CHAPTER FOURTEEN

14. Linear Programming

Linear Programming involves problems where a variable has to be maximised or minimised subject to certain conditions. The conditions are called **CONSTRAINTS** and the variable, the **OBJECTIVE FUNCTION**. They are represented in linear form.
It is best illustrated by example.

14.1 **Example 1**

A manufacturer makes products P_1 and P_2 by using machines M_1 and M_2. The time in hours required to make P_1 and P_2 on each machine is shown in the table below.

	P_1	P_2
M_1	4	$1\frac{1}{2}$
M_2	3	2

The profits on P_1 and P_2 are \$30 and \$15 each respectively. The manufacturer operates for 420 hours a month.
How many of each product P_1 and P_2 should be made per month to maximise profit?
The problem may be set up as follows.

Let x and y represent the number of P_1 and P_2 made respectively per month.
The profit = \$30x + 15y is called the **OBJECTIVE FUNCTION**.
Time on machine M_1 per month cannot exceed 420 hours.

$$\therefore 4x + 1\frac{1}{2}y \le 420$$

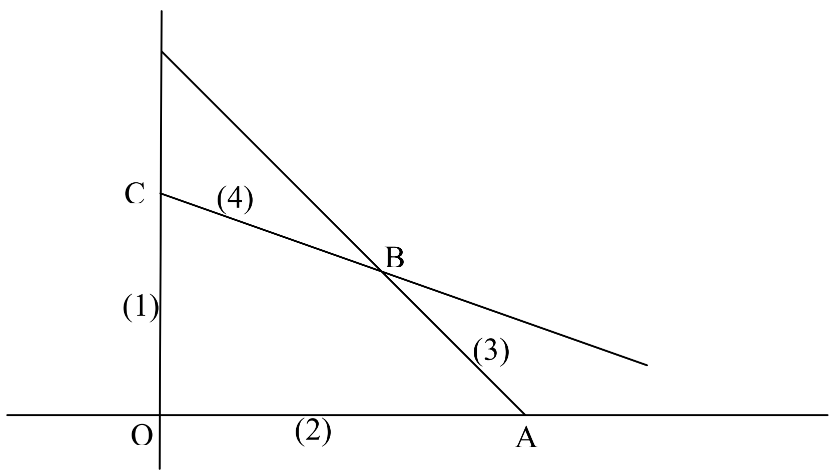
Similarly the time condition of machine M_2 leads to

$$3x + 2y \le 420$$

Clearly $x \ge 0$ and $y \ge 0$.
The problem may be summarised.
Maximise 30x + 15y OBJECTIVE FUNCTION

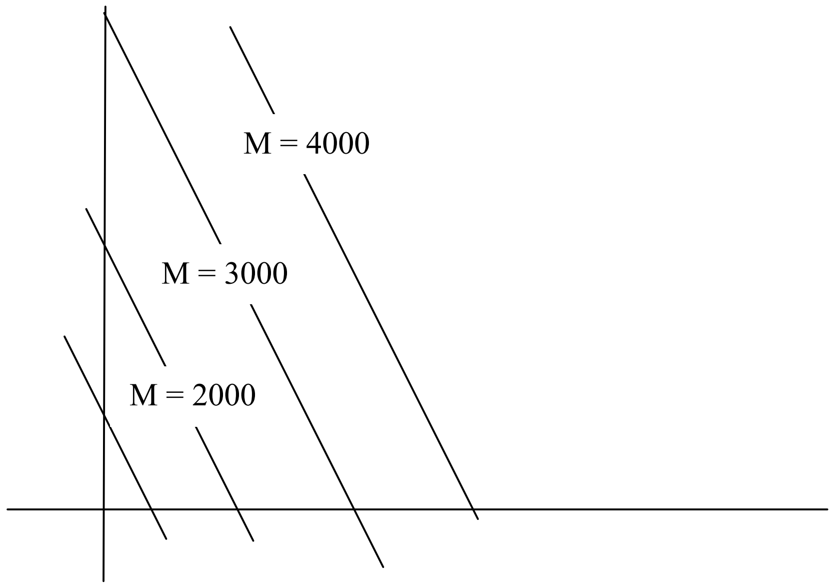
$$\left.\begin{array}{lll} \text{Subject to} & x \geq 0 & (1) \\ & y \geq 0 & (2) \\ & 4x + 1\frac{1}{2}y \leq 420 & (3) \\ & 3x + 2y \leq 420 & (4) \end{array}\right\} \text{CONSTRAINTS}$$

There are many ways of solving linear programming problems like this but for simple two variables cases the simplest and most intuitive are by graphical methods. The constraint may be drawn as follows.



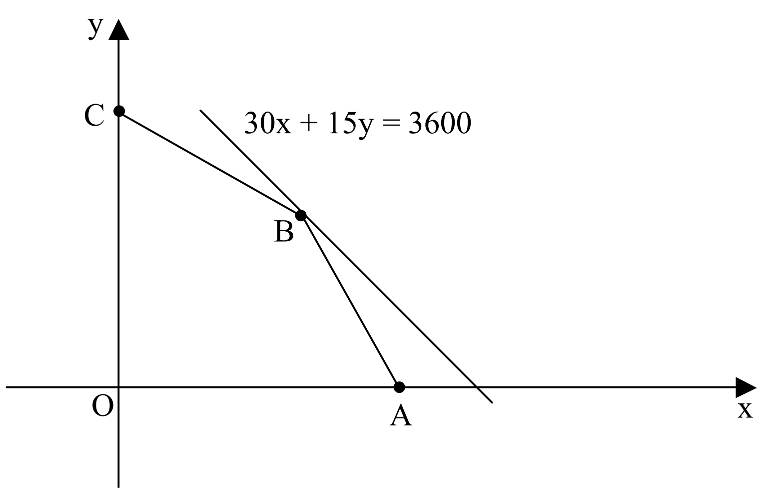
The shaded area OABC represents the intersection of the four constraints (1), (2), (3), (4). We call OABC the **FEASIBLE REGION.** This means that any point (with integer co-ordinates) inside OABc is a feasible solution to the problem – not necessarily the optimum solution. For example, (10,20) is a possible solution, i.e. we could produce 10 or P_1 and 20 of P_2. Clearly however we can do better by making more of each.

The objective function is represented by $30x + 15y = M$ where M is the profit. $30x + 15y = M$ is a line with slope -2 whose position is determined by the value of M.



The graph shows profit lines for different values of M. What we need to do is to find a point in the feasible region lying on a profit line whose profit (M) is greatest.

Superimposing the profit lines on the feasible region produces the fact that point B lies on a profit line which is “farthest out” from the origin, i.e. having the maximum value for M, i.e. maximum profit.



Point B has co-ordinates (60,120) (i.e. the intersection of $4x + 1\frac{1}{2}y = 420$ and $3x + 2y = 420$) and hence the manufacturer does best by producing 60 of product P_1a nd 120 of product P_2 for a total profit of \$3600.

Example 2

Two workers, A and B, ear \$6 per hour and \$8 per hour respectively. A makes 6 products of P_1 and 4 products of P_2 per hour. B makes 10 products of P_1 and 2 products of P_2 per hour. It is necessary to manufacture 148 of P_1 and 52 of P_2 to complete an order. How many hours should each work to fill the order at minimum labour cost? Find the minimum labour cost.

Solution

Let x,y, be hours A and B work respectively.

