrangement of the objects where order, rank, status etc matters.
e.g. $A C B$ and $A B C$ are permutations of the set $\{A, B, C\}$.

Calculations of the number of permutations available arise for example when people are assigned job classifications.

For example consider the question:
"How many permutations are there (i.e. how many ways) so that a committee of 2 people can be chosen from \{Susan, Harry, Sharon, Fred\} where the committee has a President and Secretary."

Note that if we tabulate the possibilities we get

| President | Secretary |
| :---: | :---: |
| Susan | Harry |
| Susan | Sharon |
| Susan | Fred |
| Harry | Susan |
| Harry | Sharon |
| Harry | Fred |
| Sharon | Susan |
| Sharon | Harry |
| Sharon | Fred |
| Fred | Susan |
| Fred | Harry |
| Fred | Sharon |

Note that there are 4 ways of choosing the President and with each of these possibilities there are 3 ways of choosing the Secretary and hence there are $4 \times 3$ (=12) permutations available.

If we wished to choose a committee of 3 people including a President, Secretary and Treasurer then the number of permutations would be $4 \times 3 \times 2=24$.
This is written ${ }_{4} \mathrm{P}_{3}$.

In general, the number of permutations of $r$ objects chosen from a set of $n$ objects is written ${ }_{n} P_{r}$ and equals $n \cdot(n-1) \cdot(n-2) \ldots(n-r+1)$.

For example the number of permutations of 5 objects chosen from a set of 8 people is

$$
8 \times 7 \times 6 \times 5 \times 4=6720
$$

A notation which is often used involves using factorials, written like an exclamation mark! This means by definition for example that $7!=7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1=5040$

Note that ${ }_{\mathrm{n}} \mathrm{P}_{\mathrm{r}}=\frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{r})!}$
For example
${ }_{8} \mathrm{P}_{5}=\frac{8!}{(8-5)!}=\frac{8!}{3!}=\frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}=8 \times 7 \times 6 \times 5 \times 4(=6720)$ as seen before

## Question

Two hundred raffle tickets are sold. When the first, second and third prizes are drawn, in how many different ways can the prizes be won?

Answer

| $1^{\text {st }}$ Prize | $2^{\text {nd }}$ Prize | $3^{\text {rd }}$ Prize |
| :---: | :---: | :---: |
| 200 | 199 | 198 |

i.e. there are $200 \times 199 \times 198$ ways

$$
=7880400
$$

## Question

How many four digit integers (greater than 3000) can be formed using digits $1,3,5,7$ if no digit is repeated?

Answer

The 1000s digit can be 3 or 5 or 7. i.e there are 3 possibilities
The 100s digit can be 1 or either of the two digits not chosen for the 1000s digit
The 10s digit can be either of the 2 digits not chosen for the 1000s digit or 100s digit The units is then automatically chosen.
i.e there are $3 \times 3 \times 2 \times 1$ possibilities.

These are $3157,3175,3517,3571,3715,3751,5137,5173,5317,5371,5713,5731$, 7135, 7153, 7315, 7351, 7513, 7531.

## Question

How many different 'words' can be made using a) all five letters b) four letters c) three letters from the word TRAIN.

## Answer

a) 5 ! $=5 \times 4 \times 3 \times 2 \times 1=120$
b) Note that the answer here is the same as a_ because when we chose 4 letters the letter omitted is automatic i.e. \# of 'words' $=5 \times 4 \times 3 \times 2=120$
c) $5 \times 4 \times 3=60$

## Question

How many different 'words' can be made using a) all five letters b) four letters c) three letters from the word TRAIT.

