CHAPTER ELEVEN

11. Proof

Proof forms a central theme in the study of mathematics. Many mathematicians consider the proofs of theorems based on a set of axioms which are accepted as true as being the real essence of mathematics.

Unfortunately, there still abounds in mathematics text books and necessary school curriculum the presence of anachronistic elements. I speak of such as the abhorrence of dividing by zero, (which can be logically correct), rationalising denominators, trigonometric identity "proofs", formalised 'proofs' of Euclidean Geometry presented as an arcane religion with its "Givens, CPCTC, ITT" and holy writ. Future secondary school teachers, but certainly nobody else, need to master this obscure theology masquerading as mathematics.

Mathematics is a system of thought based on the principle "If …. then". It is not a set of skills though its practice requires and inculcates skills. It is not simply a science. Thus, especially in this age of calculators, computers and the like, what matters is that the student learns to think rationally and be capable of expressing these logical thoughts to an uninitiated reader. Mathematics is something you do, not merely something you learn.

My hope is that this chapter will help the student to master the art of writing proofs and thinking rationally.

Statements whose truth remains unproven are known as conjectures so, for greater example, the statement that every even number greater than or equal to six is the sum of two odd prime numbers is a conjecture because, although no one has ever found an even number which is not the sum of two odd primes, it has never been proven.

Two famous conjectures were proved only relatively recently:

The four colour conjecture: This stated that only four colours were needed to colour any map so that no countries with a common border had the same colour. This conjecture was finally proven in 1976, having first been put forward in 1852.

Fermat's Last Theorem: This stated for n, an integer greater than two, there are no positive integer values x, y, and z such that $x^n + y^n = z^n$. It was called a theorem since the seventeenth century when it was first stated because Fermat claimed to have proved it even though no one since was able to do so until 1995 when Andrew Wiles of Princeton produced a fantastic 200 page proof ending 350 years of intense study.\

In general, a statement is not considered to be a theorem simply because it is true. A theorem carries a mathematical significance by virtue of its generality. For example, Pythagoras' theorem is important because it is true for all right triangles rather than a few specific ones. For that reason it is not usually possible to prove a theorem by enumerating some specific examples. Noting that $3^2 + 4^2 = 5^2$ does not prove that $x^2 + y^2 = z^2$ for a right triangle with z as hypotenuse.

However, it is possible to disprove a false theorem by quoting a counter-example. Suppose that it is conjectured that

$$(x + y)^2 = x^2 + y^2$$

then the conjecture can be shown to be false by simply noting a single counter-example e.g. x = 2 and y = 3. Then L.H.S. = 25 and R.H.S = 13 which is clearly false.

<u>Fallacy</u> is an incorrect result which has an apparently logical explanation of why the result is correct, or a correct result obtained through incorrect reasoning.

e.g. $\frac{16}{64} = \frac{1}{4}$ " by canceling sixes"

OR

de Morgan's Fallacy

Let x = 1 then x = 0

"Proof"

$x = 1$

$\therefore x^2 = x$ (multiplying both sides by x)

$\therefore x^2 - 1 = x - 1$ (subtracting 1 from both sides)

$\therefore \frac{x^2 - 1}{x - 1} = \frac{x - 1}{x - 1}$ (dividing both sides by $x - 1$)

$\therefore x + 1 = 1$ (canceling $x - 1$ on both sides)

$\therefore x = 0$

An Infinite Series Fallacy

Consider

$S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots.$

Then $S = (1 - 1) + (1 - 1) + (1 - 1) + \ldots.$

$\therefore S = 0 + 0 + 0 \ldots$

$\therefore S = 0$

Also $S = 1 - (1 - 1) - (1 - 1) - (1 - 1) - \ldots.$

$\therefore S = 1 - 0 - 0 - 0 - \ldots.$

$\therefore S = 1$

Also $S = 1 - (1 - 1 + 1 - 1 + 1 - 1 \ldots.)$

$\therefore S = 1 - S$

$\therefore 2S = 1$

$\therefore S = \frac{1}{2}$

The fallacy here comes from the fact that S does not actually have a sum.