Then, since there has to be at least 148 of P_1 produced,

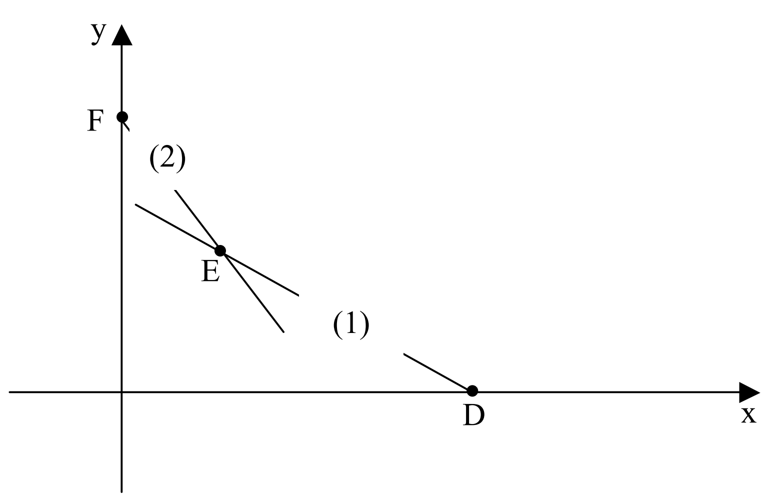
$6x + 10y \geq 148$ (1)

Similarly with respect to P_2,

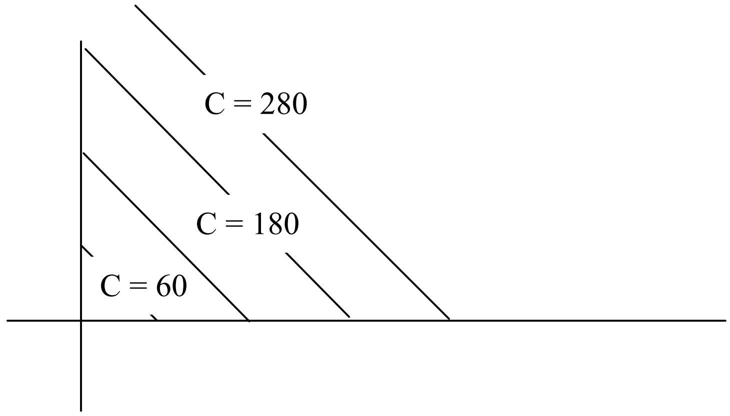
$4x + 2y \geq 52$ (2)

Also $x \geq 0$ and $y \geq 0$.

Therefore the feasible region is as shaded below, i.e. x axis – DEF – y axis.



The cost lines are $6x + 8y = C$ where C is the cost.



We need to find the point in the feasible region lying on a cost line which has minimum value for C.

A visual superimposing of the cost lines upon the feasible region shows us that point E ist he point required. E has co-ordinates (8,10) – the intersection of $6x + 10y = 148$ and $4x + 2y = 52$.

i.e. Worker A works for 8 hours, B for 10 hours with a total cost of
8 x \$6 + 10 x \$8, i.e. \$128

A problem involving more constraints follows.

Example 3

Two warehouses W_1 and W_2 have 12 cartons and 8 cartons respectively of a product which has to be shipped to three stores S_1, S_2, S_3 which need 8, 6 and 6 cartons respectively, Shipping costs vary according to the table below.

		To		
		S_1	S_2	S_3
From	W_1	\$7	\$3	\$2
	W_2	\$3	\$1	\$2

What is the most economical way to ship the cartons?

Solution

Let x be the number of cartons shipped from W_1 to S_1.

Let y be the number of cartons shipped from W_1 to S_2

Then the table showing the shipping of cartons is as below.

		S_1	S_2	S_3
From	W_1	x	y	$12 - x - y$
	W_2	$8 - x$	$6 - y$	$x + y - 6$

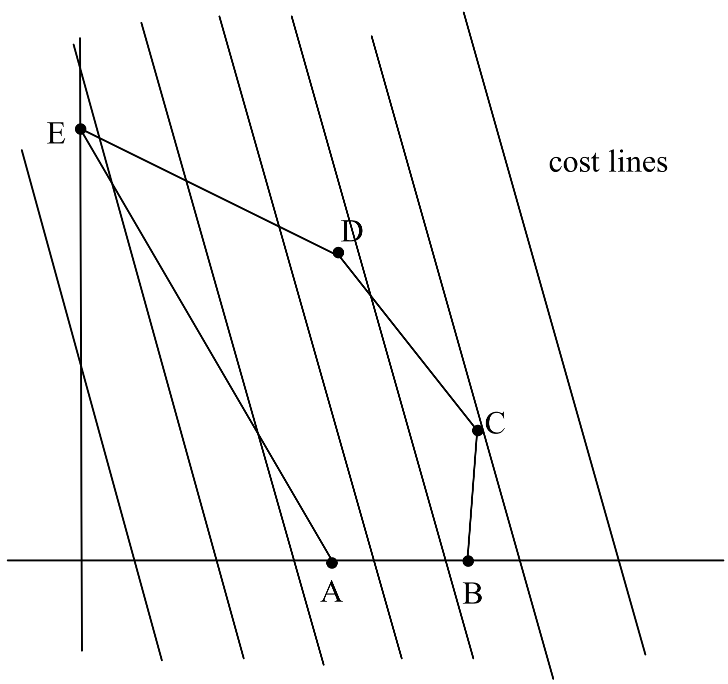
Clearly each of these values is not negative hence

$$\left.\begin{array}{l} x \geq 0,\ y \geq 0,\ 12 - x - y \geq 0 \\ 8 - x \geq 0,\ 6 - y \geq 0,\ x + y - 6 \geq 0 \end{array}\right\}$$

i.e. $0 \leq x \leq 8$, $0 \leq y \leq 6$, $6 \leq x + y \leq 12$. Constraints

The most is $\$(7x + 3y + 2(12 - x - y) + 3(8 - x) + 1(6 - y) + 2(x + y - 6))$

i.e. cost is $\$4x + 2y + 42$ Objective Function



Drawing the constraints produces the feasible region ABCDE. The cost lines superimposed indicate that E is the optimum solution, i.e. at (0,6).
i.e. the optimum solution occurs when x = 0 and y = 6.
i.e. the shipping table is

	S_1	S_2	S_3
W_1	0	6	6
W_2	8	0	0

The total cost is $54.

From the examples quoted it appears that a vertex of a feasible region is an optimum solution and in fact this is true in all cases. The proof is beyond the scope of this text but note that the optimum solution is not necessarily unique. For example, the objective function may be parallel to a boundary of the feasible region leading to the result that all points on the boundary are optimum solutions.

From the examples quoted it might also appear true that the optimum solution is the intersection point of the two boundary lines of the feasible region having slopes one, the smallest of the slopes greater than the objective function and the other, the largest of the slopes less than the objective function. This is not true and the following example shows the advantage of drawing a diagram rather than relying on the algebraic data.

Example 4

A man buys 100 pills to satisfy his vitamin requirements of 700 units of B_1, 600 units of B_2 and 260 units of B_6.

Pills 1, 2 and 3 contain units of B_1, B_2 and B_6 as in the table below.

	B_1	B_2	B_6	Cost
Pill 1	10	5	3	$0.05 each
Pill 2	10	2	8	$0.06 each
Pill 3	6	7	2	$0.04 each

Find the number of each pill he should buy to satisfy his vitamin requirement at least cost.

Solution

Let x be the number bought of Pill 1.

Let y be the number bought of Pill 2.

Then $100 - x - y$ is the number bought of Pill 3.

The cost function is $5x + 6y + 4(100 - x - y)$¢

i.e. Cost = $x - 2y + 400$¢

Constraints

$x \geq 0$, $y \geq 0$, $100 - x - y \geq 0$.

B_1 requirement constraints $10x + 10y + 6(100 - x - y) \geq 700$ (1)

B_2 requirement constraints $5x + 2y + 7(100 - x - y) \geq 600$ (2)

B_6 requirement constraints $3x + 8y + 2(100 - x - y) \geq 360$ (3)

