CHAPTER TWELVE

12. Linear Transformations

12.1 A matrix is an example of a general set of transformation called **linear transformations** (or linear maps or **homomorphisms**).

A linear transformation T is a mapping from a vector space to a vector space such that

i) it preserves vector addition

ii) it preserves scalar multiplication

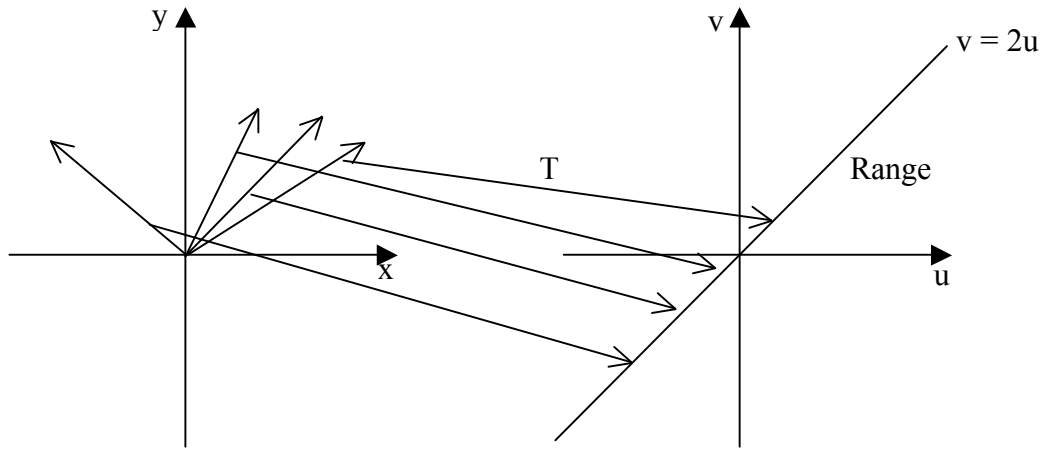
i) This means that for any two vectors $\mathbf{v_1}$, $\mathbf{v_2}$ in the domain of T, $T(\mathbf{v_1} + \mathbf{v_2}) = T(\mathbf{v_1}) + T(\mathbf{v_2})$. i.e., adding vectors in the domain and then mapping the result by T produces the same result as mapping the two vectors separately and adding their images.

ii) This means that for any vector **v** and any scalar c, $T(c\mathbf{v}) = cT(\mathbf{v})$
i.e. multiplying a vector by a scalar and then mapping by T gives the same result as mapping T first and multiplying the image by that same scalar.

Linear transformation will be abbreviate to l.t. If T, a l.t., maps from $V \rightarrow W$, then V is called the **domain** and W is called the **Range Space**.

The set of all images of mapping of vectors in V, loosely written T(V), is called the **Range**.

It is important to distinguish between Range and Range Space. For example, for $T(\mathbf{x},\mathbf{y}) = (\mathbf{x+y}, \mathbf{2x + 2y})$, the Range Space is R^2 but the Range is $\{(\mathbf{u},\mathbf{v}) | v = 2u\}$



Statement

"All matrices are linear transformations"

Proof

A 'proof' will be given in the 2 x 2 case but it will be clear that the proof can be generalised to any matrix (square or otherwise).

Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $\mathbf{v_1} = (\mathbf{x_1},\mathbf{y_1})$ and let $\mathbf{v_2} = (\mathbf{x_2},\mathbf{y_2})$ and m be a scalar.

Then $T(\mathbf{v_1} + \mathbf{v_2}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = (\mathbf{ax_1 + ax_2 + by_1 + by_2, cx_1 + cx_2 + dy_1 + dy_2})$

$= (\mathbf{ax_1 + by_1, cx_1 + dy_1}) + (\mathbf{ax_2 + by_2, cx_2 + dy_2})$

$= T(\mathbf{v_1}) + T(\mathbf{v_2})$

Also $T(m\mathbf{v_1}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} mx_1 \\ my_1 \end{bmatrix} = (\mathbf{amx_1 + bmy_1, cmx_1 + dmy_1})$

$= m(\mathbf{ax_1 + by_1, cx_1 + dy_1})$

$= mT(\mathbf{v_1})$

$\therefore$ T is a l.t. □

This result explains why a translation is not a matrix. For example T(**x,y**) = (**x +1, y + 2**) is not a linear transformation because T(**1,0**) = (**2,2**) and T(**0,1**) = (**1,3**) but T(**1,1**) = (**2,3**) which is **not** (**2,2**) + (**1,3**).

Note that any linear transformation always leaves **0** unchanged because T(m**v**) = mT(**v**) applies equally when m = 0.

i.e. T(**0**) = **0** for all linear transformations, T.

Definition: 1-1

A function is said to be 1-1 if for each element in the Range there is a **unique** element in the domain which maps to it.

e.g. f(x) = 2x + 3 is 1-1.