During World War II the Swedish Broadcasting Company made the following radio announcement

> "A civil – defence exercise will be held this week. In order to make sure that the civil-defence units are properly prepared, no one will know in advance on what day this exercise will take place"

A paradox occurs when there is a contradiction but no apparent fallacy. This statement above is a paradox because;

Assume the announcement was made on a Monday. Then the exercise must take place before the next Monday. It cannot take place on Sunday because by then people will know it will take place. Because it cannot take place on Sunday, then it cannot take place on Saturday by the same reasoning. And so on. Therefore, it cannot take place. Another example of a paradox is attributable to Bertrand Russell who developed the following in 1918.

> "A man of Seville is shaved by the Barber of Seville if and only if the man does not shave himself. Does the Barber of Seville shave himself?"

If the answer is "yes", then the answer is "no". If the answer is "no" then the answer is "yes"

<u>Direct Proof</u>

A direct proof is one in which statements and results are used which have previously been proven true and accepted as such.

<u>Example</u>

Prove "If M is a 2 x 2 matrix and $M^2 = M + I$ then $M^4 = 3M + 2I$."

Proof

$$
\begin{aligned}
M^4 &= (M^2)^2 \\
&= (M + I)^2 \qquad \text{(given } M^2 = M + I\text{)} \\
&= (M + I)(M + I) \\
&= M^2 + MI + IM + I^2 \text{ (distributivity)} \\
&= M^2 + M + M + I \text{ (Identity leaves a matrix unchanged)} \\
&= (M + I) + 2M + I \text{ (given } M^2 = M + I\text{)} \\
&= 3M + 2I
\end{aligned}
$$

One difficulty in writing a proof occurs because the axioms governing mathematical structure are not usually clearly itemized until rather later in the career of a mathematics student and so it is not always easy for the student to know what can be accepted and what needs to be explained or proved. For example, in the above proof $I^2 = I$ was already stated. Should it have been proved? Also the associativity and commutativity of matrix addition was assumed. Should there have been some explanation? There are no simple answers to these questions and students can only learn what steps may be assumed and what steps need to be proved by experience.

Example 2

Prove $n^3 - n$ is divisible by 6 for all positive integers n.

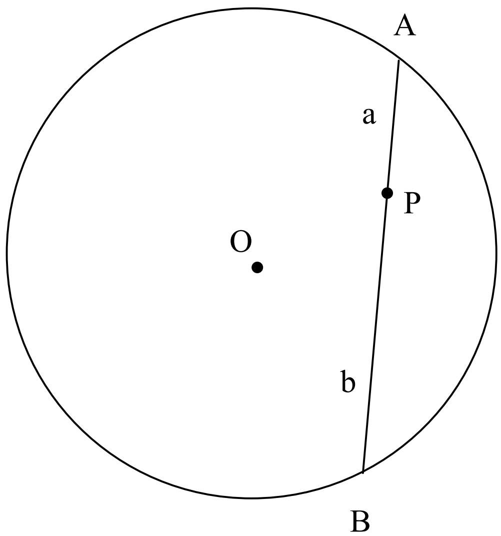
Proof

$$
\begin{aligned}
&n^3 - n \\
&= n(n^2 - 1) \\
&= n(n - 1)(n + 1) \\
&=(n - 1)(n)(n + 1)
\end{aligned}
$$

Now, $n - 1$, n and $n + 1$ are three consecutive integers and hence at least one is divisible by 2 and at least one is divisible by 3.

Therefore $(n - 1)n(n + 1)$ is divisible by 6.

Example 3



Let P be a point inside a circle different from the centre O. Let AB be a chord containing point P. Prove that the shortest chord AB passing through P is such that P is the mid-point of the chord AB.