Simplified, the problem becomes

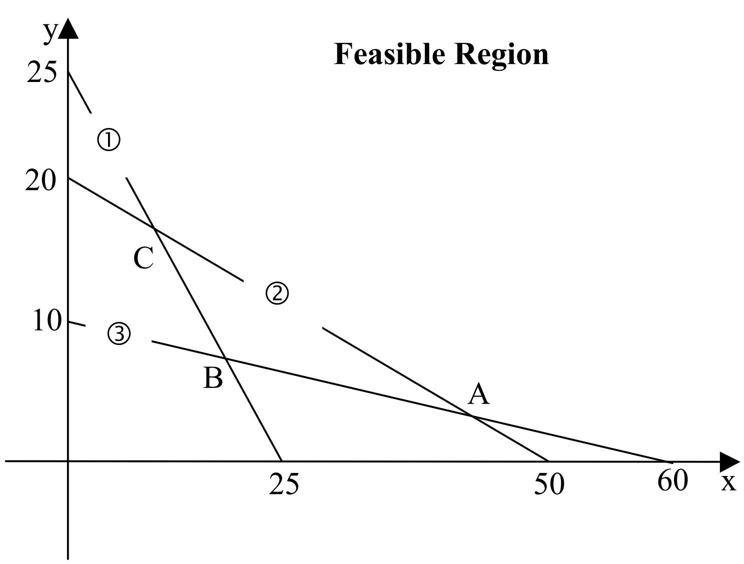
Minimise $x + 2y + 400$

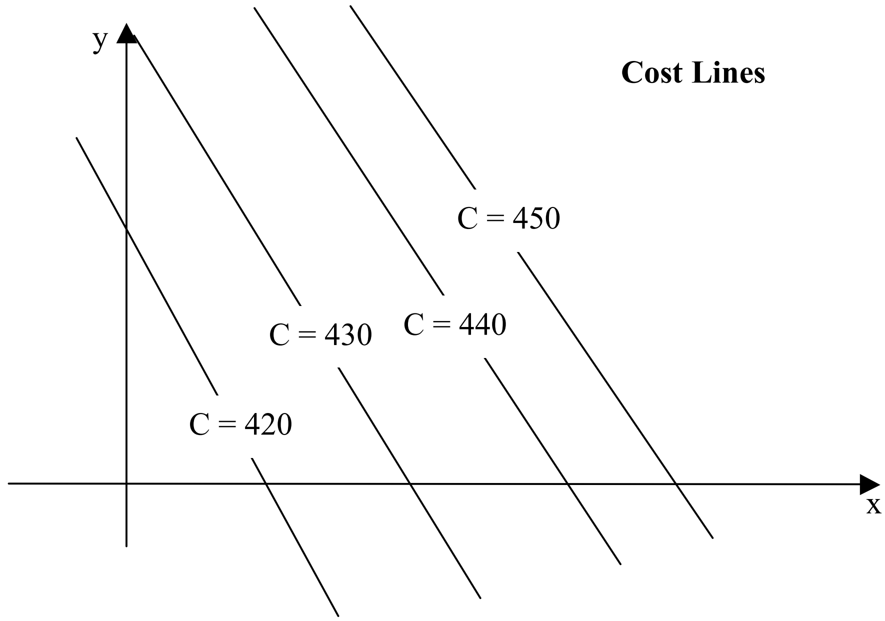
Subject to $x \geq 0$, $y \geq 0$, $x + y \leq 100$.

$$4x + 4y \geq 100 \quad (1)$$

$$2x + 5y \leq 100 \quad (2)$$

$$x + 6y \geq 60 \quad (3)$$





The cost lines shown above when superimposed upon the feasible region lead to the result that B is the vertex of the feasible region lying on a cost line which has smallest cost.

B has co-ordinates (18,7) – the intersection of $4x + 4y = 100$ and $x + 6y = 60$.

i.e. the man should buy 18 of Pill 1, 7 of Pill 2 and hence 75 of Pill 3 for a total cost of 432¢, i.e. \$4.32.

Exercise 14.1

1. A manufacturer is to make an unknown number of two models A and B. The models require machine work by three machines M_1, M_2 and M_3 as indicated in the table.

Time in hours

	M_1	M_2	M_3
Model A	1	2	1.6
Model B	2	1	1.6

No machine may work more than 48 hours per week and profit on Model A is \$4 and on Model B is \$3. How many should be produced per week of each to maximise profit?

2. A linear programming problem leads to the constraints $x \geq 0$, $y \geq 0$, $x + 2y \leq 18$, $2x + y \leq 12$. Find the point (x,y) such that k is maximised in the cases below.
   i) $y = k$
   ii) $y = -\frac{1}{4}x + k$
   iii) $y = \frac{1}{2}x + k$
   iv) $y = -x + k$
   v) $y = -2x + k$
   vi) $y = -3 + k$
3. Use the same details as in 14.1, Example 3 except the cost table is: -

	S_1	S_2	S_3
W_1	\$2	\$1	\$3
W_2	\$5	\$3	\$4

   W_1 and W_2 have 12 cartons each and S_1, S_2, S_3 need 8 cartons each.
4. Two mines A and B produce three grades of ore: - High, Medium and Low. Together they have to produce 12 unites of High, 8 unites of Medium and 24 units of Low to complete an order.
   The production table per day is:

	High	Medium	Low
A	6	2	4
B	2	2	12

   The running costs per day are \$2,000 for A and \$1,600 for B. Find the optimum working arrangement of the mines so that the total cost is minimised when completely the order.
5. A man has a diet whereby he has to have a **minimum** weekly requirement of 32 units of protein, a **minimum** of 25 units of carbohydrates and a **maximum** of 61 units of fats. He can choose between three types of food – A, B or C. The prices and contents of the foods, in units are in the table on the next page. –

Food

	A	B	C
Protein	3	1	2
Carbohydrates	2	1	1
Fats	5	2	1
Cost per pound	\$3.00	\$1.50	\$2.00

He is to buy a weekly total of exactly 20 pounds. How many pounds of each food should he buy each week to minimise his cost? What is that minimum cost?

6. Maximise $4x + y$

   Subject to $x + y \leq 7$, $x \geq 0$, $y \geq 0$, $2x - y \leq 3$, $2x + 4y \leq 24$, $4x - y \leq 8$.

7. A manufacturer produces two types of products A and B using three machines – lathe, grinder and drill. The machine requirements in minutes for manufacturing each product is set in the table below.

	A	B
Lathe	2	4
Grinder	6	2
Drill	6	3

The lathe can work at most 400 mins per day. The grinder can work at most 450 mins per day. The drill can work at most 480 mins per day.

The profits on products A and B are \$34 and \$27 respectively.

i) Find the optimum production plan.

ii) The manufacturer has enough money to buy one more machine and by so doing he will increase the capacity of that type of machine by 40 minutes per day. Which machine should he buy?

8. A dietician wishes to mix two types of food so that the vitamin content of the mixture contains at least 9 units of vitamin A, 7 units of vitamin B, 10 units of vitamin C and 12 units of vitamin D. Foods 1 and 2 contain vitamins in units per pound as shown in the table.

	Vitamin A	B	C	D
Food 1	2	1	1	1
Food 2	1	1	2	3

Food 1 costs \$5 per pound and Food 2 costs \$7 per pound. find the minimum cost of a mixture satisfying the vitamin requirements.

Excerise 14.1 Answers

1. 18 of Model A. 12 of Model B Profit = \$108.
2. i) (0,9) ii) (0,9) iii) {(0,9), (2,8)} iv) (2,8) v) {(2,8), (3,6), (4,4), (5,2), (6,0)} vi) (6,0)
3.

	S_1	S_2	S_3
W_1	8	4	0
W_2	0	4	8

Minimum cost = \$64

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4. A opens for 1 day, B for 3 days.
5. 5 pounds of A, 13 pounds of B, 2 pounds of C, at a total costs of \$38.50.
6. x = 3, y = 4 Maximum value of 16.
7. i) Manufacture 40 of A and 80 of B, for total profit of \$3520.
   ii) Drill (New optimum production is 50 of A and 75 of B with profit of \$3725.)
8. 4 pounds of Food 1 and 3 pounds of Food 2, for total cost of \$41.

10.2 Linear Programming With More Than 2 Variables.

Where more than two variables is involved, a graphical approach to solutions of linear programming problems is not recommended since, with two variables for example, it requires graphs of planes in R^3 with three dimensional feasible regions of irregular shapes with planes as boundaries. Furthermore, the objective function is a plane: all rather difficult to visualise and draw.

However, the result that a vertex is still the optimum solution still holds true and a crude but effective method of solution is to find the vertices of the feasible region and find the optimum vertex by direct substitution into the objective function.

While this may be time-consuming to do by hand it is, of course, an easy task for the computer which is ideally suited for solving linear programming problems.

Example

A manufacturer makes three types of cabinets A,B and C. These products use three kinds of wood – oak, teak and plywood. He has available 18- feet of oak, 180 feet of teak and 250 feet of plywood. Cabinet styles A, B and C require wood as in the table set out below.

Styles

	A	B	C
Oak	2	0	3
Teak	0	5	2
Plywood	3	4	2

Profits on Cabinets A, B, C are \$30, \$40, \$50 respectively. How many cabinets of each type should he make to maximise his profit?

Solution

Let z_1, z_2, z_3 be the number of cabinets made of styles A, B, C respectively.

Then $z_1 \geq 0$, $z_2 \geq 0$ $z_3 \geq 0$.

The oak constraint is $2z_1 + 3z_3 \leq 180$.

The teak constraint is $5z_2 + 2z_3 \leq 180$

The plywood constraint is $3z_1 + 4z_2 + 2z_3 \leq 250$.

