CHAPTER THIRTEEN

13. Eigenvectors

13.1 An eigenvector for a given matrix A is any non–zero vector **v** such that A(**v**) = λ**v** where λ is a scalar.

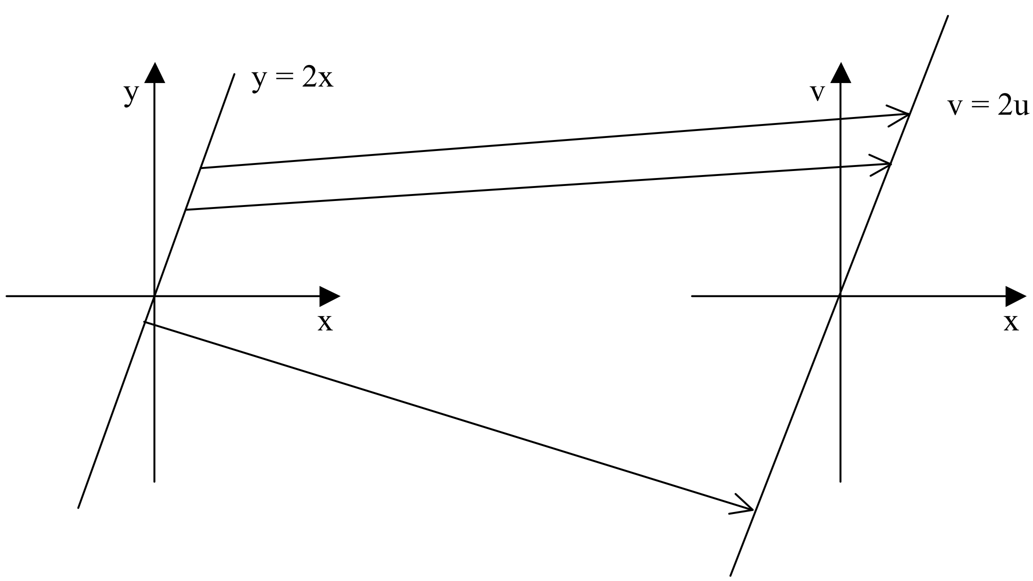
i.e. an eigenvector is one which A maps into a multiple of itself.
The possible values for λ are called eigenvalues (or characteristic values).

Example

Note that $\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ maps (**1,2**) to (**8,16**). This means that (**1,2**) is an eigenvector for $\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ with eigenvalue 8. Furthermore since $\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ is a l.t. it preserves scalar multiplication and hence it follows that all scalar multiples of (**1,2**) are mapped to 8–multiple of themselves.

e.g. $\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ maps (**2,4**) to (**16,32**)

This means that the set {m(**1,2**) | m∈ R}, i.e. y = 2x is left invariant by the matrix $\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$.



To find eigenvalues in general

We will consider the 2x2 case but the argument readily generalises to the nxn case.

Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

We wish to investigate $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$

i.e $ax + by = \lambda x$

$cx + dy = \lambda y$

i.e. $(a – x) + by = 0$

$cx + (d – \lambda)y = 0$

i.e. the equations may be written: $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ is 1–1 then Kernel $= \{\mathbf{0}\}$ only and there are no eigenvectors for A.

(**0** is not an eigenvector for reasons to be found later).

If $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ is **not** 1–1, then there will be non–zero vectors in the Kernel which is what we are looking for.

i.e. If the determinant of $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ is zero, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ will have eigenvectors with corresponding eigenvalues λ.

Note that notationally $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ may be written $\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ i.e $A - \lambda I$.

i.e. $\boxed{\text{A has eigenvalue } \lambda \text{ if } \det(A - \lambda I) = 0.}$

Example

To find eigenvalues and eigenvectors for $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

We need to look at det $\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = 0$

i.e. $(1-\lambda)(2-\lambda) - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4$ or -1

i.e. the eigenvalues are 4 and –1.

The set of all eigenvalues is called the **SPECTRUM**.

Spectrum for above matrix is {4,1}.

To find the eigenvectors corresponding to $\lambda = 4$.

We wish to solve $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \end{bmatrix}$

$$\left.\begin{aligned} \text{i.e. } x + 2y &= 4x \\ 3x + 2y &= 4y \end{aligned}\right\} \quad \text{i.e. } 2y = 3x$$

This means that all position vectors for points on 2y = 3x are eigenvectors for $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ with eigenvalue 4.