Proof

Let PA = a and PB = b

Then since PA x PB is constant regardless of the positions of points A and B then

$a \cdot b \equiv k$ (a constant) (Incidentally k is called the power of point P) We wish to minimize

$\ell = a + b = a + \dfrac{k}{a}$

Then $\dfrac{d\ell}{da} = 1 - \dfrac{k}{a^2}$

When ℓ is minimum, $\dfrac{d\ell}{da} = 0$ i.e. $a^2 = k$

but $a \cdot b = k$

$\therefore\ a = b$

and hence P bisects the chord

(incidentally OP will be perpendicular to AB)

Example 4

To prove $\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$

Proof

We will use matrix multiplication

A rotation of θ in R^2 is represented by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider the operations rotating by θ followed by rotating by α. Clearly this is equivalent to rotating by $(\theta + \alpha)$.

Hence

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) \\ \sin(\theta+\alpha) & \cos(\theta+\alpha) \end{bmatrix}$$

$$\therefore \begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha & * \\ * & * \end{bmatrix} = \begin{bmatrix} \cos(\theta+\alpha) & * \\ * & * \end{bmatrix}$$

$\therefore$ $\cos\theta\cos\alpha - \sin\theta\sin\alpha = \cos(\theta + \alpha)$ as required.

Example 5

To prove that if z is a root of a polynomial equation (with real co-effiicents) then $\bar{z}$ (z conjugate) is also a root of the polynomial equation.

Note that, as an example of this theorem, $x^3 - 3x^2 + 4x - 2 = 0$ has roots of 1, 1 + i, 1 – i.

To prove this theorem we need to show the truth of some lemmas; a lemma is a statement whose truth needs to be shown prior to proving a theorem.

Lemma 1

$(\bar{z})^2 = (\overline{z^2})$

Let $z = a + bi$

Then $\bar{z} = a - bi$

and $(\overline{z}^2) = a^2 - 2abi + b^2i^2$

$= a^2 - b^2 - 2abi$

$= \overline{a^2 - b^2 + 2abi}$

$= (\overline{z}^2)$

By extension it follows that $(\overline{z})^n = (\overline{z^n})$

Lemma 2

$c \bullet \overline{z} = \overline{c \bullet z}$ where c is a constant.

As before let $z = a + bi$

Then $c\overline{z} = c(a - bi)$

$= ca - cbi$

$= \overline{ca + cbi}$

$= \overline{cz}$

Lemma 3

$\overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}$

Let $z_1 = a + bi$ and let $z_2 = c + di$

Then $\overline{z}_1 + \overline{z}_2 = a - bi + c - di$

$= a + c - (b + d)i$

$= \overline{a + c + (b + d)i}$

$= \overline{z_1 + z_2}$

Now let's consider our theorem

Let x be a root of $a_n x^n + a_{n-1}x^{n-1} + ...a_0 = 0$

Then $a_n z^n + a_{n-1}z^{n-1} + ...a_0 = 0*$

Consider

$a_n \overline{z}^n + a_{n-1}\overline{z}^{n-1} + ...a_0 = 0$

Note that if we can show that this expression does equal zero then we have completed the proof because it follows that $\overline{z}$ must be a root also.

$a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \ldots a_0$

$= a_n(\overline{z^n}) + a_{n-1}(\overline{z^{n-1}}) + \ldots a_0$ (by Lemma 1)

$= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \ldots \overline{a_0}$ (by Lemma 2)

$\overline{a_n z^n + a_{n-1} z^{n-1} + \ldots a_0}$ (by Lemma 3)

$= \overline{0}$ from $*$

$= 0$

$\therefore \overline{z}$ is a root of the polynomial equation.