Objective Function Maximise $P = 30z_1 + 40z_2 + 50z_3$.

We need to find the vertices of the feasible region. This is done by noting that a vertex of the feasible region is the intersection of three planes satisfying the other three inequalities. There will be many points of intersection (20 in fact for this problem) but many of these will not lie in the feasible region because they will not satisfy the other inequalities. For example, the intersection of $z_1 = 0$, $z_2 = 0$ and $3z_1 + 4z_2 + 2z_3 = 250$ is (0,0,125). But (0,0,125) is **not** a vertex of the feasible region because (0,0,125) does not satisfy $2z_1 + 3z_3 \geq 180$.

As noted, the process of finding the vertices of the feasible region is time-consuming. Students with knowledge of computer languages and techniques could use this knowledge to facilitate the finding of the vertices.

The vertices of the feasible region with the corresponding profits are listed below.

	O	A	B	C	D	E	F	G
Vertices	(0,0,0)	(0,0,60)	(0,36,0)	$(83\frac{1}{3},0,0)$	(0,12,60)	(78,0,8)	$(35\frac{1}{3},36,0)$	(30,20,40)
Profit	0	3000	1440	2500	3480	2740	2500	3700

Point G (30,20,40) is hence the optimum vertex providing the maximum profit of $3,700.

The best production plan is to make 30 of cabinet A, 20 of Cabinet B and 40 of Cabinet C.

The fact of non-integer co-ordinates for some vertices of the feasible region is a minor difficulty overcome by rounding off to the nearest valid integer co-ordinate.

More sophisticated techniques will be developed later for investigating problems with three or more variables.

The following problem will be analysed later very thoroughly but we show now the vertex substitutions methods.

Example

Minimise $12x_1 + 5x_2 + 2x_3$.

Subject to constraints –

$x_1, x_2, x_3 \geq 0$.

$x_1 + x_2 + x_3 \geq 5$.

$x_1 - x_2 + x_3 \geq 3$.

$x_1 - 2x_2 - x_3 \geq 4$.

The vertices of the feasible region are (5,0,0) $(4\frac{1}{2},0,\frac{1}{2})$ and $(4\frac{2}{3},\frac{1}{3},0)$. The feasible region is an unbounded one. the corresponding values of the objection function are 60, 55, $57\frac{2}{3}$ respectively. The minimum value of the objective function is hence obtained at the vertex $(4\frac{1}{2},0,\frac{1}{2})$.

i.e. Solution is $x_1 = 4\frac{1}{2}$, $x_2 = 0$, $x_3 = \frac{1}{2}$ yielding a minimum of 55. We will refer back to this problem later on.

Example

A manufacturer uses one or more of four production processes involving labour in man-hours and tons of raw material. The manufacturer wishes to determine the optimum daily production schedule with the information as below.

	Process 1	Process 2	Process 3	Process 4
Man hours	2	1	2	1
Tons of raw material	2	3	5	6
Profit per unit	\$60	\$40	\$70	\$50

There is available a total of 100 man-hours daily and 500 tons of raw material are available daily.

Solution

Let z_1, z_2, z_3, z_4 be the number of units produced using processes 1,2,3,4 respectively.

Then $z_1, z_2, z_3, z_4 \geq 0$

$$2z_1 + z_2 + 2z_3 + z_4 \leq 100$$

$$2z_1 + 3z_2 + 5z_3 + 6z_4 \leq 500$$

Maximise $60z_1 + 40z_2 + 70z_3 + 50z_4$

The vertices of the feasible region and corresponding profits are:

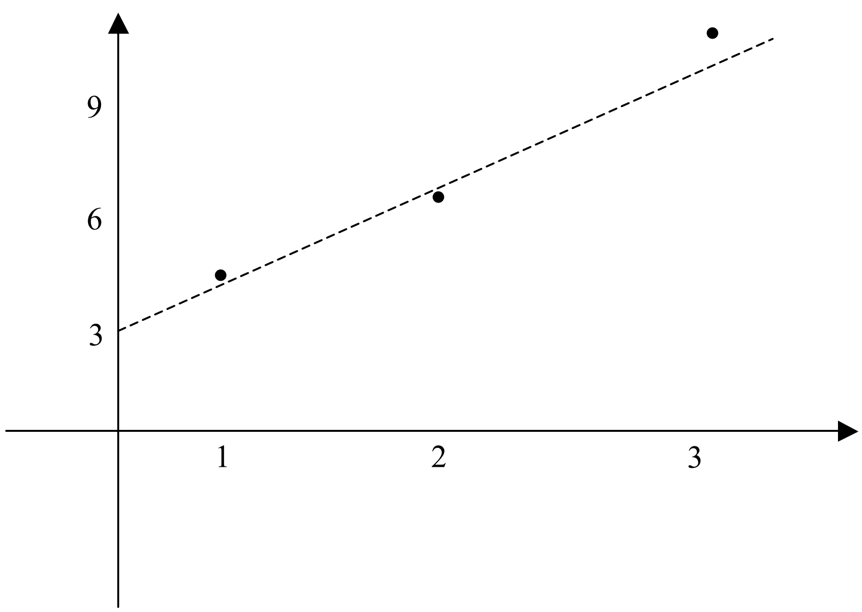
	O	A	B	C	D	E	F	G
Vertices	(0,0,0,0)	(0,0,0,83)	(0,0,50,0)	(0,0,14,71)	(0,100,0,0)	(0,34,0,66)	(50,0,0,0)	(10,0,0,80)
Profit	0	4150	3500	4530	4000	4660	3000	4600

Some of the co-ordinates have been rounded off. The optimum vertex is point E (0,34,0,66) with a profit of $4,660. The optimum production schedule is 34 units using process 2 and 66 units using process 4.

Certain problems involve finding a **linear** functional relationship between two variables, for which data are available, which best approximates the true relationship.

Example **MINIMAX**

We know from an experiment that when x = 1, y = 4.5, when x = y, y = 6.5, and x = 3, y = 9.5. Find the "best" linear relationship between x and y using the **MINIMAX** method. Minimax means minimising the maximum difference between the true value and the value estimated by the linear relationship.



i.e. the dotted line is the line of best fit where the maximum vertical distance between any of the three points and the dotted line is minimised.

Solution

Let $y = mx + b$ be the line of "best fit" A visual inspection of the data leas us to deduce that $m \geq 0$ and $b \geq 0$ (not completely necessary, but desirable).

Let $d = \max$ {vertical distance}

$= \max \{|m + b - 4.5|, |2m + b - 6.5|, |3m + b - 9.5|\}$

We wish to minimise d

subject to the following constraints.

$|m + b - 4.5| \leq d$

$|2m + b - 6.5| \leq d$

$|3m + b - 9.5| \leq d$

$m, b, d \geq 0.$

But $|z| \leq d$ is equivalent to $z \leq d$ and $-z \leq d$.

i.e. our constraints are:

$$\left.\begin{array}{l} m + b - 4.5 \leq d \\ 4.5 - m - b \leq d \\ 2m + b - 6.5 \leq d \\ 6.5 - 2m - b \leq d \\ 3m + b - 9.5 \leq d \\ 9.5 - 3m - b \leq d \\ m \geq 0,\ b \geq 0,\ d \geq 0\ . \end{array}\right\}$$

Rearranging these constraints yields:

$m + b - d \leq 4.5$

$m + b + d \geq 4.5$

$2m + b - d \leq 6.5$

$2m + b + d \geq 6.5$

$3m + b - d \leq 9.5$

$3m + b + d \geq 9.5$

$m \geq 0, b \geq 0, d \geq 0$ Minimise d.

The vertices (m,b,d) of the feasible region are (0,0,9.5), (0,7,2.5), $(2\frac{1}{2}, 0,2)$, $(3\frac{1}{2}, 0,1)$, $(2,3, \frac{1}{2})$, $(3,1, \frac{1}{2})$, $(2\frac{1}{2}, 1\frac{3}{4}, \frac{1}{4})$.

It therefore follows that the vertex having the minimum value for d is $(2\frac{1}{2}, 1\frac{3}{4}, \frac{1}{4})$.

i.e. the optimum solution occurs when $m = 2\frac{1}{2}$ and $b = 1\frac{3}{4}$ yielding the maximum difference of $\frac{1}{4}$.

i.e. $y = 2\frac{1}{2}x + 1\frac{3}{4}$ is the line of "best fit"

This means that we could extrapolate the conjecture that when x = 4, $y = 11\frac{3}{4}$.