We call this S_4. i.e. $S_4 = \{(\mathbf{x},\mathbf{y}) \mid 2y = 3x\}$.

In general the set of eigenvectors corresponding to an eigenvalue λ is called S_λ.

To find S_{-1}.

i.e $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$

$$\left.\begin{aligned} \text{i.e } x + 2y &= -x \\ 3x + 2y &= -y \end{aligned}\right\} \Rightarrow y = -x$$

i.e. any position vector on $y = -x$ is an eigenvector with eigenvalue –1.

i.e. $S_{-1} = \{(\mathbf{x},\mathbf{y}) \mid y = -x \}$.

Example

To find spectrum and S_λ for $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

det $\begin{bmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix} = 0$

i.e. $\lambda^2 - 5\ \lambda + 6 = 0$

i.e. $\lambda = 2$ or 3.

i.e. Spectrum is $\{2,3\}$

To find S₂.

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\text{i.e. } \left.\begin{aligned} x - 2y &= 2x \\ x + 4y &= 2y \end{aligned}\right\} \Rightarrow y = -\frac{1}{2}x$$

i.e. $S_2 = \{(\mathbf{x},\mathbf{y}) \mid y = -\frac{1}{2}x\}$

To find S₃.

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

$$\text{i.e. } \left.\begin{aligned} x - 2y &= 3x \\ x + 4y &= 3y \end{aligned}\right\} \Rightarrow y = -x$$

i.e. $S_3 = \{(\mathbf{x},\mathbf{y}) \mid y = -x\}$

For the purposes of calculating eigenvalues, the zero vector is not considered as an eigenvector because A(**0**) = **0** for **all** matrices A.

Hence every matrix **would** have **0** as an eigenvector. Furthermore, since $\lambda\mathbf{0} = \mathbf{0}$ for all λ, every real number **would** be an eigenvalue for every matrix, clearly an undesirable situation. Hence **0** is not usually an eigenvector.

The special case where $\lambda = 1$, i.e, where vectors remain unchanged by the matrix, is of interest in the study of Stochastic Processes, Markov Chains and Probability Theory in general.

Exercise 13.1

1. Find the spectrum and S_λ for i) $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$

   v) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ vi) $\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$ vii) $\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$

2. If m is a **double** root of det $(A-\lambda I) = 0$, does it follow that S_m is a plane (as opposed to a line). (Hint –consider $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$).

3. If **v** is an eigenvector for A show that **v** is an eigenvector for A^n, for all n.

4. If λ is an eigenvalue for A, show that λ^2 is an eigenvalue for A^2. Is λ^n an eigenvalue for A^n?

5. i) Show that if λ is an eigenvalue for A then $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}. (Assume A^{-1} exists, of course)

   ii) Does (S_λ for A) = ($S_{\frac{1}{\lambda}}$ for A^{-1})?

6. Find λ so that $A(\mathbf{v}) = A^{-1}(\mathbf{v}) = \lambda v$.

7. Show that if A is an idempotent matrix then the spectrum of λ is a subset of {0,1}.

8. Show that if λ is a nilpotent matrix, then the spectrum of λ is {0}.

9. Explain why a rotation of any angle other than 180 in R^2 cannot have eigenvalues.

10. What are the possible eigenvalues for:

    i) A reflection?

    ii) An orthogonal matrix?

11. Find a matrix having eigenvalues –2 and +3. (Assume matrix is 2x2)

12. If A is $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ find the eigenvalues λ.

Exercise 13.1 Answers

1. i) $\{1,4\}$ $S_1 = \{(\mathbf{x},\mathbf{y}) \mid 2y + x = 0\}$, $S_4 = \{(\mathbf{x},\mathbf{y}) \mid y = x\}$.
   ii) $\{6,-1\}$ $S_6 = \{(\mathbf{x},\mathbf{y}) \mid y = x\}$, $S_{-1} = \{(\mathbf{x},\mathbf{y}) \mid 5y = -2x\}$.
   iii) $\{6, -1\}$ $S_6 = \{(\mathbf{x},\mathbf{y}) \mid 5x = 2y\}$, $S_{-1} = \{(\mathbf{x},\mathbf{y}) \mid y = -x\}$.
   iv) $\{7,-2\}$ $S_7 = \{(\mathbf{x},\mathbf{y}) \mid y = x\}$, $S_{-2} = \{(\mathbf{x},\mathbf{y}) \mid x + 2y = 0\}$
   v) $\{1\}$ $S_1 = \{(\mathbf{x},\mathbf{y}) \mid y = -x\}$
   vi) $\{0,12\}$ $S_0 = \{(\mathbf{x},\mathbf{y}) \mid 3x + 2y = 0\}$, $S_{12} = \{(\mathbf{x},\mathbf{y}) \mid 2y = 3x\}$.
   vii) ϕ i.e. no eigenvalues.