Example 6

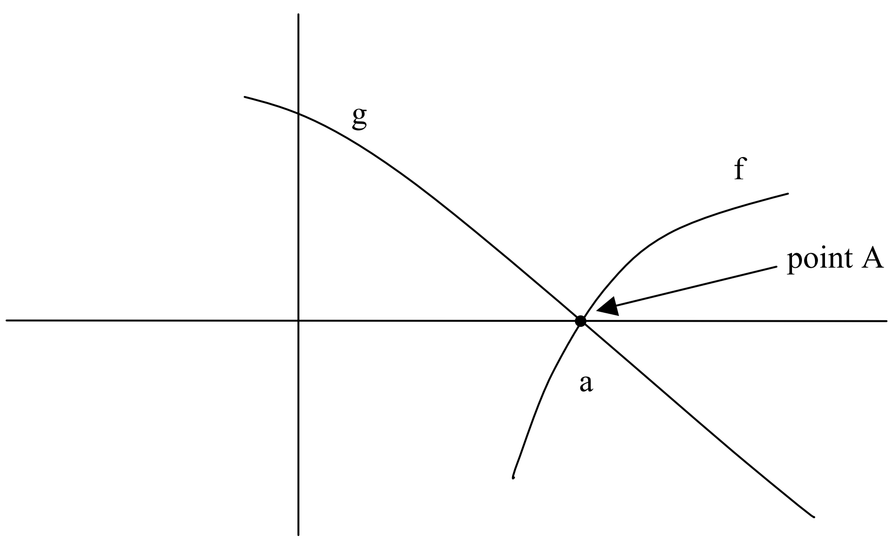
In calculus when studying limits a well known result called L'Hôpital's Rule states that if f(a) = g(a) = 0 then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ (assuming f and g are smooth continuous functions)

[For example $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x^2 - 9} = \lim_{x \to 3} \frac{\frac{1}{2\sqrt{x+1}}}{2x} = \frac{1}{24}$]

A proof attributable to Bernoulli is shown below.

Let f and g be functions where f(a) = g(a) = 0



Consider $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ where f(a) = g(a) = 0

In the limiting case as x $\longrightarrow$ a we can consider f and g to be linear functions using the tangent approximation idea.

Then the equation of the tangent at point A to the f function is y = f '(a)(x – a)

Similarly the equation of the tangent at point A to the g function is y = g`(a)(x – a)

$$\text{Then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f`(a)(x-a)}{g'(a)(x-a)} = \lim_{x \to a} \frac{f`(a)}{g`(a)}$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

□

It is true also that L'Hôpital's Rule applies when f(a) = g(a) = ∞

A proof of this case follows using the $\dfrac{0}{0}$ case as a Lemma.

Consider $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ where f(a) = g(a) = ∞.

Call the value of the limit L

Define $F(x) = \dfrac{1}{f(x)}$ and $G(x) = \dfrac{1}{g(x)}$ and hence F(a) = G(a) = 0.

$$\text{Then } L = \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{G(x)}{F(x)} = \lim_{x \to a} \frac{G'(x)}{F'(x)} \quad \text{(using the lemma)}$$

$$= \lim_{x \to a} \frac{(-1)[g(x)]^{-2} g'(x)}{(-1)[f(x)]^{-2} f'(x)} = \lim_{x \to a} \frac{[f(x)]}{[g(x)]}^{2} \frac{g'(x)}{f'(x)}$$

$$= L^2 \lim_{x \to a} \frac{g'(x)}{f'(x)}$$

$$\text{i.e. } L = L^2 \lim_{x \to a} \frac{g'(x)}{f'(x)}$$

$$\therefore \frac{1}{L} = \lim_{x \to a} \frac{g'(x)}{f'(x)}$$

$$\therefore L = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{as required}$$

Ex 11.1

1. Prove $n^4 + 6n^3 + 11n^2 + 6n$ is divisible by 24.
2. Find a counter-example to the conjecture that $n^2 + n + 41$ is always a prime number for any positive integer n.

3. Find the value of $\lim_{x\to\infty} x(2^{\frac{1}{x}} - 1)$

4.Using any method prove that the angle in a semi-circle is 90°

5. Given the quadratic polynomial equation

$x^4 - ax^3 + bx^2 - cx + d = 0$ has **ONE** root of multiplicity 4, prove that bc = 6ad.