Example

Matrix Games

Players A and B play a game where A and B simultaneously choose a playing card Jack, Queen or King and compare their choices. Player A then pays player B in dollars according to the following chart.

B

A		Jack	Queen	King
	Jack	0	-2	3
	Queen	2	0	-1
	King	-3	1	0

This chart means, for example that if A chooses the Queen and B chooses the Jack then A pays B \$2. The negative entries indicate that B has to pay the corresponding amount, e.g. if A chooses the King and B the Jack then B plays A \$3.

A wishes to determine his optimum strategy.

A assigns probabilities p_1, p_2, $1 - p_1 - p_2$ to his choices Jack, Queen, King respectively and then tries to determine p_1 and p_2 so that the amount he pays B is a minimum (of course, he would like it to be a negative payment if possible).

If B always chooses the Jack, then, **IN THE LONG RUN**, A expects to pay B, $0p_1 + 2p_2 - 3(1 - p_1 - p_2)$ dollars, i.e. \$($3p_1 + 5p_2 - 3$) each time.

If B always chooses the Queen, then in the long run, A expects to pay B, $-2p_1 + 0p_2 - 1(1 - p_1 - p_2)$ dollars, i.e. \$($-3p_1 - p_2 + 1$) each time.

If B always chooses the King, then in the long run, A expects to pay B, $3p_1 - 1p_2 + 0(1 - p_1 - p_2)$ dollars, i.e. \$($3p_1 - 5p_2$) each time.

Since A wishes to minimise the amount he expects to pay B each time regardless of B's strategy, then the problem can be set up as follows.

Let M be the **maximum** amount A expects to pay B each time in the long run. Then A wishes to: -

Minimise M

subject to

$$\left.\begin{array}{l} 3p_1 + 5p_2 - 3 \le M \\ -3p_1 - p_2 + 1 \le M \\ 3p_1 - 5p_2 \le M \\ p_1 \ge 0,\ p_2 \ge 0,\ 1 - p_1 - p_2 \ge 0 \end{array}\right\}$$

These constraints can be simplified and rearranged below.

$$\left.\begin{array}{l} 3p_1 + 5p_2 - M \le 3 \\ 3p_1 + p_2 + M \ge 1 \\ 3p_1 - 5p_2 - M \le 0 \\ p_1 \ge 0 \\ p_2 \ge 0 \\ p_1 + p_2 \le 1 \end{array}\right\}$$

The vertices (p_1, p_2, M) of the feasible region are –

$(\frac{1}{6},\frac{1}{2},0)$, $(0, \frac{2}{3}, \frac{1}{3})$, $(\frac{1}{2},\frac{1}{2},0)$, $(0,1,2)$, $(\frac{1}{6},0,\frac{1}{2})$, $(0,0,1)$ and $(1,0,3)$.

i.e. the optimum vertex is $(\frac{1}{6},\frac{1}{2},0)$ since this has the smallest value for M, i.e. zero. This means that A's optimum strategy for the long run is to choose the Jack, $\frac{1}{6}$ th of the time, the Queen, $\frac{1}{2}$ of the time and hence to choose the King, $\frac{1}{3}$ of the time.

This result seems a little counter-intuitive when one re-examines the payment chart. It seems at first glance that A should select the King more frequently – the reason he does not is that in fact B's strategy is also to choose the Jack $\frac{1}{6}$ th of the time, the Queen, $\frac{1}{2}$ of the time and hence to choose the King, $\frac{1}{3}$ of the time and hence A does not collect \$3 from B very often and pays out \$1 far more often when (A) selects the King.

14.2 ii) Gaussian Elimination

To find solutions to equations in three or more variables a method is called **GAUSSIAN ELIMINATION** is readily applicable and is illustrated by the following example:

Solve -

$$x + 2y + 3z = 9 \quad (1)$$
$$4x + 3y + 2z = 11 \quad (2)$$
$$5x + 5y + z = 9 \quad (3)$$

We eliminate x from equations (2) and (3).

i.e. multiply (1) by 5 and subtract (3). Replace (3) with this resulting equation.

Multiply (1) by 4 and subtract (2). Replace (2) with this resulting equation.

i.e.

$$x + 2y + 3z = 9 \quad (1)$$
$$5y + 10z = 25 \quad (2)$$
$$5x + 4y + z = 9 \quad (3)$$

Simplify (2) and (3), then multiply (2) by 3 and subtract (3). Replace (3) with the result.

i.e. $x + 2y + 3z = 9$ (1)

$y + 2z = 5$ (2)

-z = 3 (3)

We now solve the resulting system of equations by back substitution, i.e. z = 3 from (3). Substituting in (2) yields y = - 1 and substituting in (1) yields x = 2.

i.e. (2, -1, 3) is the solution.

What has happened is that the original equation –

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 5 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

i.e. the system of equations is solvable providing the co-efficients matrix is diagonalisable.

Difficulties may arise when leading co-efficient of the first equation at the first stage is zero of when the lead co-efficient of the second equation at the second stage is zero and so on. Interchanging the equation with a later one will overcome this difficulty if the system of equations has a solution.

The Gaussian Elimination method may be generalised to four or more variables and is recommended in those cases.

Exercise 14.2

1\. i) Solve $2x - y + z = 4$ by Gaussian Elimination.

$x - 2y - z = -1$

$x + y - 2z = 1$

ii) repeat for:

$2x - y + z = 4$

$x - 2y - z = -1$

$x + y - 2z = 1$

2. Solve:

$w + x + 2y + z = 5$

$2w + 2x - y + z = 4$

$2w+ 3x + y - z = 5$

$3w + 2x + y - 2z = 4$

3. Maximise $2x + 2y + 6z$

subject to $6x + 6y + 10z \leq 25$

$3x + 8y + 10z \leq 20$

$x \geq 0, y \geq 0, z \geq 0.$

4. Maximise $2x + 3y + 4z$

subject to $2x + y + z \leq 13$

$x + 2y + 2z \leq 17$

$x \geq 0, y \geq 0, z \geq 0.$

5\. A manufacturer can produce four types of cabinets A, B, C or D using oak, teak and plywood. He has available 2000 feet of oak, 2000 feet of teak and 10,000 feet of plywood. Cabinet require wood as in the table.

	A	B	C	D	Available Supply
Oak	2	1	1	0	2000
Teak	0	3	1	1	2000
Plywood	12	3	4	2	10000
Profit	$45	$15	$30	$15	

Find the optimum production plan.

6. See previous example on matrix games.

   Find the optimum strategy for player A where the payment chart is as below:

B

		Jack	Queen	King
A	Jack	0	4	-5
	Queen	-4	0	3
	King	5	-3	0

7. Find the optimum strategy for both player A and B in the following matrix game:

B

		Jack	Queen	King	Ace
A	Jack	5	3	4	5
	Queen	3	7	6	4
	King	2	8	1	1

Find the amount that A has to pay B each time, in the long run.

8. Using the set of data below, estimate the value of y when x = 4.5. Use the minimax method to find the line of best fit.

x	0	2	3
y	2	4	6.5

Exercise 14. 2 Answers

1. i) (2,1,1) ii) ϕ 2. (1,1,1,1)   3. $(\frac{5}{3},0,\frac{3}{2})$.   Maximum value is 14.

4. (3,0,7) or (0,0, . Maximum value is 34.   5. 250 of A, none of B, 1500 of C, 500 of D. Profit is \$63,750.   6. A chooses Jack $\frac{1}{4}$ of the time, Queen $\frac{5}{12}$ of the time and King $\frac{1}{3}$ of the time.   7. Strategy for A is Jack, $\frac{2}{3}$rds of time, Queen $\frac{1}{3}$rd of the time, King never. Strategy for B is Jack, $\frac{2}{3}$rds of time, Queen $\frac{1}{3}$rd of the time, King never.

8. y = 1.5x + 1.5. When x = 4.5, y = 8.25

14. 3 (i) Duality

Linear programming problems are such that for a minimising problem there is a corresponding maximising problem (and vice versa) such that the objective functions have the same optimum value. The problems are called **DUALS** of each other. Although the underlying proof and theory is beyond the scope of this book, the result is remarkable.