But $g(x) = x^2$ is not 1-1 because g(3) = g(-3) = 9.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is 1-1

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not 1-1 because $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 26 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ is also $\begin{bmatrix} 13 \\ 26 \end{bmatrix}$.

Definition: Onto

A linear transformation T is said to be **ONTO** if for every element in the **Range Space** there is an element in the Domain mapping to it i.e. if Range and Range space are the same set of vectors (points).

For example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is onto.

But $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not onto because no element in the Domain maps to (1,3) for example.

Question

Is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ onto?

Let (a,b) be an arbitrary element in the Range Space. We need to investigate whether $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has a solution (x,y,z) for **all** (a,b). (We are no longer concerned with uniqueness remember).

i.e. $x + 2y + 3z = a$

$2x + 3y + 4z = b$

These two equations have a solution for any values of a and b since they are two non-parallel, and hence intersecting, planes.

$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ is onto.

(Note however, that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ is not onto since for example, (1,3) is not in the Range).

Question

Is $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ onto?

Let (a,b,c) be an arbitrary element in the Range Space. We need to examine the existence of a solution (x,y) to $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ is not onto because, for example, the range is the plane $x - 2y + z = 0$ and not R^3.

Definition

An **ISOMORPHISM** is a l.t. which is both 1-1 and onto. The word isomorphism, from its Greek roots, means 'equal shape'. This means that if an isomorphism exists between two vector spaces V and W then V and W have the same 'basic structure'. Operations within V and W are preserved by the isomorphism. We say V is isomorphic to W.

e.g. i) R^2 is isomorphic to C (set of complex numbers)

$(x,y) \longleftrightarrow x + yi$

ii) The set of all quadratic functions is isomorphic to R^3

$ax^2 + bx + c \longleftrightarrow (a,b,c)$

A l.t. (i.e. homomorphism) has an inverse if and only if it is both 1-1 and onto, i.e. an isomorphism.

For example, $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not an isomorphism since its determinant is zero and therefore has no inverse.

Note as shown earlier that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is neither 1-1 nor onto.

Exercise 12.1

1. Find a matrix mapping (1,1) to (1,1,1)

2. State whether the following are homomorphisms or not.

   i) $T: R^3 \longrightarrow R^3 \ni T(x,y,z) = (\sin x, \sin y, \sin z)$

   ii) $T: R^3 \longrightarrow R^3 \ni T(x,y,z) = (z + xy, x + yz, y + xz)$

   iii) $T: R^3 \longrightarrow C \ni T(x,y,z) = x + y + zi$

   iv) $T: R^3 \longrightarrow R^3 \ni T(x,y,z) = (x, x + y, x - y)$.

3. Let A be a 2 x 2 matrix such that det A = 0

   i) Does A represent an onto function?

   ii) Does A represent a 1-1 function?

4. True or False?

   i) A is onto → A is a square matrix

   ii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is 1-1.

   iii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is onto.

   iv) det A = 0 → A has no inverse

   v) det A = 2 → A is an isomorphism

   vi) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ has no inverse.

vii) det A = 2 → A is 1-1.

viii) A is a square matrix → A is 1-1

ix) A is a square matrix → A is onto.

x) A is onto → A is not a square

xi) det A ≠ 0, det B ≠ 0 → (AB) is 1-1.

5. Fill in the blank.

   A:V ⟶ W is a l.t. such that for any **w** ∈ W, there exists a **v** ∈ V such that

   A(**v**) = **w**. ∴ A is ____?____.

6. Convince yourself that the definition for 1-1 is equivalent to (f(x) = f(y) → x = y)

   In fact, the standard procedure for 1-1 proofs is to let f(x) = f(y) and show that this only occurs when x = y.

7. Find a matrix representing a reflection in the x-axis in R^3.

8. Give an example of a 3x2 matrix which is not 1-1.

9. Find the image of the set of vectors {(**x,y**)|y = 2x} after being mapped by the homomorphism $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$.

10. Investigate whether $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix}$ is i)1-1 ii) onto

11. Write f as a matrix where f(**x,y,z**) = (**2x+y,z**).

12. a) Investigate whether $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ is i) 1-1 ii) onto.

b) Repeat for $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{bmatrix}$

13. Find the inverse of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

14. Give an example of a matrix mapping form R^3 to R^2 which is neither 1-1 nor onto.

15. Let f: $R^2 \longrightarrow R^2$ be such that f(x,y) = (x+y, x – y). Prove f is a linear transformation.

16. Give an example for A where $A^{12} = I$.

17. f is a function such that f(x,y) = (x + y + 1, x + y – 1).

    i) Can f be represented by a matrix?

    ii) Is f 1-1?

Exercise 12.1 Answers

1. $\begin{bmatrix} 1 & a \\ 1 & b \\ 1 & c \end{bmatrix}$ for any a,b,c.