6. Find the error in the following "proof"

$$\frac{1}{2} > \frac{1}{4}$$

$$\therefore \frac{1}{2} > \left(\frac{1}{2}\right)^2$$

$$\therefore \ln\frac{1}{2} > \ln\left(\frac{1}{2}\right)^2$$

$$\therefore \ln\frac{1}{2} > 2\ln\left(\frac{1}{2}\right)$$

$$\therefore 1 > 2$$

7. Given the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, ….

Where each term is the sum of the two previous terms i.e. $t_n = t_{n-1} + t_{n-2}$ Prove that

$$\lim_{n\to\infty} \frac{t_n}{t_{n-1}} = \frac{1+\sqrt{5}}{2} \text{ (the Golden Mean)}$$

8. Find a) $\lim_{x\to\infty} \frac{\sqrt{x^2 + 2x}}{\sqrt{x^2 - 2x}}$

b) $\lim_{x\to\infty} \sqrt{x^2 + 2x} - \sqrt{x^2 - 2x}$

9. Prove that if p is an odd integer and q is an odd integer that p + q + pq is also odd.

10. Prove that if we are given 10 different integers each below 100 then there always exists two sets of numbers chosen from these 10 which have the same sum.

(Answer proof is given below)

Exercise 11.1 Answers

2. n = 41
3. ln 2
6. $\ln\frac{1}{2}$ is negative
8. a) 1 b) 2
10. There exists 2^{10} (= 1024) subsets of this set of numbers. If each of these subsets had a different sum then that means there exists 1024 **different** sums. (i.e. numbers) **below** the sum of **all** the ten numbers in the set. This is false since the sum of all numbers in the set is less than 1000.

Contra-Positive Proof

A contra-positive proof involves the idea of assuming that the required result is not true and showing that that assumption leads to a contradiction hence negating the assumption and establishing the validity of the theorem.

In logical terms it can perhaps be explained by the following

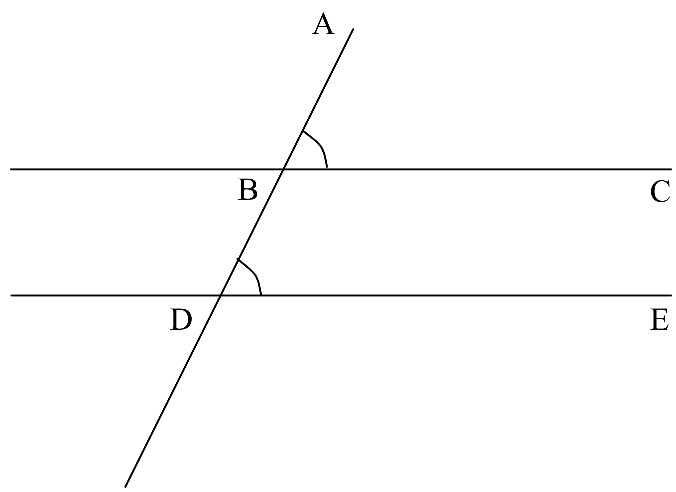
"If it is Sunday then the Post-Office is closed." Is completely equivalent to the implication

"If the Post-office is open then it is not Sunday"

e.g. "p" implies "q" is equivalent to "not q" implies "not p"

Example 7

Theorem

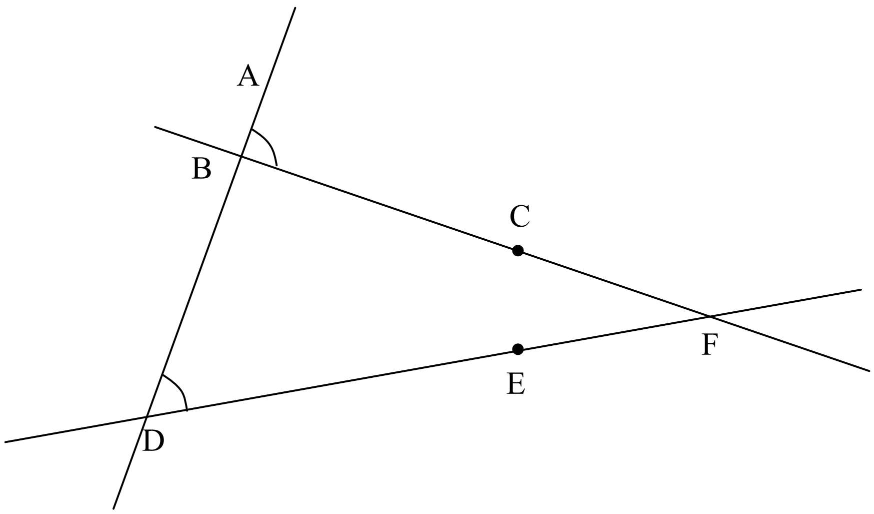


Given

$\angle ABC = \angle BDE$ prove that BC is parallel to DE.