Note on previous example is

Minimise $12x_1 + 5x_2 + 2x_3$

subject to –

$$x_1 + x_2 + x_3 \geq 5$$

$$x_1 - x_2 + x_3 \geq 3$$

$$x_1 - 2x_2 - x_3 \geqslant 4$$

$$x \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0.$$

This could be written in the matrix form

$$\text{Minimise } [12 \quad 5 \quad 2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Subject to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geqslant \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geqslant \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we label the matrices as follows

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \end{bmatrix} \quad R = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} \quad \text{and } C = [12 \quad 5 \quad 2]$$

We have Minimise CX subject to

$AX \geqslant R$ and $X \geqslant 0$

The corresponding dual problem is set up using matrices A, R and C and a new matrix Z where Z is a corresponding one row matrix with the correct number of elements to make the following matrix multiplication possible.

i.e. Maximise ZR

subject to $ZA \leqslant C$ and $z \geqslant 0$.

Here $Z = [z_1 \quad z_2 \quad z_3]$

i.e. the corresponding maximising problem is: -

$$\text{Maximise } [z_1 \quad z_2 \quad z_3]\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$\text{Subject to } [z_1 \quad z_2 \quad z_3]\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \end{bmatrix} \leqslant [12 \quad 5 \quad 2]$$

$$\text{and } [z_1 \quad z_2 \quad z_3] \geqslant [0 \quad 0 \quad 0]$$

i.e. Maximise $5z_1 + 3z_2 + 4z_3$

subject to $z_1 \geqslant 0$, $z_2 \geqslant 0$, $z_3 \geqslant 0$

and

$$z_1 + z_2 + z_3 \leqslant 12$$

$$z_1 - z_2 - 2z_3 \leqslant 5$$

$$z_1 + z_2 - z_3 \leqslant 2$$

Solving the dual maximizing problem using techniques developed earlier in the chapter gives us a feasible region whose vertices are as listed below.

Vertex	(0,7,5)	(7,0,5)	(0,0,12)	(0,2,0)	(2,0,0)	(0,0,0)
Values of objection F'n	41	55	48	6	10	0

From this we deduce that the vertex (7,0,5) yields the maximum value 55 for the objective function. Note that 55 is also the optimum value of the objective function in the original minimizing problem.

A urther remarkable aspect of this duality relationship is that if $AX \geqslant R$ is compared with $z \geqslant 0$ and $X \geqslant 0$ is compared with $ZA \leqslant C$, then, if then problems have solutions, taking inequalities in corresponding pairs, **at least one** must be an **equality** for the relevant solution point. This may be summarised as:

$$\boxed{\begin{array}{c} AX \geqslant R \leftrightarrow Z \geqslant 0 \\ X \geqslant 0 \leftrightarrow ZA \leqslant C \\ \text{Minimise } CX \leftrightarrow \text{Maximise } ZR \end{array}}$$

For our example

Minimising Problem		Maximising Problem
Solution point $(4\frac{1}{2},0,\frac{1}{2})$		Solution point (7,0,5)
E $x_1 + x_2 + x_3 \geqslant 5$	$\longleftrightarrow$	$z_1 \geqslant 0$ I
I $x_1 + x_2 + x_3 \geqslant 3$	$\longleftrightarrow$	$z_2 \geqslant 0$ E
E $x_1 - 2x_2 - x_3 \geqslant 4$	$\longleftrightarrow$	$z_3 \geqslant 0$ I
I $x_1 \geqslant 0$	$\longleftrightarrow$	$z_1 + z_2 + z_3 \leqslant 12$ E
E $x_2 \geqslant 0$	$\longleftrightarrow$	$z_1 - z_2 - z_3 \leqslant 5$ I
I $x_3 \geqslant 0$	$\longleftrightarrow$	$z_1 + z_2 - z_3 \leqslant 2$ E

Objective Function Value for both problems

$12x_1 + 5x_2 + 2x_3 \quad = 55 = \quad 5z_1 + 3z_2 + 4z_3$

I and E refer to the fact of whether the relevant constraint is an inequality of Equality for the respective solution point. Note that for the solutions, comparing corresponding constraints yields exactly one is an inequality and the other is an equality. This is true for the majority of linear programming problems and gives us an alternative approach to solutions for these types of problems. In problems where the solution is not unique, e.g.

where any point on a whole line segment may represent the solution then in the dual problem the solution may or may not be unique and comparing corresponding constraints may produce the result that **both** are equalities.

Nevertheless it is always true that at least one constraint is an equality.

In a previous example we learned that the problem could be summarized.

$z_1 \geqslant 0, z_2 \geqslant 0, z_3 \geqslant 0$

$$\begin{aligned} 2z_1 \qquad + 3z_3 &\leqslant 180 \\ 5z_2 + 2z_3 &\leqslant 180 \\ 3z_1 + 4z_2 + 2z_3 &\leqslant 250 \end{aligned}$$

Maximise $30z_1 + 40z_2 + 50z_3$

With a solution (30,20,40).

In matrix form this is –

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \geqslant \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 4 \\ 3 & 2 & 2 \end{bmatrix} \leqslant \begin{bmatrix} 180 & 180 & 250 \end{bmatrix}$$

Maximise $\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \\ 50 \end{bmatrix}$

The dual minimising problem is therefore

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 4 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geqslant \begin{bmatrix} 30 \\ 40 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geqslant \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Minimise $\begin{bmatrix}180 & 180 & 250\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

i.e. problems are:

$z_1 \geqslant 0$	$2x_1 \quad + 3x_3 \geqslant 30$
$z_2 \geqslant 0$	$+ 5x_2 + 5x_3 \geqslant 40$
$z_3 \geqslant 0$	$3x_1 + 2x_2 + 2x_3 \geqslant 50$
$2z_1 \quad + 3z_3 \leqslant 180$	$x_1 \geqslant 0$
$+ 5z_2 + 3z_3 \leqslant 180$	$x_2 \geqslant 0$
$3z_1 + 4z_2 + 2z_3 \leqslant 250$	$x_3 \geqslant 0$
Maximise $30z_1 + 40z_2 + 50z_3$	Minimise $180x_1 + 180x_2 + 250x_3$
Solution (30,20,40)	Solution $(\frac{450}{41}, \frac{240}{41}, \frac{110}{41})$ ✪
Maximal value 3700	Minimal value 3700

The solution ✪ of the minimising problem was obtained by checking the vertices of the feasible region and is merely quoted here without derivation for simplicity's sake.

Note: i) The two objective functions have a common value of 3700

ii) The constraints are matched

I ---- E

I ---- E

I ---- E

E ---- I

E ---- I

E ---- I

Example

Consider a minimising problem which leads to the constraints

$2x_1 + 4x_2 + 4x_3 + 18x_4 \geqslant 105$

$5x_1 + 3x_2 + 3x_3 + 14x_4 \geqslant 140$

$x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0$

Minimise $90x_1 + 110x_2 + 160x_3 + 630x_4$

Clearly a graphical approach is not possible but its dual problem is considerably more simple, viz:

$z_1 \geqslant 0$ $z_2 \geqslant 0$ (1) – (2)

$2z_1 + 5z_2 \leqslant 90$ (3)

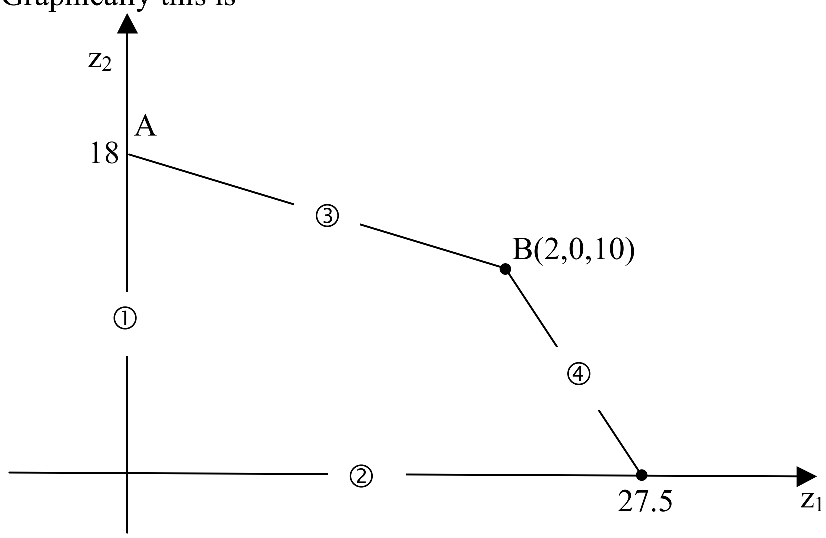
$4z_1 + 3z_2 \leqslant 110$ (4)

$4z_1 + 3z_2 \leqslant 160$ (5)

$18z_1 + 14z_2 \leqslant 630$ (6)

Maximise $105z_1 + 140z_2 = P$

Graphically this is



This feasible region is easy to arrive at because it becomes readily apparent that in fact constraints (5) and (6) are irrelevant. The objective function lines are marked as $\diagdown\diagdown\diagdown$, hence the optimum point is B, (20,10) – the intersection of $2z_1 + 5z_2 = 90$ and $4z_1 + 3z_2 = 110$.