2. No. $\lambda = 3$ is a double root of $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ but S_3 is a line $\{(\mathbf{x},\mathbf{y}) \mid y = -x\}$.

4. Yes.

5. ii) Yes.

6. $\lambda = \pm 1$.

9. All rotations (except I) change directions of vectors.

10. i) ± 1. ii) ± 1

11. $\begin{bmatrix} m & n \\ \dfrac{m - m^2 + 6}{n} & 1 - m \end{bmatrix}$ e.g. $\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

12. $\left\{-\dfrac{1}{12}, 1\right\}$

13.2 Eigenvalues for 3x3 matrices

Consider $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A$

This maps $(\mathbf{x},\mathbf{y},\mathbf{z})$ to λ $(\mathbf{x},\mathbf{y},\mathbf{z})$, as in the R^3 case, when $\begin{bmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

And as before, we obtain the eigenvalues λ from the equation $\det(A - \lambda I) = 0$.

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -2 & 1 \end{bmatrix} \text{ has eigenvalue } \lambda \text{ when det} \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ -2 & -2 & 1-\lambda \end{bmatrix} = 0.$$

i.e. $(1-\lambda)(3-\lambda)(1-\lambda) = 0$

i.e. $\lambda = 1$ or 3 Note $\lambda = 1$ is a double root.

To find S_1

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ occurs when}$$

$$\left.\begin{aligned} x &= 3x \\ 2x + 3y &= 3y \\ -2x - 2y + z &= 3z \end{aligned}\right\} \Rightarrow y = -x$$

Note that S_1 is a **plane** $\{(\mathbf{x,y,z}) \mid y = -x\}$

To find S_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \\ 3z \end{bmatrix}$$

i.e.

$$\left.\begin{aligned} x &= x \\ 2x + 3y &= y \\ -2x - 2y + z &= z \end{aligned}\right\} \Rightarrow x = 0;\ y = -z$$

i.e. S_3 is a line $\{m(\mathbf{0,1,-1}) \mid m \in R\}$

Example

To find eigenvalues and S_λ for $\begin{bmatrix} 3 & -2 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$

$$\det \begin{bmatrix} 3-\lambda & -2 & 3 \\ 1 & 2-\lambda & 1 \\ 1 & 3 & -\lambda \end{bmatrix} = 0$$

i.e. $-\lambda^3 + 5\lambda^2 - 2\lambda - 8 = 0$

i.e $(-1)(\lambda + 1)(\lambda - 2)(\lambda - 4) = 0$

i.e. $\lambda = -1, 2$ or 4.

To find S_{-1}

$$\begin{bmatrix} 3 & -2 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$$

i.e. $\left.\begin{aligned} 3x - 2y + 3z &= -x \\ x + 2y + z &= -y \\ x + 3y \quad &= -z \end{aligned}\right\} \Rightarrow \dfrac{x}{11} = y = \dfrac{z}{-14}$

i.e. S_{-1} is a line $\{m(\mathbf{11,1,-14}) \mid m \in R\}$

Similar techniques show that –

$S_2 = \{m(\mathbf{1,-1,-1}) \mid m \in R\}$

and

$S_4 = \{m(\mathbf{1,1,1}) \mid m \in R\}$

This means each of the three lines of vectors

S_{-1}, S_2 and S_4 remains invariant after $\begin{bmatrix} 3 & -2 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$

The previous examples suggest the following theorem –

Theorem 13.1

If λ is an eigenvalue for matrix A then S_λ is a subspace of the domain A.

Proof

Let $\mathbf{v_1}$ and $\mathbf{v_2} \in S_\lambda$

Then $A(\mathbf{v_1}) = \lambda\mathbf{v_1}$ and $A(\mathbf{v_2}) = \lambda\mathbf{v_2}$

i.e. $A(\mathbf{v_1} + \mathbf{v_2}) = A(\mathbf{v_1}) + A(\mathbf{v_2})$ since A is a l.t.

$$= \lambda\mathbf{v_1} + \lambda\mathbf{v_2}$$

$$= \lambda(\mathbf{v_1} + \mathbf{v_2})$$

$\therefore\ \mathbf{v_1} + \mathbf{v_2} \in S_\lambda$ i.e. S_λ is closed under vector addition.