## Answer

a) Since there are two Ts which are indistinguishable then for example $\mathrm{T}_{1} \mathrm{RAIT}_{2}$ is the same as $\mathrm{T}_{2} \mathrm{RAIT}_{1}$ and hence the total \# of ways is $\frac{5!}{2}=60$.
b) We need to consider two cases (i.e whether the letter not chosen is a T or not)

| Case 1 | Case 2 |
| :---: | :---: |
| If we chose TRAI then | If we choose TRAT then |
| there are 4! ways | there are $\frac{4!}{2}$ ways |

Similarly if we choose TRIT or TAIT then there are $\frac{4!}{2}$ ways.
The total \# of ways is therefore
$4!+3$ times $\frac{4!}{2}$
$=24+36$
$=60$. Incidentally it is not a coincidence this is the same answer as a) (see previous question)
c)

| $\underline{\text { Case 1 }}$ | $\underline{\text { Case 2 }}$ | $\underline{\text { Case 3 }}$ | $\underline{\text { Case 4 }}$ | $\underline{\text { Case 5 }}$ | $\underline{\text { Case 6 }}$ | $\underline{\text { Case 7 }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TRA | TRI | RAI | AIT | TTI | TTA | TTR |
| $3!$ | $3!$ | $3!$ | $3!$ | $\frac{3!}{2}$ | $\frac{3!}{2}$ | $\frac{3!}{2}$ |

Total \# of ways $=3!+3!+3!+3!+\frac{3!}{2}+\frac{3!}{2}+\frac{3!}{2}$

$$
\begin{aligned}
& =6+6+6+6+3+3+3 \\
& =33
\end{aligned}
$$

## Exercise 9.1

## Permutations

1. Evaluate

$$
\text { a) }{ }_{6} \mathrm{P}_{3} \quad \text { b) }{ }_{10} \mathrm{P}_{4} \text { c) } \frac{10!}{8!} \text { d) } \frac{20!}{18!}
$$

2. How many permutations are there of the letters of the set $\{A, B, C, D\}$ take three at a time?
3. A club has 28 members. In how many ways can the president, treasurer and secretary be chosen to form a committee of three?
4. How many permutations are there of the letters of the word SANDWICH? How many seven letters permutations of the word SANDWICH are there?
5. Express 9 times 8 times 7 in factorials.
6. In a class of 24 students a first and a second prize are to be awarded. In how many different ways can this be done?
7. How many different 'words' can be made using all five letters of
a) the word CAUSE? b) the word CEASE?
8. Solve the following equation for n :

$$
\frac{(\mathrm{n}-2)!}{\mathrm{n}!}=\frac{1}{12}
$$

9. How many four digit integers (greater than 3000) can be formed with the digits $2,3,4,5$ if no digit is repeated?
10. How many even numbers can be formed from the digits $3,4,5,6$ if reptitions are allowed? e.g. 3356 is one possibility.
11. In how many ways can 9 different books be arranged on a shelf if two particular books have to be separated?
12. Find the number of arrangements of the letters of the word STRENGTH if the order of the consonants must not be changed?
13. In how many ways can an 8 question multiple choice exam be answered if each question has five possible choices (A,B,C,D,E)? In how many ways can the exam be answered if no two consecutive answers can be the same?

## Exercise 9.1 (cont'd)

14. How many 3 digit numbers can be formed from the digits of the set $\{2,3,4,5,6,7,8,9\}$
a) if no repetitions are allowed?
b) if no repetitions are allowed and the number must be odd?
c) if repetition is allowed and the number must be even?
d) If no repetition is allowed and the number must be $\geq 500$ ?
e) if repetitions is allowed and the number must be $\leq 300$ ?
f) if no repetition is allowed and the number must be divisible by 5 ?
$\mathrm{g})$ if no repetition is allowed and the number must be divisible by 3 ?
15. Find the value of $n$ if $2\left({ }_{n} \mathrm{P}_{2}\right)+50={ }_{2 \mathrm{n}} \mathrm{P}_{2}$ ?
16. How many integers between 100 and 999 inclusive have repeated digits? e.g. 337
17. How many arrangements are there of all the letters of the word BASEBALL if
a) there are no restrictions
b) the word must begin with S
c) the word must being with B
d) the 2 L's must be together
e) the $S$ and $E$ must be together
f) the 2 B's must be apart
18. One black die and one red die and one blue die are rolled (each numbered one to six). In how many different way scan a total score of 7 be obtained?