The maximum value of the objective function is 105 x 20 + 140 x 10 i.e. 3500.

Cross-referencing the constraints in the two problems and using the 'at least one is an equality' condition leads to the result that –

i) Minimum value of $90x_1 + 130x_2 + 160x_3 + 630x_4$ is 3500

ii) $x_3 = 0$ and $x_4 = 0$ (because (5) and (6) are **in**equalities)

iii) $x_1 = 17\frac{1}{2}$ and $x_2 = 17\frac{1}{2}$ (solving $2x_1 + 4x_2 + 0 + 0 = 105$ and

$5x_1 + 3x_2 + 0 + 0 = 140$)

i.e. solution to the original minimising problem is $(17\frac{1}{2}, 17\frac{1}{2}, 0, 0)$.

14.3 (ii) Simplex Method

The simplex method for solving linear programming problems, introduced by C. Dantzig in 1947, is a long and cumbersome one by hand, but in the age of the high speed computer it is readily programmable for solving problems with any number of variables.

There are many revisions of the basic simplex method for dealing with non-standard problems but this text will attempt to deal with the standard maximising problem only. Students interested in further linear programming should refer to the wealth of text available on specialised linear programming problems.

It should be remembered that any standard minimising problem has its dual standard maximising problem so that the method shown here will suffice for any standard problem.

Example 1

Maximise $4x_1 + x_2$

subject to $x_1 + x_2 \leqslant 7$

$2x_1 - x_2 \leqslant 3$

$2x_1 + 4x_2 \leqslant 24$

$4x_1 - x_2 \leqslant 8$

$x_1 \geqslant 0, x_2 \geqslant 0$

The data are set out in a table with constraint co-efficients in the main body of the table and the constraint terms entered in the right hand column.

The bottom row constraints the negatives of the co-efficients of the objective function.

1	1	7
2	-1	3
2	4	24
4	-1	8
-4	-1	0
(1)	(2)	

The (1) and (2) are labels referring to x_1 and x_2. This initial table refers to the situation at (0,0) wit the corresponding value of the objective function, i.e. zero entered in the bottom right hand corner.

The simplex method involves moving from the vertex to the vertex of the feasible region in a productive way, improving the value of the objective function each time, until no further improvement can be made. A new table is set up each time corresponding to the new vertex.

Method

Step 1 Locate the column having the least value in the bottom row, i.e. – 4. Call this the pivotal column

Step 2 For each **positive** entry in the pivotal column find the quotient of the entry in the right hand column divided by the entry in the pivotal column. The **pivot** is the entry in the pivotal column corresponding to the smallest quotient, i.e. 2. (Since $3 \div 2$ is the smallest of $7 \div 1$, $3 \div 2$, $24 \div 2$, $8 \div 4$) Call the pivot p.

Step 3 A new table is then produced using the following rules.

i) Each entry in the pivotal column is divided by – p
ii) Each entry in the pivotal row is divided by + p
iii) p is replaced by $\frac{1}{p}$
iv) All other entries are transformed by the diagonal rule.

i.e. p …………c
.
.
.
b…………..a

a is replaced by $a - \frac{b x c}{p}$ where b and c are entries forming the two other corners of "rectangle" formed by a and p.

Step 4 The label of the pivotal column is reassigned to the pivotal row.

The new table becomes

$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{11}{2}$	
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	(1)
-1	5	21	
-2	1	2	
2	-3	6	
	(2)		

This table represents the situation at ($\frac{3}{2}$, 0) (note (1) is next to $\frac{3}{2}$) where the value of the objective function is 6. Steps 1 to 4 are then repeated until the bottom row contains no negative entries.

The readjustment of the table at each stage is, in fact, the result of a Gaussian Elimination to arrive at the new vertex.

the new pivot is 1 (row 4 – column 2) and then rearranged table is

$\frac{5}{2}$	$-\frac{3}{2}$	$\frac{5}{2}$	
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$	(1)
9	-5	11	
-2	1	2	(2)
-4	3	12	

This last table represents the situation at ($\frac{5}{2}$, 2) (note (1) and (2) labels) with an objective functional value of 12.

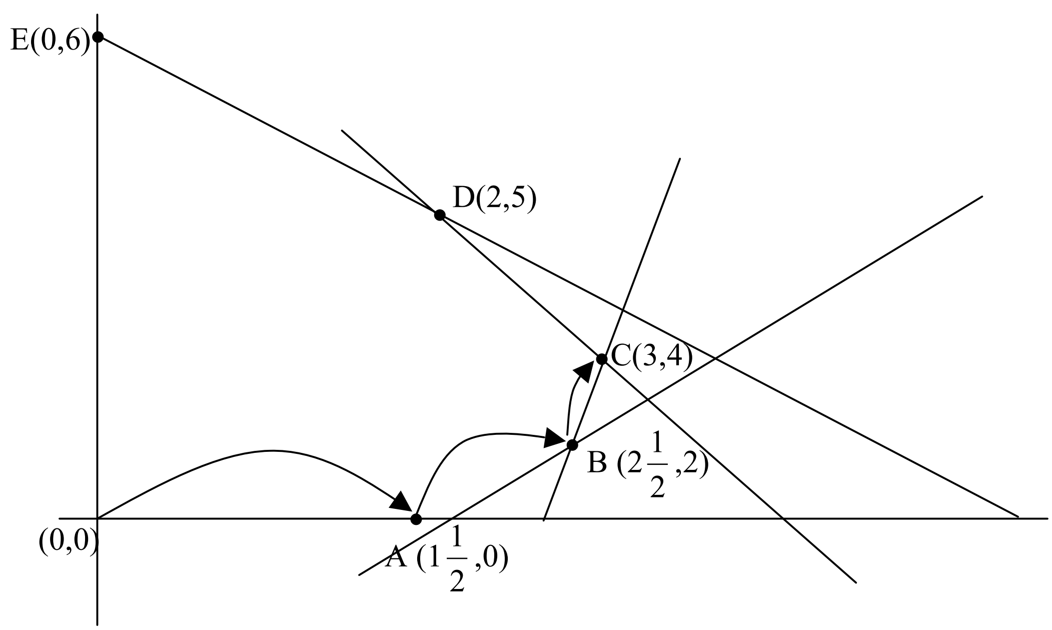
The -4 in the bottom rows tells us that this can be improved upon and the new pivot is

$\frac{5}{2}$ (row 1 – column 1)

The next (and final) table is

$\frac{2}{5}$	$-\frac{3}{5}$	1	
$\frac{1}{5}$	$\frac{1}{5}$	3	(1)
$-\frac{18}{5}$	$-\frac{2}{5}$	2	
$\frac{4}{5}$	$-\frac{1}{5}$	4	(2)
-4	3	12	

This is the final table because each entry in the bottom row is positive. The solution to the problem occurs when $x_1 = 3$ and $x_2 = 4$ (see (1) and (2) labels) yielding a maximum objective function of 16. Graphically what has happened is shown below.



The feasible region is OABCDE. The initial table represents the situation at O.
Subsequent tables reflect the situations at A, B, C in order.

Example

Minimise $12x_1 + 3x_2 + 7x_3$

subject to

$$6x_1 - 2x_2 + 5x_3 \geqslant 3$$

$$2x_1 + 3x_2 - 4x_3 \geqslant -2$$

$$3x_1 + 9x_2 + x_3 \geqslant 8$$

$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0.$

Normally we could not deal with this problem by Simplex because it is both a minimising problem and has a negative constant term in one of the constraints.

However, the dual of this problem is

$z_1 \geqslant 0,\ z_2 \geqslant 0,\ z_3 \geqslant 0$

$$6z_1 + 2z_2 + 3z_3 \leqslant 12$$

$$-2z_1 + 3z_2 + 9z_3 \leqslant 3$$

$$5z_1 - 4z_2 + z_3 \leqslant 7$$

Maxmimise $3z_1 - 2z_2 + 8z_3$

which we can handle using a standard simplex method.