2. i) No ii) No iii) Yes iv) Yes

3. i) No. ii) No.

4. i) True ii)True iii) True iv) True v) True vi) False

   vii) True viii) False ix) False x) False xi) True

5. onto.

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

8. $\begin{bmatrix} a & ma \\ b & mb \\ c & mc \end{bmatrix}$ e.g. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 6 \end{bmatrix}$

9. Zero vector.

10. i) No ii) Yes.

11. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. a) i) Yes ii) No b) i) Yes ii) No

13. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} a & b & c \\ ma & mb & mc \end{bmatrix}$ e.g. $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 12 \end{bmatrix}$

16. A can be a rotation of 30° a.c. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

17. i) No ii) No. iii) R^2 is a domain $\{(u,v) \in R^2 : u - v = 2\}$.

Theorem 12.1

"The Range of a linear transformation is a vector space". i.e. Let A:V → W be a l.t. then U = {A(**v**) | **v** ∈ V} is a vector space. Note U is a subset of W.

Proof

i) **To show U is closed under Vector addition**

Let $\mathbf{u_1}$ and $\mathbf{u_2} \in$ U. Then there exists $\mathbf{v_1}$ and $\mathbf{v_2} \in$ V such that $A(\mathbf{v_1}) = \mathbf{u_1}$ and

$A(\mathbf{v_2}) = \mathbf{u_2}$.

Then $\mathbf{u_1} + \mathbf{u_2} = A(\mathbf{v_1}) + A(\mathbf{v_2})$

$= A(\mathbf{v_1} + \mathbf{v_2})$ since A is a l.t.

But $\mathbf{v_1} + \mathbf{v_2} \in$ V since V is a vector space and hence closed under vector addition.

$\therefore A(\mathbf{v_1} + \mathbf{v_2}) \in$ U

$\therefore \mathbf{u_1} + \mathbf{u_2} \in$ U and hence U is closed under vector addition.

ii) **To Show U is closed under scalar multiplication.**

Let $\mathbf{u_1} \in$ U and c be scalar.

As before $A(\mathbf{v_1}) = \mathbf{u_1}$

Then $c\mathbf{u_1} = c(\mathbf{v_1})$

$= A(c\mathbf{v_1})$ since A is a l.t.

But $c\mathbf{v_1} \in$ V since V is a vector space and hence closed under scalar multiplication.

$\therefore A(c\mathbf{v_1}) \in$ U and hence $c\mathbf{u_1} \in$ U.

This establishes that U is closed under scalar multiplication.

$\therefore$ U is a vector space.

Note that this means for our consideration that the Range of a matrix is $\{\mathbf{0}\}$, a line through the origin, a plane through the origin, R^2 or R^3.

The dimension of the Range of a matrix is call its **RANK**.

e.g. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has Range $\{ (\mathbf{u},\mathbf{v}) \mid v = 2u\}$ i.e. its Rank is **<u>1.</u>**

Theorem 12.2

Let $T: R^3 \longrightarrow R^3$ be a l.t. Then $\{T(\mathbf{i}), T(\mathbf{j}), T(\mathbf{k})\}$ spans the Range of T.

This theorem means that to investigate the Range of a l.t., all we need do is investigate the mappings of **i**, **j**, **k**.

Proof

Let **u** be some arbitrary vector in the Range of T.

The $\mathbf{u} = T(\mathbf{v})$ for some **v** in R^3.

i.e. $\mathbf{u} = T(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$ for some scalars a,b,c.

$= aT(\mathbf{i}) + bT(\mathbf{j}) + cT(\mathbf{k})$ since T is a l.t.

i.e. $T(\mathbf{i}), T(\mathbf{j}), T(\mathbf{k})$ space the Range of T.

One might think that $\{T(\mathbf{i})), T(\mathbf{j}), T(\mathbf{k})\}$ is a basis for the Range of T. This is not true, however, unless T is 1-1, since they may not be linearly independent.

Question

Find rank of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = T$

T(**i**) = (**1,1,1**) (first column of T)

T(**j**) = (**2,2,2**) (second column of T)

T(**k**) = (**3,3,3**) (third column of T)

And hence by Theorem 12.2 the Range of T is the set of all linear combinations of (**1,1,1**), (**2,2,2**), (**3,3,3**) which is clearly the line x = y = z. Rank of T is 1.

Question

$$\text{Find the Range of } \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} = T$$

T(**i**) = (**1,2,3**)

T(**j**) = (**2,3,4**)

∴ The Range of T is the vector space spanned by the vectors (**1,2,3**) (**2,3,4**), i.e. a plane. (**1,2,3**) x (**2,3,4**) is normal to this plane, i.e. (**-1,2,-1**). Plane passes through the origin since Range of T is a vector space by Theorem 8.1

∴ Range of T is $-x + 2y - z = 0$. Rank of T is 2.