Proof (by Contra-Positive)

Assume BC is not parallel to DE then BC intersects DE at some point (say) F. Without loss of generality let's say BC and DE intersect as shown.



Now $\angle ABC$ is an exterior angle of ΔBDF and hence $\angle ABC = \angle BDE + \angle F$ contradicting the given fact since $\angle F$ is clearly not zero degrees.

Hence the assumption is false and BC **is** parallel to DE.

Example 8

To prove $\sqrt{2}$ is irrational

Proof (by contradiction)

Assume $\sqrt{2}$ is rational.

Then $\sqrt{2} = \frac{a}{b}$ for some integers a and b

i.e. $2 = \frac{a^2}{b^2}$

and $2b^2 = a^2$ *

Now let's consider the prime factorisation of each side of this equation.
For example if 2 divides into b, then 2^2 divides into b^2 and hence the number of factors of 2 which do divide into b^2 must be even.

By reference to * the number of factors of 2 in the left-hand side is ODD and the number of factors of 2 in the right-hand side is **EVEN**.

This is a contradiction, hence our assumption is false and hence $\sqrt{2}$ is irrational.

Example 9

Theorem

The number of prime numbers is infinite.

Proof (by contra-positive)

Assume that the number of prime numbers is **finite**. Therefore there exists a **largest** prime number. Call it P.

Now consider the number N where

$N = (P_1)(P_2)(P_3)\ldots..(P) + 1$

Where P_1 is the first prime number, P_2 is the second number etc. i.e. N is 1 more than the product of **all** prime numbers. Clearly N is larger than P and hence N cannot be a prime number. Therefore N has some factor (say) P_i where P_i is not 1.
Then $N = k P_i$ for some integer k.
$\therefore\ k P_i = (P_1)(P_2)(P_3)\ldots..(P_i)\ldots. (P) + 1$
$\therefore\ k P_i - (P_1)(P_2)(P_3)\ldots..(P_i)\ldots. (P) = 1$
$\therefore\ P_i$ is a factor of the left-hand side and clearly P_i is not a factor of 1 on the right-hand side.
Contradiction.

Hence the assumption is false and there is an infinite number of primes.

Example 10

Given a set $\underline{\overline{X}}$ then P($\underline{\overline{X}}$) is defined to be the set of all subsets of $\underline{\overline{X}}$

e.g. If $\underline{\overline{X}}$ = {4, Susan, tree}

then P($\underline{\overline{X}}$) = {0, {4}, {Susan}, {tree}, {4, Susan}, {4, tree}, {Susan, tree}, {4, Susan, tree}}

WE have seen earlier that if $\underline{\overline{X}}$ has n elements then P($\underline{\overline{X}}$) has 2^n elements.

Theorem

The number of elements in P($\underline{\overline{X}}$) is always **larger** than the number of elements in $\underline{\overline{X}}$.

This theorem seems self-evident since surely 2^n is larger than n for all finite integers n.

But suppose $\underline{\overline{X}}$ is an infinite set. What happens for example if $\underline{\overline{X}}$ is the set of all real numbers? This clearly is not a high-school problem and yet the proof can be understood by able high-school students.

To establish that two sets have an equal number of elements, even when the sets are infinite, we say that if we can establish a 1 – 1 correspondence between the sets then the sets have the same number of elements.

For example the set of all positive even integers and the set of all positive odd integers have an equal number of elements as shown below.

odd		even
1	$\longleftrightarrow$	2
3	$\longleftrightarrow$	4
5	$\longleftrightarrow$	6
7	$\longleftrightarrow$	8
9	$\longleftrightarrow$	10
.		.
.		.
.		.

A 1 – 1 function mapping an odd integer to an even integer is $f(n) = n + 1$.
Let's return to our theorem.