## Exercise 9.1 Answers

1. a) 120
b) 5040
c) 90
d) 380
2. 24
3. 19656
4. $8!, 8!5 \cdot \frac{9!}{6!}$
5. 552
6. 120, 60
7. $n=4$
8. $18 \quad 10.128$
9. 7 times 8 ! i.e. 282240
10. 8
11. 390625,81920
12. 210, 120, 147, 120, 49, 30, 72
13. $n=5$
14. 252
15. 5040, 630, 1260, 1260, 1260, 3780
16. 15

## Combinations

A combination is simply a choice of a \# of objects from a set of objects where order, position, rank etc has no relevance. In effect a combination is a choice of a set whereas a permutation is a choice of an ordering of objects.


In other words the \# of combinations is the \# of permutations divided by $r$ ! where $r$ is the \# of objects being chosen.

## Example

The \# of combinations of 2 people chosen from $\{$ Susan, Harry, Sharon, Fred $\} \backslash$ \# of permutations divided by $2!=\frac{12}{2!}=6$.

In general, the \# of combinations of $r$ objects chosen from a set of $n$ objects is $\frac{n!}{(n-r)!r!}$.
This is written ${ }_{n} \mathrm{C}_{\mathrm{r}}$ or more commonly $\binom{\mathrm{n}}{\mathrm{r}}$

## Example

Question Find the \# of combinations of 3 people chosen from a set of 7 objects.
Answer

$$
\binom{7}{3}=\frac{7!}{(7-3)!3!}=\frac{7!}{4!3!}=\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(3 \times 2 \times 1)}=\frac{7 \times 6 \times 5}{3 \times 2 \times 1}=35
$$

For calculation purposes it is easier not to think in terms of factorials. For solving algebraic equations or proofs use factorials.

For example

$$
\binom{20}{3}=\frac{20 \times 19 \times 18}{3 \times 2 \times 1}=10 \times 19 \times 6=1140
$$

Question

In how many ways can 3 people be chosen from a set of 5 people?

Answer

$$
\binom{5}{3}=\frac{5 \times 4 \times 3}{3 \times 2 \times 1}=10
$$

Note that it is not a coincidence that $\binom{5}{3}=\binom{5}{2}$ since clearly $\frac{5 \times 4 \times 3}{3 \times 2 \times 1}=\frac{5 \times 4}{2 \times 1}$
(by canceling the 3 ) but more importantly note that when choosing 3 people there are 2 people left out and hence there is a $1-1$ correspondence between those chosen and those left out.


$$
\text { In general }\binom{n}{r}=\binom{n}{n-r}
$$

## Question

A committee of 5 men and 6 women is to be chosen from a group of 7 men and 9 women. How many possible committees can be chosen.

## Answer

$$
\binom{7}{5} \operatorname{times}\binom{9}{6}=\binom{7}{2} \times\binom{ 9}{3}=\frac{7 \times 6}{2 \times 1} \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1}=21 \times 84=1764
$$

## Question

Solve for n

$$
\binom{n+2}{6}=\binom{n}{4}
$$

Answer

$$
\frac{(n+2)!}{6!(n-4)!}=\frac{n!}{4!(n-4)!}
$$

i.e. $\quad \frac{(n+2)!}{n!}=\frac{6!}{4!}$
$\therefore(\mathrm{n}+2)(\mathrm{n}+1)=6 \times 5$

$$
\mathrm{n}^{2}+3 n+2-30=0
$$

$$
n^{2}+3 n-28=0
$$

$$
(n+7)(n-4)=0
$$

Clearly $\mathrm{n} \neq 7$ therefore $\mathrm{n}=4$.