Similarly $A(c\mathbf{v_1}) = cA(\mathbf{v_1})$ since A is a l.t.

$$= c\lambda(\mathbf{v_1})$$

$$= \lambda(c\mathbf{v_1})$$

$\therefore\ c\mathbf{v_1} \in S_\lambda$ i.e. S_λ is closed under scalar multiplication.

Therefore S_λ is a subspace of domain of A.

Theorem 13.2

If A: V→ W is a l.t. and λ is an eigenvalue then $\lambda = 0 \leftrightarrow \det A = 0$

Proof

(→) If $\lambda = 0$ then some vectors in the domain of A are mapped to zero times themselves, i.e. mapped to **0**. This mean that Kernel of A contains something other than **0**, i.e. $\mathrm{Ker}A \neq \mathbf{0}$ $\therefore$ A is not 1–1.

$\therefore \det A = 0$

(←) The above argument is completely reversible. □

Exercise 13.2

1. Find x so that $\begin{bmatrix} 1 & x \\ 3 & 1 \end{bmatrix}$ has eigenvalue 4.

2. Find eigenvalues for $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Hence deduce its inverse.

3. Find eigenvalues and the corresponding S_λ for: $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 1 & -2 \\ 3 & 4 & -5 \end{bmatrix}$.

4. Does it follow that if λ is a **double** root of det (A – λI) = 0 where A is a 3x3 matrix then S_λ is a plane?

   Investigate by considering $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 2 & 0 \end{bmatrix}$.

5. Show that $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$ is idempotent. Find its eigenvalues and describe its geometric significance.

6. Find spectrum of $\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$.

7. Investigate $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ for eigenvalues and the resulting subspaces S_λ.

8. By considering the range of matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

   show that i) is not onto

   ii) A is not 1–1

   iii) Ker A $\neq$ 0

   iv) λ = 0 is an eigenvalue

   Furthermore show that the range is {m(**1,1,1**) | m $\in$ R}, that λ = 6 is an eigenvalue and hence that the only possible subspace for S_6 is {(**x,y,z**) | x = y = z}.

9. Find examples λ_1 and λ_2 which are eigenvalues for matrices A_1 and A_2 respectively but such that $\lambda_1 + \lambda_2$ is not an eigenvalue for $A_1 + A_2$.

10. i) If λ is an eigenvalue for A, is λ an eigenvalue for mA ($m \neq 1$) ?

    ii) Is $m\lambda$ an eigenvalue for mA?

11. If λ_1 and λ_2 are eigenvalues for A_1 and A_2 respectively, is $\lambda_1\lambda_2$ an eigenvalue for A_1A_2?

12. Find a matrix other than I or the zero matrix which has every vector as an eigenvector. How many eigenvalues does the matrix have?

Exercise 13.2 Answers

1. $x = 3$

2. $\{5, 1\}$ $A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\{2,-2,-3\}$ $S_2 = \{(\mathbf{x,y,z}) \mid x = y = z\}$ $S_{-2} = \{(\mathbf{x,y,z}) \mid x = -\frac{y}{3} = -\frac{z}{3}\}$

   $S_{-3} = \{(\mathbf{x,y,z}) \mid \frac{x}{2} = \frac{y}{2} = \frac{z}{7}\}$

4. No. $\lambda = 1$ is a double root but S_1 is a line $\{(\mathbf{x,y,z}) \mid x = y = z\}$.

5. $\lambda = 0$ or 1. It maps everything to $y = x$.

6. $\{0, \sqrt{3}, -\sqrt{3}\}$

7. $\lambda = 1$ or 3. 1 is a double root. $S_1 = \{(\mathbf{x,y,z}) \mid x = -y\}$ $S_3 = \{(\mathbf{x,y,z}) \mid x = 0; y = -z\}$.

10. i) No. ii) Yes. 11. No.

    A magnification $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ which has a single eigenvalue a.

13.3 If A is a matrix then $\det(A - \lambda I) = 0$ is called the **characteristic polynomial of A.**

e.g. If A is $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ then $\det\begin{bmatrix} 2-\lambda & -1 \\ 3 & 4-\lambda \end{bmatrix} = 0.$

i.e. $\lambda^2 - 6\lambda + 11 = 0$ is the characteristic polynomial

Hamilton–Cayley Theorem

If A is a matrix with characteristic polynomial $F(\lambda)$, then $F(A) = [0]$ where $[0]$ is the zero matrix.