Question

$$\text{Find Range of } \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 3 & 4 & 8 \end{bmatrix} = T$$

T(**i**) = (**1,2,3**)

T(**j**) = (**2,3,4**)

T(**k**) = (**4,6,8**)

The Range of T is spanned by (**1,2,3**), (**2,3,4**), (**4,6,8**). Since however (**4,6,8**) is a multiple of (**2,3,4**) it does not contribute any extra vectors to the Range of T from those spanned by (**1,2,3**) and (**2,3,4**). Thus the Range of T is as before in the last question. –x + 2y – z = 0. Its Rank is 2.

From Theorem 12.2 and the three questions it becomes clear that the Rank of a Matrix = the number of linearly independent column vectors of the Matrix.

e.g. $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ has Rank 1 but $\begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$ has Rank 2.

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix}$ has Rank 1 but $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 9 \end{bmatrix}$ has Rank 2.

Also $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ has Rank 2 and $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 6 \end{bmatrix}$ has Rank 3 (since {(**1,1,1**), (**2,3,4**), (**3,4,6**)} is a l.i. set)

Similarly we may derive a matrix using these techniques.

Question

Find the matrix representing a rotation of 90° a.c. about the x axis followed by a reflection in the y axis **in $\mathbf{R}^3$**.

i is mapped to –**i**.

j is mapped to –**k**

k is mapped to –**j**.

$\therefore$ The matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

Definition of Kernel

Let $T:V \longrightarrow W$ be a l.t. The Kernel of T (written KerT) is the set of all vectors in V mapping to **0** in W.

i.e. $\text{Ker}T = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\}$. i.e. KerT is a subset of the Domain of T.

Examples

i) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then Ker A = $\{(\mathbf{x},\mathbf{y})\} | x + 2y = 0\}$

ii) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then Ker A = $\{\mathbf{0}\}$

iii) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then Ker A = the set of all vectors $(\mathbf{x},\mathbf{y},\mathbf{z})$ satisfying

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x + 2y + 3z = 0$

$4x + 5y + 6z = 0$

$7x + 8y + 9z = 0$

Solving these yields a line of intersection $x = \frac{y}{-2} = z$.

i.e. Ker A = $\{ m(\mathbf{1,-2,1}) \mid m \in R\}$.

Theorem 12.3

Let $T:V \longrightarrow W$ be a l.t. then T is 1-1 if and only if Ker T = $\{\mathbf{0}\}$.

Proof

The proof is split in two parts since it is an "if and only if" theorem.

(➔)

If T is 1-1, then only one vector maps to the zero vector. Since T(**0**) = **0** then Ker T = {**0**}

(🡐)

Let KerT = {**0**}. To show T is 1-1.

[See Exercise 12.1 #6 We will let T($\mathbf{v_1}$) = T($\mathbf{v_2}$) and try to show $\mathbf{v_1} = \mathbf{v_2}$]

Let T($\mathbf{v_1}$) = T($\mathbf{v_2}$)

Then T($\mathbf{v_1}$) - T($\mathbf{v_2}$) = **0**

$\therefore$ T($\mathbf{v_1} - \mathbf{v_2}$) = **0**

$\therefore$ $\mathbf{v_1} - \mathbf{v_2} \in$ KerT definition of Kernel

$\therefore$ $\mathbf{v_1} - \mathbf{v_2}$ = **0** since KerT = {**0**}

$\therefore$ $\mathbf{v_1} = \mathbf{v_2}$

$\therefore$ T is 1-1

□

Question

Is $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ 1-1?

Kernel of this matrix is found by considering $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e. $x + 2y = 0$

$2x + 3y = 0$

$3x + 4y = 0$

This clearly only has (**0,0**) as a solution. i.e. Kernel = {**0**}.

i.e. Matrix is 1-1.

Question

$$\text{Is } A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 3 & 4 & 8 \end{bmatrix} \text{ 1-1?}$$

We examine
$$\left.\begin{aligned} x + 2y + 4z &= 0 \\ 2x + 3y + 6z &= 0 \\ 3x + 4y + 8z &= 0 \end{aligned}\right\}$$
and note that (**0,0,0**) is **not** the only solution.

($\mathbf{n_1} \bullet \mathbf{n_2} \times \mathbf{n_3} = 0$) hence KerA $\neq \{\mathbf{0}\}$

i.e. A is not 1-1.

Theorem 12.4

Let $A: V \longrightarrow W$ be a l.t. The KerA is a vector space (i.e. subspace of V).

Proof

The proof is left as an exercise for the reader.

(See Theorem 12.1 as a guide).

Example

For $\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$ the Kernel is $6x + 4y = 0$, i.e. a line through the origin, i.e. a vector space.