## Question

Find the number of ways in which 7 different objects can be disturbed among 3 people so that Mr. A receives 3 objects, Mr. B receives 3 objects and Mr. C receives 2 objects.

## Answer

$$
\begin{aligned}
& \binom{8}{3} \text { times }\binom{5}{3} \text { times }\binom{2}{2} \\
& =\frac{8 \times 7 \times 6}{3 \times 2 \times 1} \text { times } \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \text { times } \frac{2 \times 1}{2 \times 1} \\
& =56 \text { times } 10 \times 1=560
\end{aligned}
$$

## Question

Solve $(2 n)!=20(n!)^{2}$

$$
\begin{aligned}
& \text { i.e. } \frac{2 n!}{n!n!}=20 \\
& \therefore\binom{2 n}{n}=5 \times 4
\end{aligned}
$$

This equation cannot be solved by regular high school methods because n is a positive integer by experimentation (trial and error) we can deduce that $\mathrm{n}=3$.
Clearly $f(n)=\binom{2 n}{n}$ is an increasing function for all integers $n$ and hence $n=3$ is the unique solution.

## Exercise 9.2

## Combinations

1. Evaluate a) ${ }_{6} \mathrm{C}_{3}$ b) ${ }_{9} \mathrm{C}_{4}$ c) $\binom{9}{2}$ d) $\binom{11}{9}$
2. In how many ways can a soccer team (of 11 players) be chosen from 15 people?
3. In how many ways can a baseball manager choose a starting line-up of 9 players out of a total of 15 players? (No order of batting is required)
4. In how many ways can three cards be chosen from a desk of 52 cards?
5. A committee of 4 men and 3 women is to be chosen to form a committee of 7 people from a group of 10 men and 6 women. How many possible committees can be chosen?
6. Of 10 electric light bulbs, 3 are faulty but it is not known which ones are faulty. In how many ways can 3 bulbs be chosen from the 10 ? How many of these selections will contain no faulty bulbs? How many of the selections will contain exactly 1 faulty bulb?
7. How many ways are there of choosing a committee of 4 people from a group of 7 people? How many ways are there of choosing a committee of 3 people from 7 people? Is it a coincidence that the number of choices are equal?
8. How many matches are required to play a round-robin tournament involving 37 people? How many matches would be required if a knock-out tournament were played?
9. a) Twelve straight lines are drawn on a piece of paper. No two lines are parallel and no three lines are concurrent (i.e. no three lines pass through the same point). How many points of intersection are there of pairs of the 12 lines?
b) How many triangles are formed by the vertices when all 12 sides of a polygon are joined?
10. Determine the number of bridge hands (of 13 cards) which contain
a) exactly 3 spades
b) exactly 5 diamonds and 8 clubs
c) exactly 2 five card suits
d) no hearts.

## Exercise 9.2 (cont'd)

11. In how many ways can a committee of 4 be selected from 6 men and 7 women if at least 1 woman must be on the committee?
12. Find the number of ways in which 12 different objects can be distributed among 3 people so that Mr. Smith receives 3 objects, Mr. Jones receive 4 objects and Mr. Allen receives 5 objects.
13. Find the number of ways in which 8 people can be assigned to 2 different rooms if each room must have at least 2 people in it.
14. Solve the following equations for $n$ :
a) $\binom{\mathrm{n}}{4}=\binom{\mathrm{n}}{8}$
b) $\frac{n!}{(n-2)!}=42$
c) $\binom{2 \mathrm{n}}{2}=6\binom{\mathrm{n}}{2}$
15. In how many ways can 5 cards be selected from a pack of 52 cards if exactly 2 aces must be included?
16. A selection of 5 books has to be made from a collection of 8 different books where order of choice does not matter:
a) how many selections are there?
b) how many selections are there if one particular book has to be selected?
c) how many selections are there if one particular book has to be excluded?
17. Fill in the blank