Proof (by contra-positive)

Assume there exists some function f which is a 1 – 1 mapping $\underline{\overline{X}}$ to $P(\underline{\overline{X}})$

[e.g. if $\underline{\overline{X}} = \{1,2,3\}$ then $P(\underline{\overline{X}}) = \{\{0\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ then possibly $f(1) = \{1,2\}$ (say) and $f(2) = \{0\}$]

Consider the set $A = \{x \in \underline{\overline{X}}$ such that $x \notin f(x)\}$

For the example above $2 \in A$ but $1 \notin A$.

Now A is a subset of $\underline{\overline{X}}$ and therefore there must exist some $y \in \underline{\overline{X}}$ so that $f(y) = A$.

Now let's consider whether $y \in A$.

Case 1

If $y \in A$

Then $y \in f(y)$

$\therefore$ by definition of A, $y \notin A$

Case 2

If $y \notin A$ then by definition of A

$y \in A$

Clearly these are contradictions and hence our own assumptions is false and hence no 1-1 correspondence exists.

$\therefore$ clearly the # of elements in $P(\underline{\overline{X}})$ is larger than the # of elements in $\underline{\overline{X}}$

<u>Exercise 11.2</u> (All questions can be proven using a contra-positive method)

1. Prove that the shortest distance from a point to a line is the perpendicular distance.
2. Given four points A,B,C,D (in that order) in a circle it is well known that $\angle ABD = \angle ACD$ (inscribed angles in a circle). Prove the converse of this theorem.
3. Prove that no integer value of m exists such that $(2m - 1)$ and $(4m^2 - 4m + 2)$ are both perfect squares.
4. Prove that if $\vec{v}$ in R^2 is expressed as a linear combination of vectors $\vec{i}$, $\vec{j}$ then that representation is unique.
5. Prove that if A is a 2 x 2 matrix such that detA = 2 then there is only **<u>one</u>** vector which A maps to $(\overrightarrow{3,4})$,
6. Three boxes A,B,C are coloured red, white and blue not necessarily respectively. Of the following statements only **<u>ONE</u>** is true

   1. A is red

   2. B is not red

   3. C is not blue.

What colour is each box?

7. a) Prove $\sqrt{3}$ is irrational.

   b) Find where the "proof" of $\sqrt{4}$ is irrational breaks down in comparison with a)
8. Prove that if A,B,C are 2 x 2 matrices and AB = BC and $B \neq C$ then det A = 0.

Mathematical Induction

Induction is a method of proof used where it is required to show that a statement is true for a countably infinite number of values.

For example it could be used to prove

$1 + 2 + 3 + \ldots. + n = \frac{1}{2}n(n + 1)$ where n is any positive integer.

The method of proof depends upon the idea of showing FIRSTLY that the statement is true for some small value of the variable (usually 1). SECONDLY we show that **if** we assume the statement is true for some arbitrarily chosen (but fixed) value of the variable then it will be true for the next larger value of the variable.

This concept can be compared to saying that we can climb up an infinite ladder if

1. we can get on the first rung and
2. we are capable of going from any rung to the next higher rung.

Example 11

Prove that $1 + 3 + 5 + \ldots. (2n - 1) = n^2$ for all positive integers n.

Proof

FIRST when $n = 1$, L.H.S. = 1 and R.H.S. = $1^2 = 1$

$\therefore$ statement is true when $n = 1$.

SECONDLY

Assume that the statement is true when $n = k$. (k is randomly chosen but is a fixed value.)

Then $1 + 3 + 5 + \ldots (2k - 1) = k^2$.

It suffices now to show, on the basis of the assumption stated above, that

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

Proof:

L.H.S. $= 1 + 3 + 5 + \ldots (2k - 1) + (2k + 1)$

$= [1 + 3 + 5 + \ldots + (2k - 1)] + 2k + 1$

by Assumption

$= k^2 + 2k + 1$

$= k^2 + 2k + 1$

$= (k + 1)^2$ as required.

$\therefore$ statement is true for **all** positive integers n.

Example 12

The number of diagonals of a polygon with n sides is $\frac{n(n-3)}{2}$.