The dimension of the Kernel is called the **NULLITY** of the matrix.

e.g. $\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$ has nullity 1.

Theorem 12.5

Given $T: V \longrightarrow W$ is a linear transformation then

Rank + Nullity = Dimension of Domain.

i.e. Dimension of Range + Dimension of Kernel = Dimension of Domain

A proof will be given in n-space as an example of a general proof for more advanced work.

Proof

Let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\}$ be a basis for Ker T. i.e Nullity = n.

Since the Kernel is a subspace of V we can extend this basis for Ker T to a basis for V.

i.e. let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}, \ldots \mathbf{v_m}\}$ be a basis for V. i.e. dimension of domain = m.

It suffices now to show dim (Range T) = m – n.

Let S = $\{T(\mathbf{v_{n+1}}), T(\mathbf{v_{n+2}}), .. T(\mathbf{v_m})\}$

Claim

S is a basis for Range of T.

Proof of Claim

We need to show S spans Range of T and is a l.i. set.

i) **To show S spans Range of T.**

Let $\mathbf{w} \in$ Range T. Then $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V.

But $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_m}\}$ is a basis for V.

$\therefore \mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \ldots a_m\mathbf{v_m}$

$\therefore \mathbf{w} = T(\mathbf{v}) = T(a_1\mathbf{v_1} + a_2\mathbf{v_2} + \ldots a_m\mathbf{v_m})$

$= a_1T(\mathbf{v_1}) + a_2T(\mathbf{v_2}) + \ldots a_mT(\mathbf{v_m})$ since T is a l.t.

$= \mathbf{0} + \mathbf{0} + \ldots a_{n+1}T(\mathbf{v_{n+1}}) + a_{n+2}T(\mathbf{v_{n+2}}) + \ldots a_mT(\mathbf{v_m})$

Since $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_m}\} \in$ KerT

i.e. $\mathbf{w}$ is a linear combination of $T(\mathbf{v_{n+1}}), T(\mathbf{v_{n+2}}), .. T(\mathbf{v_m})$

i.e. S spans Range of T.

ii) **To show S is a linearly independent set**

Let $b_{n+1}T(\mathbf{v_{n+1}}) + b_{n+2}T(\mathbf{v_{n+2}}) + \ldots + b_mT(\mathbf{v_m}) = \mathbf{0}$

By Theorem 12.4 it suffices to show $b_{n+1} = b_{n+2} = \ldots. = b_m = 0$.

Since T is a l.t. then

$T(b_{n+1}\mathbf{v_{n+1}} + b_{n+2}\mathbf{v_{n+2}} + \ldots b_m\mathbf{v_m}) = \mathbf{0}$

$\therefore b_{n+1}\mathbf{v_{n+1}} + b_{n+2}\mathbf{v_{n+2}} + \ldots b_m\mathbf{v_m} \in$ KerT

$\therefore b_{n+1}\mathbf{v_{n+1}} + b_{n+2}\mathbf{v_{n+2}} + \ldots b_m\mathbf{v_m} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots c_n\mathbf{v_n}$ for some c_i.

$\therefore c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots c_n\mathbf{v_n} - b_{n+1}\mathbf{v_{n+1}} - Tb_{n+2}\mathbf{v_{n+2}} - \ldots b_m\mathbf{v_m} = \mathbf{0}$

But $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_m}\}$ is a l.i. set $\therefore$ by Theorem 12.4

$c_1 = c_2 = \ldots c_n = b_{n+1} = b_{n+2} = \ldots b_m = 0$

In particular, $b_{n+1} = b_{n+2} = \ldots b_m = 0$

$\therefore$ S is a l.i. set.

Since S spans Range of T and is a l.i. set, then S is a basis for Range of T. Therefore since S has m-n elements and is a basis for Range of T, then Range of T has dimension m-n.

i.e. Rank of T = m – n.

$\therefore$ Rank + Nullity = Dimension (domain).

□

This theorem has an important corollary.

Corollary

For square matrices 1 -1 is equivalent to onto.

Proof

Let matrix be mxm. Then Dim(Domain) = m.

If function is 1-1, Kernel = {**0**} Theorem 8.3.

i.e. Nullity = 0. By Theorem 8.5 it follows that Rank = m.

$\therefore$ Range of matrix has dimension m. Since Range Space is of dimension m (matrix is m x m) then matrix is onto. The argument is completely reversible establishing the equivalence of 1-1 and onto for square matrices.

i.e. A square matrix is an isomorphism iff it is 1-1.

This corollary explains why $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ for example, is not onto because it is not 1-1.

Furthermore if det A = 0, then A is neither 1-1 nor onto.

Example

Investigate the Kernel, Range of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$

Is it 1-1, onto, an isomorphism?

i) The Kernel is the intersection of $x + 2y + 3z = 0$

$$2x + 4y + 6z = 0$$

$$4x + 8y + 12z = 0$$

i.e. three overlapping planes.