$$
\binom{8}{5}=\binom{7}{4}+?
$$

18. Fill in the blank

$$
\binom{\mathrm{n}}{\mathrm{r}}=\binom{\mathrm{n}-1}{\mathrm{r}-1}+?
$$

## Exercise 9.2 (cont'd)

19. In how many ways can 5 different Latin books, 4 different Greek books and 3 different French books be arranged on a shelf so that the books in each language come together?
20. How many arrangements of the letters of the word TOMATO are such that the T's are separated?
21. How many diagonals does a 23 sided polygon have?
22. In how many ways can a tennis mixed doubles pair be chosen from 5 married couples if no husband can play with his wife?
23. In how many ways can a committee of 7 be chosen from 9 married couples if:
a) no restrictions are made
b) the committee must have 4 men and 3 women
c) a husband and a wife cannot serve on the same committee
d) Mrs. Smith refuses to serve on a committee with Mrs. Jones

## Exercise 9.2 Answers

1. $20,126,36,55$
2. $\binom{15}{11}$
3. $\binom{15}{9}$
4. $\binom{52}{3}$
5. $\binom{10}{4}\binom{6}{3}$
6. $\binom{10}{3} \cdot\binom{7}{3} 3\binom{7}{2}$
7. 35,35 , no
8. $\binom{37}{2} 36$
9. $\binom{12}{2},\binom{12}{3}$
10. a) $\binom{13}{3}\binom{39}{10}$
b) $\binom{13}{5}\binom{13}{8}$
c) $\binom{4}{2}\binom{13}{5}\binom{13}{5}\binom{26}{3}$
d) $\binom{39}{13}$
11. $\binom{13}{4} \operatorname{minus}\binom{6}{4}$
12. $\binom{12}{3}\binom{9}{4}$
13. 238
14. a) 12
b) 7
c) 2
15. $\binom{4}{2}\binom{48}{3}$
16. a) 56
b) 35
c) 21
17. $\binom{7}{5}$
18. $\binom{n-1}{r}$
19. 103680
20. 120
21. 230
22. 20
23. a) $\binom{18}{7}$
b) $\binom{9}{4}\binom{9}{3}$
c) $2^{7} \operatorname{times}\binom{9}{7}$
d) $\binom{18}{7}-\binom{16}{5}$

## Question

How many subsets does a set of three objects have? (including the empty set and the set itself).

## Answer

Consider the set $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$
Each object is either chosen or it is not. Therefore we can tabulate the \# of subsets as below

| A | B | C |
| :---: | :---: | :---: |
| Yes or No | Yes or No | Yes or No |

## For example

| A | B | C |
| :---: | :---: | :---: |
| No | Yes | Yes |

Corresponds to the subset $\{B, C\}$

Since there are 2 possibilities for each element of the set of 3 objects then there are $2^{3}=8$ possible subsets.
These are $\phi,\{A\},\{B\},\{C\},\{A, B\},\{A, C\},\{B, C\},\{A, B, C\}$

In general, the number of subjects of a set of $n$ objects is $2^{n}$.

## Pascal's Theorem

## Example

In how many ways can 4 people for a committee be chosen from $\{A, B, C, D, E, F\}$ if
a) person A must be included
b) person A must be excluded
a) Since person A is included on the committee then we have essentially to choose 3 more people from $\{B, C, D, E, F\}$ i.e. $\#$ of committees $=\binom{5}{3}=10$
b) Since A is excluded then essentially we have to choose 4 people from $\{\mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$
i.e. $\#$ of committees $=\binom{5}{4}=5$
c) Since a) and b) together represent the total number of ways of choosing a committee of 4 people from the set of 6 people then

$$
\binom{6}{4}=\binom{5}{3}+\binom{5}{4}
$$

Note L.H.S $=15$ and R.H.S $=10+5$.