Table 1

6	2	3	12
-2	3	9	3
5	-4	1	7
-3	2	-8	0
(1)	(2)	(3)	

The initial pivot is 9

Table 2

$6\frac{2}{3}$	1	$-\frac{1}{3}$	11	(3)
$-\frac{2}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{3}$	
$5\frac{2}{9}$	$-4\frac{1}{3}$	$-\frac{1}{9}$	$6\frac{2}{3}$	
$\frac{43}{9}$	$\frac{42}{9}$	$\frac{8}{9}$	$\frac{24}{9}$	
(1)	(2)			

The new pivot is $5\frac{2}{9}$

Table 3

$-\frac{60}{47}$	$\frac{307}{47}$	$-\frac{27}{141}$	$\frac{117}{47}$	
$\frac{2}{47}$	$\frac{7}{47}$	$\frac{5}{47}$	$\frac{29}{47}$	(3)
$\frac{47}{9}$	$-\frac{39}{47}$	$-\frac{1}{47}$	$\frac{60}{47}$	(1)
$\frac{43}{47}$	$\frac{33}{47}$	$\frac{37}{47}$	$\frac{412}{47}$	
	(2)			

This is the final table. The label (2) left unmoved in the final row indicates $z_2 = 0$.

The optimum solution for the dual is therefore $(\frac{60}{47}, 0, \frac{29}{47})$ with an objective function value of $\frac{412}{47}$.

Comparing this with the original minimising problem and relying on the idea in the duality section that at least one of the two corresponding constraints must be an Equality, we deduce that the optimum vertex for the minimising problem occurs at the intersection of $x_1 = 0$, $6x_1 - 2x_2 + 5x_3 = 3$ and $3x_1 + 9x_2 + x_3 = 8$.

i.e. optimum vertex is $(0, \frac{37}{47}, \frac{43}{47})$

Incidentally it is **not** a coincidence that $\frac{37}{47}$ and $\frac{43}{47}$ appear in the bottom row of the final simplex table. They are obtained as follows.

Table 1

			①
		9	②
			③
(1)	(2)	(3)	

①,②,③ refer to x_1, x_2, x_3 in the **original** problem.
(1), (2), (3) refer to z_1, z_2, z_3 in the dual problem.

Since 9 is the pivot then (3) and ② interchange giving

Table 2

			①
		9	②
$5\frac{2}{9}$			③
(1)	(2)	(3)	

In Table 2, $5\frac{2}{9}$ is the new pivot since labels (1) and ③ interchange.

Table 3

x	x	x	$\frac{117}{47}$	①
x	x	x	$\frac{29}{47}$	(3)
x	x	x	$\frac{60}{47}$	(1)
$\frac{43}{47}$	$\frac{33}{47}$	$\frac{37}{47}$	$\frac{412}{47}$	
③	(2)	②		

Since ① remains unmoved in the final column, $x_1 = 0$ but $x_2 = \frac{37}{47}$ and $x_3 = \frac{43}{47}$ as indicated in the final row to yield the optimum value of $\frac{412}{47}$. In other words, Simplex solves the two problems simultaneously.

For further reading, students are recommended G.B. Dantzig, Linear Programming and its Extensions (Princeton University Press) 1963, and W.A. Spivey, Linear Programming – An Introduction**.** (Macmillan – New York) 1968.

(Author's note)

A recent (1980) mathematical paper by a Russian mathematician L. G. Khachian, suggest that a much more efficient method for solving linear programming problems will soon be possible.

At present the Simplex method is said to be exponential time one. This means that the computer time required to solve a problem increases **exponentially** as the number of variables increases. Khachian's method has a **polynomial** time solution. The dime difference between the two is illustrated by the fact that a polynomial time algorithm requiring x^3 steps takes about $\frac{1}{5}$ second of computer time when $x = 60$ whereas an exponential time algorithm requiring 3^x steps takes billions of centuries of computer time when $x = 60$.

The Simplex method involves a polygon in multi-dimensional space whose boundaries are determined by the constraints in the problem and whose vertices are possible solutions.

Khachian's method involves the construction of a sequence of "ellipsoids" in multi-dimensional space that close in automatically on the optimum solution.

Exercise 14.3

1. Find the dual problem of

   Minimise $2x_1 + 3x_2 + 4x_3$

   subject to $5x_1 + 6x_2 + 7x_3 \geqslant 8$

   $9x_1 + 10x_2 + 11x_3 \geqslant 12$

   $x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0$

   and solve by any method.

2. Find the dual problem of

   Maximise $5z_1 + 3z_2$

   subject to $z_1 + 2z_2 \leqslant 3, 4z_1 + 5z_2 \leqslant 6,$

   $7z_1 + 8z_2 \leqslant 9, 10z_1 + 11z_2 \leqslant 12$

   $z_1 \geqslant 0, z_2 \geqslant 0$

3. A manufacturer has 240, 360, and 420 pounds of wood, plastic and steel respectively. Products of A, B, C require quantities of these materials as outlined in the table below.

	A	B	C
Oak	1	3	2
Teak	3	2	1
Steel	2	1	3

Profits on products A, B, C are \$3, \$4, \$5 respectively. Find the optimum production plan.

The manufacturer is able to obtain free, from the supplier, 60 pounds of wood, plastic or steel. Which should he choose?

4. Use Simplex Method to solve.

   Minimise $5x_1 + 12x_2 + 10x_3$

   subject to $x_1 + x_2 + 2x_3 \geqslant 4$

   $x_1 - 2x_2 + 3x_3 \geqslant 7$

   $x_1 + 2x_2 - x_3 \geqslant 5$

$$-x_1 + 4x_2 + 4x_3 \geqslant -1$$

$$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0$$

Exercise 14.3 (cont'd)

5. Use the Simplex method to find the optimum strategy for A in the follow matrix game. What is the expected payout reach time in the long run?

B

A		Jack	Queen	King
	Jack	0	-3	4
	Queen	-3	0	-5
	King	4	-5	0

6. Determine the straight line y = mx + b which best fits the following data in the minimax sense.

x	0	1	2	3	4	5
y	6.2	6.8	7.0	7.8	8.5	9.4

7. Maximise $z_1 + z_2 + z_3 + z_4$

   subject to $z_1 - z_2 + z_3 - z_4 \leqslant 2$

   $z_1 + z_2 - z_3 + z_4 \leqslant 4$

   $-z_1 + z_2 + z_3 - z_4 = 0$

   $z_1 \geqslant 0$, $z_2 \geqslant 0$, $z_3 \geqslant 0$, $z_4 \geqslant 0$.

   (Hint: $z = 0 \Rightarrow z \leqslant 0$ and $-z \leqslant 0$)

Exercise 14.3 Answers

1. Maximise $8z_1 + 12z_2$

   Subject to $5z_1 + 9z_2 \leqslant 2$

   $6z_1 + 10z_2 \leqslant 3$

   $7z_1 + 11z_2 \leqslant 4$

   $z_1 \geqslant 0$

   $z_2 \geqslant 0$

   Solution for minimising problem is $(\frac{8}{5},0,0)$ with minimum value of $\frac{16}{5}$.

2. Minimise $3x_1 + 6x_2 + 9x_3 + 12x_4$.

   Subject to $x_1 + 4x_2 + 7x_3 + 10x_4 \geqslant 5$

   $2x_1 + 5x_2 + 8x_3 + 11x_4 \geqslant 5$

   $x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0$

   Solution to maximizing problem is $(\frac{6}{5}, 0)$ with maximum value of 6.

3. 96 of A, none of B, 72 of C.

   60 pounds of wood.

4. $(\frac{47}{9},\frac{5}{18},\frac{7}{9})$ Minimum value is $\frac{335}{9}$. note solution is obtained by solving the dual problem by simplex.

5. i) A chooses Jack $\frac{5}{14}$ of the time, Queen $\frac{8}{14}$ of the time and King $\frac{1}{14}$ of the time.

   ii) $\frac{10}{7}$

6. $y = 0.64x + 6.18$ (max. d is 0.02)

7. $z_1 = 0, z_2 = 1\frac{1}{2}, z_3 = 0, z_4 = 1\frac{1}{2}$ .Max. value is 3.