$\therefore$ Kernel is the plane $x + 2y + 3z = 0$. i.e. Nullity = 2.

Since Kernel $\neq \{\mathbf{0}\}$, matrix is not 1-1, hence not onto and not an isomorphism, i.e. determinant = 0.

ii) The Range is the subspace spanned by the column vectors of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix} \text{(Theorem 12.2)}$$

$\{(\mathbf{1,2,4}), (\mathbf{2,4,8}), (\mathbf{3,6,12})\}$. It is the line $x = \frac{y}{2} = \frac{z}{4}$.

Rank = 1.

Note: Rank + Nullity = Dim(Domain) as required.

Example

Investigate $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix}$ for Kernel, Range, 1-1, onto, isomorphism.

i) The Kernel is the intersection of $x + 2y + 3z = 0$ and $2x + 5y + 7z = 0$, i.e. a line. Therefore Nullity = 1. Since Ker $\neq \{\mathbf{0}\}$, it is not 1-1 and hence not an isomorphism.

By Theorem 8.5, since Dim(Domain) = 3, Rank = 2.

Therefore Range is a plane. Since the Range Space is R^2, i.e. dimension 2. Range = Range Space and matrix is **onto**.

i.e. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix}$ is not 1-1, is onto, not an isomorphism.

Exercise 12.2

1. Find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

2. Investigate the following matrices fully. Find their Kernels, Ranges. State whether they are 1-1, onto or isomorphisms.

i) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ v) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{bmatrix}$ vi) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$

vii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{bmatrix}$ vii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

3. Explain why, in your own words, that if det A = 0, then A is neither 1-1 nor onto.

4. Find a vector mapping to (**1,0**) for transformation $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$.

5. **True or False?**

   i) A is 1-1 → A is a square matrix

   ii) det A = 2 → A is a square matrix

   iii) det A = 0 → A is a square matrix

   iv) Ker A = {(x,y) | y = 2x} → A is not 1-1.

   v) det $(A^2) = (\det A)^2$

   vi) Ker A = {**0**} → A is a square matrix.

   vii) Ker A is a subspace of domain of A.

   viii) Determinant of a shear can be zero.

   ix) Determinant of a projection can be 1.

   x) det A = 0 → A is onto. (Is this the same question as A^{-1} does not exist implies A is onto?)

   xi) Dimension Domain of A ≥ Rank A.

   xii) The Range of a 2x3 matrix may be R^3.

   xiii) The Range of a 3x2 matrix may be R^3.

xiv) A maps a basis for Domain to a basis for the Range.

xv) $\{(\mathbf{2,3,1})\}$ is a basis for the Range of $\begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 1 & 2 \end{bmatrix}$

xvi) A^{-1} exists → A is an isomorphism.

xvii) $f(x,y,z) = (x + y, y - z, y + z)$ is a homomorphism.

xviii) A, a 3x3 matrix, and det $A \neq 0$ → Rank A = 3.

xix) A, a 3x3 matrix, and Rank A = 2 → det A = 0.

xx) $\det(P^{-1} AP) = \det A$.

xxi) $A^3 = A^2$ → det A = 0 or 1.

xxii) $A^2 = A$ and A^{-1} exists → A = I.

xxiii) A projection is idempotent.

xxiv) A reflection is an isometry.

xxv) If A is a rotation, $A^T = A^{-1}$.

xxvi) If det A = 1 then A maps a basis for the domain to a basis for the range.

xxvii) A^{-1} does not exist implies A is into.

xxviii) A maps a basis for the domain to a set spanning the range.

6. Let A be a 2x2 matrix such that det A = 0.
   i) Can Ker A = $\{\mathbf{0}\}$?
   ii) Can A be onto? 1-1?
7. Let A be a 2x2 matrix such that ker A $=\{\mathbf{0}\}$.
   i) Is A necessarily onto?
   ii) Is A an isomorphism?
8. What is the dimension of the subspace of R^3 spanned by $\{(\mathbf{1,1,0}), (\mathbf{2,1,1}), (\mathbf{0,-1,1})\}$?
9. Give an example of a homomorphism mapping from $R^3 \longrightarrow R^2$ which is neither 1-1 nor onto.
10. Convince yourself that:
    i) if determinant of a 3x3 matrix $\neq 0$ then Kernel = $\{\mathbf{0}\}$ and Range = R^3.
    ii) If T: $R^3 \longrightarrow R^2$ is a l.t. then T is never 1-1.

iii) If T: $R^2 \longrightarrow R^3$ is a l.t. then T is never onto.

11. f is a l.t. from $R^3 \longrightarrow R^2$ such that

    f(**1,2,0**) = (**3,4**), f(**-2,0,3**) = (**2,-3**) and f(**4,1,0**) = (**1,1**)

    Find f(**12,1,9**). Do **not** find f as a matrix.