## In general

$$
\binom{\mathrm{n}}{\mathrm{r}}=\binom{\mathrm{n}-1}{\mathrm{r}-1}+\binom{\mathrm{n}-1}{\mathrm{r}}
$$

This is called PASCAL'S THEOREM and is the basis for PASCAL'S TRIANGLE used in the study of the BINOMIAL THEOREM (to be studied soon)

## Fundamental Principles of Counting: Basic Formulae

## 1. THE MULTIPLICATION PRINCIPLE

If operation A can be performed in a ways, and for each of these ways operation $B$ an be performed in $b$ ways, then the combined operation $A$ and $B$ can be performed in ab ways.
e.g. How many different coat-and-trouser outfits can be formed if 4 coats and 5 pairs of trouser are available? Solution: $4 \times 5=20$

## 2. THE ADDITION PRINCIPLE

If event $A$ can occur in a ways, even $B$ can occur in $b$ ways and event $A$ and $B$ cannot occur simultaneously, then A or B can occur in $\mathrm{a}+\mathrm{b}$ ways.
e.g. When two dice are rolled a total score of 7 can be obtained in six ways, and a total score of 11 can be obtained in two ways. In how many ways can a total score of 7 or 11 be obtained? Solution $6+2=8$

## 3. The "TAKE IT OR LEAVE IT" PRINCIPLE

A set containing n different objects has $2^{\mathrm{n}}$ subsets, including the original set and the empty set.
e.g. How many distinct subsets has the set $\{a, b, c\}$ ?

Solution: In construction of a subset, we examine each of the elements a, b , and c in turn. Each can be disposed of in two ways, either included or excluded. Therefore there are $2 \times 2 \times 2=8$ subsets.

## 4. THE "HOW MANY?" PRINCIPLE

If there are $p$ objects of one kind, $q$ of a second kind, $r$ of a third kind, etc. then there are $(\mathrm{p}+1)(\mathrm{q}+1)(\mathrm{r}+1)$ different selections, including the empty selection which can be made from the collection.
e.g. If there are 5 bananas, 3 oranges and 7 pears available, in how many ways may a selection of one or more fruits be made?

Solution: The selection may contain $0,1,2,3,4$ or 5 bananas. Thus the bananas may be represented in 6 different ways. Similarly, the oranges represented in 4 ways and the pears in 8 ways. Therefore, there are in theory $6 \times 4 \times 8=196$ different selections. However, included among these is the
selection consisting of non bananas, no oranges and no pears. Therefore there are 196-1 = 195 valid selections

## 5. PERMUTATIONS OF OBJECTS, ALL OF WHICH ARE DIFFERENT

IF there are $n$ different objects available and $r$ positions are to be filled $(r \leq n)$, then the number of distinct ways that the positions can be filled with distinct objects is $n(n-1)(n-2) \ldots(n-(r-1))(r$ factors). This is a direct application of the multiplication principle.

Notation and definition: ${ }_{n} \mathrm{P}_{\mathrm{r}}=\frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{r})!}$
Computational Formula: ${ }_{n} \mathrm{P}_{\mathrm{r}}=(\mathrm{n})(\mathrm{n}-1) \ldots(\mathrm{n}-\mathrm{r}+1)$
e.g. ${ }_{7} \mathrm{P}_{3}=7 \times 6 \times 5=210$

The computation formula is usually used in actual calculations or in solving equations; the definition is used in proofs.
6. COMBINATIONS OF OBJECTS, ALL OF WHICH ARE DIFFERENT Suppose we have 12 people and we wish to invite 8 of them to a party. In how many ways can the party be selected?
Answer $\binom{12}{8}=\binom{12}{4}=\frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1}=495$
7. PERMUTATIONS OF OBJECTS SOME OF WHICH ARE THE SAME

Consider the arrangements of all the letters of the word CHEESE. If all the letters were different we would have 6! Permutations. But since the 3 Es are all the same then we need to divide by e! since there are 3! Ways of arranging the Es. This means the number of permutations of all the letters of CHEESE is $\frac{6!}{3!}=120$. Similarly, the number of permutations of all the letters of MISSISSIPPI is $\frac{11!}{4!4!2!}=34650$.