We will first illustrate the meaning of this theorem.

For $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ as above, $F(\lambda) = \lambda^2 - 6\lambda + 11$.

Then F(A) means $A^2 - 6A + 11I$.

The theorem allows us to deduce that $A^2 - 6A + 11I.= [0]$

Check

$$A^2 = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 18 & 13 \end{bmatrix}$$

$$-6A = \begin{bmatrix} -12 & 6 \\ -18 & -24 \end{bmatrix}$$

$$11I = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

Then $A^2 - 6A + 11I$ does equal $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as required by the theorem.

A general proof of the Hamilton–Cayley Theorem is beyond the scope of this text but we will show the truth of the theorem in the 2x2 case by direct substitution.

Proof

Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then characteristic polynomial is given by $\det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$

i.e. $\lambda^2 - (a+d)\lambda + ad-bc = 0$

We now need to show that

$A^2 - (a+d)A + (ad - bc)I = [0]$

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc - d^2 \end{bmatrix}$$

$$-(a+d)A = \begin{bmatrix} a^2 - ad & -ab - bd \\ -ac - dc & -ad - d^2 \end{bmatrix}$$

$$(ab-dc)I = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

Hence $A^2 - (a+d)A + (ad - bc)I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as required.

Example

Consider $\begin{bmatrix} 0 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

The characteristic polynomial is det $\begin{bmatrix} -\lambda & -1 & 1 \\ 2 & -1-\lambda & 2 \\ 1 & 0 & 1-\lambda \end{bmatrix} = 0$

i.e. $-\lambda^3 + 1 = 0$ i.e. $\lambda = 1$ is the only eigenvalue.

(we do not include complex eigenvalues)

It therefore follows from the Hamilton–Cayley Theorem that

$$-A^3 + I = [0]$$

i.e. $A^3 = I$

i.e. $A^2 = A^{-1}$

Hence $\begin{bmatrix} 0 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} = A^{-1}.$

The Hamilton–Cayley Theorem can be used occasionally to find high powers of some matrices as follows.

Question

If $A = \begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$ find A^{10}. The characteristic polynomial is $\lambda^2 - 4 = 0$ i.e. $\lambda = \pm 2$.

i.e. $A^2 - 4I = 0$

i.e. $A^2 = 4I$

$$\therefore A^{10} = (4I)^5 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1024 & 0 \\ 0 & 1024 \end{bmatrix}$$

Previously in the chapter we examined $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. We found that its eigenvalues were 4 and –1 and that $S_4 = \{(\mathbf{x,y}) \mid m(\mathbf{2,3}) \in R\}$ and $S_{-1} = \{m(\mathbf{1,-1}) \mid m \in R\}$. We will now look at $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ in greater detail. Call the matrix A.

To diagonalise A

This means to multiple A by matrices so that A becomes a diagonal matrix. This can be effected as follows.

Let P be the matrix mapping **i** to any vector in S_4, (say) (**2,3**) and mapping **j** to any vector in S_{-1}, (say) (**1,–1**).

i.e. P is $\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

Now we know A(**2,3**) = (**8,12**) and A(**1,–1**) = (– **1,1**)

(since they belong to S_4 and S_{-1} respectively)

$$\therefore AP = \begin{bmatrix} 8 & -1 \\ 12 & 1 \end{bmatrix}$$

$$\text{But } \begin{bmatrix} 8 & -1 \\ 12 & 1 \end{bmatrix} = P\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{i.e. } AP = P\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

But $\det P \neq 0$, $\therefore P^{-1}$ exists.

$$\therefore P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \otimes$$

i.e. A can be diagonalised to a diagonal matrix whose non–zero elements are its eigenvalues.

We say A is **SIMILAR** to $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$

i.e. if $A = P^{-1}BP$ then A is similar to B.

Note that $\det(P^{-1}AP) = \det P^{-1} \det A \det P$

$$= \frac{1}{\det P} \det A \det P$$

$$= \det A$$

Since the determinant of $P^{-1}AP$ is clearly the product of the eigenvalues, see ⊗ above, it follows that

$$\boxed{\det A = \text{product of its eigenvalues}}$$

This is true for nxn matrices but it is not true that every square matrix is similar to a diagonal matrix.

(Exercise 13.2 Question #4 is an example illustrating this.)