12. Investigate fully for Kernel, Range, etc. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 5 & 8 \end{bmatrix}$

13. Find the vector mapping to (**3,7,11**) after transformation $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

14. State the Rank of the following matrices –

    i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

15. Find a basis for the Kernel of: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

16. For $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ the Kernel is $\{(\mathbf{x},\mathbf{y}) \mid y = \frac{1}{2}x\}$ and the Range is $\{(\mathbf{x},\mathbf{y}) \mid y = 2x\}$

    i.e. the Kernel is "perpendicular" to the Range.

    For $\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ the same is true.

    Is it true in general? If so, prove it. If not, give a counter example.

17. Find the matrix which represents a reflection in the plane $2x + y + 4z = 0$ in R^3

18. Find the matrix which represents a 60° rotation about the line $x = y = z$ in R^3.

19. Find a 3x3 matrix such that $A^4 = I$, but $A^2 \neq I$.

20. If AB = zero matrix and neither A nor B is the zero matrix, show that neither A nor B can be isomorphism.

21. Find the matrix representing a reflection about $Ax + By + Cz = 0$.

22. Let A be a matrix such that $A^2 = A$ and $A \neq I$. Prove det A = 0.

Exercise 12.2 Answers

1. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

2.

	Kernel	Range	1-1	Onto	Isomorphism
i)	$x + 2y = 0$	$y = 3x$	No	No	No
ii)	**0**	R^2	Yes	Yes	Yes
iii)	$\frac{x}{2} = \frac{y}{-1}; z = 0$	R^2	No	Yes	No
iv)	$x + 2y + 32 = 0$	$y = 2x$	No	No	No
v)	**0**	$2x - y = 0$ in R^3	Yes	No	No
vi)	$x + 2y = 0$	$x = \frac{y}{2} = \frac{z}{3}$	No	No	No
vii)	$x = y = -z$	$2x - y = 0$ in R^3	No	No	No
vii)	$x + 2y + 3z = 0$	$x = \frac{y}{2} = \frac{z}{3}$	No	No	No

4. $\{(\mathbf{-1 + t, 1 - 2t, t}): T \in R\}$
5. i) False ii) True iii) True iv) True v) True vi) False vii) True viii) False ix) False x) True xi) True xii) False xiii) False xiv) False xv) True xvi) True xvii) True xviii) True xix) True xx) True xxi) True xxii) True xxiii) True xxiv) True xxv) True xxvi) True xxvii) False xxviii) True
6. i) No ii) No. No.
7. i) Yes. ii) Yes.
8. Two.
9. $\begin{bmatrix} a & b & c \\ ma & mb & mc \end{bmatrix}$ e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

11. (5.-12)

12. Kernel = {(**x,y,z**) : $x = \frac{y}{2} = z$)

Range = {(**x,y,z**) : z = 3x - y)

13. (**1,1**)

14. i) 1 ii) 2 iii) 3

15. Any two independent vectors in x + y + z = 0. e.g. {(**1,1,-2**) (**3,-1,-2**)}

16. No e.g. $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

17. $\begin{bmatrix} \frac{13}{21} & \frac{-4}{21} & \frac{-16}{21} \\ \frac{-4}{21} & \frac{19}{21} & \frac{-8}{21} \\ \frac{-16}{21} & \frac{-8}{21} & \frac{-11}{21} \end{bmatrix}$

18. $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

19. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ i.e. 90° rotation a.c. about the x axis.

21. $\frac{1}{A^2 + B^2 + C^2} \begin{bmatrix} B^2 + C^2 - A^2 & -2AB & -2AC \\ -2AB & A^2 + C^2 - B^2 & -2BC \\ -2AC & -2BC & A^2 + B^2 - C^2 \end{bmatrix}$

Two Theorems are now given in a slightly more theoretical way.

Theorem 12. 6

Let A: V $\longrightarrow$ W be a 1-1 l.t. Then A maps l.i. sets to l.i. sets.

Proof

Let {$\mathbf{v_1}$, $\mathbf{v_2}$, … $\mathbf{v_n}$} be a l.i. set.

Assume

{A($\mathbf{v_1}$), A($\mathbf{v_2}$), … A($\mathbf{v_n}$)} is a **dependent set**

Then there exists an i, $1 \le i \le n$, such that

$A(\mathbf{v_i}) = \sum_{j=i}^{n} c_j\, A(\mathbf{v_j})$, $j \ne i$. for some scalars c_j not all zero.

i.e. $\mathbf{0} = A(\mathbf{v_i}) - \sum_{j=i}^{n} c_j\, A(\mathbf{v_j})$,

i.e. $\mathbf{0} = \sum_{j=i}^{n} A(c_j v_j)$ since A is a l.t.