## Exercise 9.3

## Permutations and Combinations

1. How many permutations of the letters $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ are there?
2. In a form of 30 students, a first and a second prize are to be awarded. In how many ways can this be done?
3. How many permutations of the letters (A,B,C,D) are there, taken 3 at a time?
4. How many 7 digit phone numbers (using digits 0 to 9 ) can be formed if 0 can't be the first digit?
5. Of 10 electric light bulbs 3 are faulty but it is not known which. In how many ways may 3 bulbs be selected? How many of these selections will include at least one faculty bulb?
6. Evaluate.
a)6!
b) ${ }_{10} \mathrm{P}_{4}$
c) ${ }_{6} \mathrm{C}_{3}$ d) ${ }_{7} \mathrm{C}_{4}$ e) ${ }_{7} \mathrm{C}_{3}$ f) $\binom{7}{3}$
g) $\binom{8}{3}$ h) $\frac{23!}{21!}$
7. Express $10 \times 9 \times 8$ in factorials.
8. How many permutations are there of all the letters of the word FOOTBALL?
9. In how many ways can 3 cards be c hosen simultaneously from a pack of 52 cards?
10. How many consecutive zeroes occur on the right hand end of 30 ! Written out in expanded form?
11. A committee of 4 men and 2 women is to be chosen from a group of 10 men of 6 women. In how many ways can the committee be chosen?
12. Show that $\binom{n}{r}=\binom{n}{n-r} \quad$ Solve $\binom{72}{r}=\binom{72}{2 r}$
13. What is the chance of getting a total of 71 in the throw of 12 dice?
14. Determine the number of matches necessary to complete a round-robin tournament involving 64 players. How many matches would be necessary if it were a knock-out tournament?
15. What is the greatest number of points of intersection of
a) 12 straight lines
b) 9 circles
c) 6 straight lines and 5 circles?
16. There are 10 staff and 20 students who wish to participate on a committee. How many ways are there to form this 5 person committee with at least 1 student and 1 staff member on it?
17. How many bridge hands of 13 cards contain
a) exactly 12 cards in a suit
b) 4 spades, 4 hears, 4 diamonds and 1 club
c) exactly 5 clubs?
18. How many diagonals does a 75 sided polygon have?
19. Find the numbers of ways in which 6 people can be assigned to two different rooms if each room must have at least 2 people in it?
20. Solve for n if: $\binom{\mathrm{n}}{2}-\mathrm{n}=\binom{8}{2}-1$
21. In how many ways can 2 Americans, 2 Frenchmen, 2 Danes and 2 Italians be seated in a row so that those of the same nationality sit together? Remember that people are always distinguishable (i.e. different)
22. From a deck of 52 cards, how many possible bridge hands of 13 cards are there containing exactly 4 hears and 6 diamonds?
23. How many subsets, each containing 8 elements, chosen from a set of 10 objects are there if a particular one must be included in the subset? How many subsets of 8 elements would there be if the particular one had to be excluded? Try to deduce a general formula from these results.
24. In how many ways can 3 cards be selected from a desk of 52 cards if at least one of them must be an ace?
25. In how many ways can a committee of 5 be chosen from 10 candidates
a) so as to include both the youngest and the oldest
b) so as to exclude the youngest if the oldest is included.
26. In how many ways can 8 prizes be distributed among 5 students? (It is not necessary that each student must win a prize).
27. In how many ways can 6 people be divided into 3 couples?
28. In how many integers between 2000 and 8999 does the digit 3 occur?