Previously, we found that $\begin{bmatrix} 3 & -2 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ had eigenvalues –1, 2, and 4 and

$S_{-1} = \{m(\mathbf{11,1,-14}) \mid m\in R\}$, $S_2 = \{m(\mathbf{1,-1,-1}) \mid m\in R\}$and $S_4 = \{m(\mathbf{1,1,1}) \mid m\in R\}$.

We may deduce from this that $P = \begin{bmatrix} 11 & 1 & 1 \\ 1 & -1 & 1 \\ -14 & -1 & 1 \end{bmatrix}$ is a matrix such that

$$P^{-1}\begin{bmatrix} 3 & -2 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

In fact every symmetric matrix can be made similar to a diagonal matrix. More advanced texts treat similar matrices more fully where they have considerable importance.

Exercise 13.3

1. Find the eigenvalues for $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 7 & 0 & -2 \end{bmatrix}$ and hence find its determinant.

2. Repeat for $\begin{bmatrix} 0 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

3. A is the matrix $\begin{bmatrix} 3 & -4 & 2 \\ 1 & 3 & -3 \\ 1 & 6 & -6 \end{bmatrix}$. Find P such that $P^{-1}AP=B$. where B is a diagonal matrix.

4. Repeat question 3 where A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

5. Use the Hamilton–Cayley Theorem to find the inverse of $\begin{bmatrix} 0 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. Note that this is the matrix in Question 2.

6. Investigate the conjecture that if A is a 3x3 matrix with eigenvalues λ_1, λ_2, λ_3 then trace A = $\lambda_1 + \lambda_2 + \lambda_3$ [trace A is the sum of the diagonal elements of A]

7. Show that $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is orthogonal and that it represents a rotation of $\theta°$ a.c. about the z–axis.

8. Show that if A is a 3x3 orthogonal matrix then {A(**i**), A(**j**), A(**k**)} is a set of mutually perpendicular unit vectors. Is the set an orthonormal basis for the range of A?

9. Show that A = $\begin{bmatrix} 1 & 3 & -2 \\ -1 & 5 & -2 \\ -1 & 4 & 1 \end{bmatrix}$ has only one real eigenvalue. Find the corresponding S_λ. Explain why it is **not** possible to find a matrix P such that $P^{-1}AP=B$ where B is a diagonal matrix.

10. Is $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ diagonalisable?

11. Find the characteristic polynomial of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ and consequently name a matrix whose characteristic polynomial is $x^3 - 5x^2 - 6x - 7 = 0$.

12. Show that if A and B are similar matrices then they have the same eigenvalues. Do they have the same eigenvectors?

13. Find the spectrum of $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Find the corresponding eigenspaces and describe the geometric significance of the matrix.

14. True or False?
    a) If $A^2 = A$ then $A = [0]$ or I
    b) If $A + B = I$ then A^2 commutes with B
    c) If $(A - I)^2 = [0]$ then A does not have an inverse
    d) If all the eigenvalues of A are zero then $A^k = [0]$ for some integer k.
    e) If A is a diagonal matrix then AB = BA for **every** matrix B.

Exercise 13.3 Answers

1. Spectrum = $\{0, 5, -4\}$. Determinant is zero.
2. Spectrum = $\{1\}$. Determinant = 1
3. A is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. P is $\begin{bmatrix} a & 2b & 0 \\ a & b & c \\ a & b & 2c \end{bmatrix}$ for any a,b,c.

4. $P = \begin{bmatrix} a & c & 0 \\ -a & -c & e \\ b & d & -e \end{bmatrix}$ where $d \neq \frac{bc}{a}$. i.e. P(**i**), P(**j**) are l.i. An example is

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 3 \\ 4 & 5 & -3 \end{bmatrix}$$

5. Since $A^3 = I$ by Hamilton–Cayley Theorem $A^{-1} = A^2 = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$.

6. True

8. Yes

9. $\lambda = 2$ $\qquad \frac{x}{5} = y = \frac{z}{-1}$

10. No.

11. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & 6 & 5 \end{bmatrix}$

12. No

13. Spectrum = $\{0,1\}$ $S_0 = \{(\mathbf{x},\mathbf{y}) \mid y = -x\}$ $S_{-1} = \{(\mathbf{x},\mathbf{y}) \mid y = x\}$. It is a projection on to $y = x$.

14. a) False b) True c) False e.g. $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ d) True (Hamilton–Cayley Theorem)

e) False