But A is 1-1 $\therefore$ Ker A = $\{\mathbf{0}\}$ $\therefore \sum_{j=i}^{n} c_j v_j = \mathbf{0}$

This contradicts $\{\mathbf{v_i}\}^{n}_{i=1}$ is a l.i. set since not all the c_j are zero. Hence the assumption is false. □

Theorem 12.7

Let A: V $\longrightarrow$ W be a l.t. If there exists $\mathbf{w} \in$ W such that $\{\mathbf{v} \in V \mid A(\mathbf{v}) = \mathbf{w}\}$ has only one element then A is 1-1.

(i.e if A is "1-1" for one element, it is a 1-1 linear transformation).

Proof

Assume A is not 1-1, then Ker A $\ne \{\mathbf{0}\}$.

Let $\mathbf{x}$ ($\ne \mathbf{0}$)$\in$ Ker A.

$$\begin{aligned} \text{Now } A(\mathbf{v} + \mathbf{x}) &= A(\mathbf{v}) + A(\mathbf{x}) \text{ (A is a l.t.)} \\ &= \mathbf{w} + \mathbf{0} \\ &= \mathbf{w}. \end{aligned}$$

Then $\mathbf{v} + \mathbf{x}$ ($\ne \mathbf{v}$) belongs to the set $\{\mathbf{v} \in V \mid A(\mathbf{v}) = \mathbf{w}\}$ as well as $\mathbf{v}$ contradicting the hypothesis. Hence, A is 1-1. □

Example of Theorem 12.7

Previously we found that (**1,1**) was the only vector mapping to (**3,5,7**) by $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$.

Therefore $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ is 1-1.

Chapter 12 Review

1. Write A(x,y,z) = (2x – y, z, -x) as a matrix. Is it 1-1?
2. Find the nullity of $\begin{bmatrix} 1 & 2 & 1 \\ 4 & -1 & 2 \end{bmatrix}$.
3. Find the inverse of $\begin{bmatrix} 1 & 3 & 4 \\ 2 & -2 & 2 \\ 4 & 4 & -5 \end{bmatrix}$
4. Review the meaning of nilpotent, singular, idempotent, isomorphism, into, homomorphism, diagonal, orthogonal.
5. Find m so that $\begin{bmatrix} 2 & 3 \\ 4 & m \end{bmatrix}$ is an into function.
6. Fill in the missing entries $\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & . & . \\ -1 & 0 & 2 \\ 4 & . & . \end{bmatrix}$
7. Can a non-square matrix represent an isomorphism?
8. Find the Kernel and Range of $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$/
9. Find a matrix mapping from $R^2 \longrightarrow R^3$ such that its range is x = y = z and its kernel is y = 3x.
10. Find a matrix which will map (1,2) to (1,2,3) and (2,3) to (2,3,4).
11. Find the inverse of $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Is this matrix an ISOMETRY?
12. Prove or disprove $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is onto.
13. Let M = $\{\mathbf{v} \in R^3 \mid A(\mathbf{v}) = (\mathbf{1,2})\}$ where A is a 2x3 matrix. Is M a subspace of R^3?
14. Find Kernel and Range of $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
15. Let A: $V \longrightarrow W$ be a l.t. Prove Ker A is a subspace of V.
16. i) Find the matrix representing a projection onto x = y = z.

ii) Find image of (1,2,3) projected onto x = y = z.

17. Prove that all idempotent matrices, except I, are singular.

18. Find a matrix whose kernel is $x = \frac{y}{2} = \frac{z}{3}$ and whose range is x + 2y = z = 0.

Chapter 12 Review Answers

1. $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ Yes.

2. 1. Kernel is the line {**x,y,z**}: $\frac{x}{5} = \frac{y}{2} = \frac{z}{9}$}

3. $\frac{1}{120}\begin{bmatrix} 2 & 31 & 14 \\ 18 & -21 & 6 \\ 16 & 8 & -8 \end{bmatrix}$

5. m = 6

6. $\begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & -2 \\ 4 & 6 & 5 \end{bmatrix}$

7. No.

8. Kernel is {**x,y,z**}: x = -z; y = 0}

9. $\begin{bmatrix} -3a & a \\ -3a & a \\ -3a & a \end{bmatrix}$ for any a.

10. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ Yes.

12. It is onto since for any (a,b) in R^2 x + y + z = a, x + y + 2z = b has a solution (x,y,z)

13. No since $\mathbf{0} \notin M$.

14. Kernel = {**0**}. Range = R^3

16. i) $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ ii) (2,2,2)

18. There are many possible matrices. They are of the form

$$\begin{bmatrix} 3a & 3b & -a-2b \\ 3c & 3d & -c-2d \\ -3a-6c & -3b-6d & a+2b+2c+4d \end{bmatrix}$$

where a,b,c,d can be any real numbers