## Exercise 9.3 Answers

1. 6
2. 2.870
3. 24
4. $9,000,000$
5. a) 120
b) 85
6. a) 720
b) 5040 c) 20 d$)$
7. a) 66
b) $72 \quad$ c) 95
8. $\binom{30}{5}-\binom{20}{5}\binom{10}{0}-\binom{10}{5}\binom{20}{0}$
9. a) 2028
b) 4715800000
10. 2700

35 f) 21
19. 50
7. $\frac{10!}{7!}$
8. $\frac{8!}{2!2!}$
9. 22100
10.7 zeros
11. 3150
12. $z=24$
13. 1 in 3 times $6^{10}$
14. a) 2016
b) 63
20. $\mathrm{n}=9$
21. 384
22. $\binom{13}{4}\binom{13}{6}\binom{26}{3}$
23. $36,9\binom{n}{r}=\binom{n-1}{r-1}+\binom{n+1}{r}$
24. 4804
25. a) $56 \quad$ b) 196
26. 390625
27. 15
28. 2626

## Exercise 9.4

## Permutations and Combinations

1. In how many ways can 4 prizes (one each in Math, History, French and Science) be distributed among 3 students? It is not necessary that each student wins a prize.
2. In how many ways can 9 different books be arranged on a shelf if 2 particular books must be separated.
3. Determine the number of matches necessary to complete a round robin tournament of 16 players. How many matches would be necessary if it were a knock-out tournament? How many matches would be necessary for a knock-out tournament containing 25 players? (Some players would have to have a bye in opening round of course).
4. In how many ways can 3 cards be selected from a deck of 52 cards so that the 3 cards are of different suits.
5. How many subsets are there of ( $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}$ ) ?
6. How many positive integers solutions $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ are there to the following equation

$$
x+y+z=12
$$

7. Given an $n$-gon with all its diagonals drawn, find the number of internal intersections points of the diagonals if no 3 or more diagonals are concurrent.
8. In how many ways can a man who has 10 different chairs put 5 in one room, 3 in a second and 2 in a third room?
9. In how many ways can a committee of eight be arranged at a round table? In how many of these does the chairman sit between the secretary and the treasurer?
10. In how many ways can a committee of 9 be formed from 10 men and their wives, if no husband serves on it with his wife?
11. There are 6 ornaments on my mantelpiece. In how many ways can I put 3 more on it without changing the order of those already there?
12. Eight boys and two girls sit on a bench. If the girls may sit neither at the ends nor together, in how many ways can they be arranged?

## Exercise 9.4 (cont'd)

13. In selecting a hand of 5 cards from a deck of 52 cards find:
a) The number of hands containing exactly one pair e.g. 55 K Q J
b) The number of hands containing 5 cards in the same suit.
c) The number of hands containing at least one Ace
14. A language is to be constructed by forming 'words' from the letters a,e,i,o,u (e.g. iiow is a 'word')
a) How many 'words' can be created?
b) How many 'words' containing no repeated letters have the letters appearing in the English alphabetical order (e.g. aei).
15. How many arrangements of the letters of the word CALENDAR have CA (C AND A) together and the N AND D NOT together?

## Exercise 9.4 Answers

1. 81
2. 7 times 8 !
3. $120,15,24$
4. a) $\frac{\binom{13}{1}\binom{4}{2}\binom{48}{1}\binom{44}{1}\binom{40}{1}}{3!}$
5. 8788
6. 128
b) $\binom{4}{1}\binom{13}{5}$
7. 55
8. $\binom{\mathrm{n}}{4}$
c) $\binom{52}{5}-\binom{48}{5}\binom{4}{0}$
9. 2520
10. a) $5+5^{2}+5^{3}+5^{4}+5^{5}$
11. 5040, 240
b) $5+5 \cdot 4 \cdot 3+5 \cdot 4 \cdot 3 \cdot 2+5$ !
12. 5120 ?
c) 31
13. 504
14. 1,693,440
15. 6720