Proof

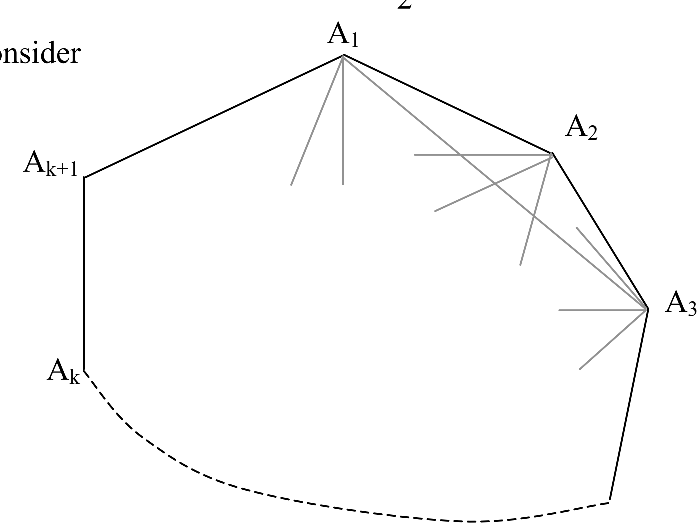
FIRSTLY, when n = 4 we have a quadrilateral which has 2 diagonals. Note that $\frac{4(4-3)}{2}$ does equal 2.

Therefore the statement is true when n = 4.

SECONDLY,

Assume that a k-gon has $\frac{k(k-3)}{2}$ diagonals. It suffices to show on the assumption above that a k + 1-gon has $\frac{(k+1)(k-2)}{2}$ diagonals.

Consider



In the k-gon $A_1A_2\ldots A_k$ by Assumption are $\frac{k(k-3)}{2}$ diagonals. In the k+1-gon $A_1A_2\ldots A_kA_{k+1}$ there are the diagonals described above plus all the diagonals emanating from the A_{k+1} i.e. k-2 diagonals plus the new diagonal A_1A_k which was originally an edge of the k-gon.

The number of diagonals in the k+1-gon is therefore $\frac{k(k-3)}{2}+(k-2)+1$

$$=\frac{k^2-3k+2k-4+2}{2}$$
$$=\frac{k^2-k-2}{2}$$
$$=\frac{(k+1)(k-2)}{2} \quad \text{as required}$$

<u>Example 13</u>

To prove de Moivre's Theorem by Induction.

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

FIRSTLY

When n = 1 clearly the theorem is true

SECONDLY

Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Show, on the basis of this assumption, that

$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\,\theta + i\sin(k+1)\,\theta$

L.H.S. $= (\cos\theta + i\sin\theta)^{k+1}$

$=(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

by Assumption

$= (\cos k\theta + i\sin k\theta)\,(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i^2\sin k\theta\sin\theta + \sin k\theta\cos\theta + i\cos k\theta\sin\theta$

$= \cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)$

$= \cos(k + 1)\,\theta + i\sin(k + 1)\,\theta$

□

This completes the proof of DeMoivre's Theorem.

Example 14

Given the Fibonacci sequence 1,1,2,3,5,8,13, …. Where $t_{n+2} = t_{n+1} + t_n$

Show that $S_n = t_{n+2} - 1$ where S_n means the sum of the first n terms.

FIRSTLY

When n = 4, $S_4 = 1 + 1 + 2 + 3 = 7$ and $t_6 = 8$

hence the theorem is true when n = 4.

SECONDLY

Assume $S_k = t_{k+2} - 1$

We need to show on the basis of this assumption that

$S_{k+1} = t_{k+3} - 1$

$S_{k+1} = S_k + t_{k+1}$ by definition of a sum.

by Assumption

$= t_{k+2} - 1 + t_{k+1}$

$= t_{k+1} + t_{k+2} - 1$

$= t_{k+3} - 1$ by definition of the Fibonacci Sequence

Exercise 11.3

1. Prove that 1 + 5 + 9 + …. (4n – 3) = n(2n – 1)
2. Prove by induction that $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
3. Prove that the number of lines determined by n points is $\frac{1}{2}n(n-1)$. (Assume that no three points are collinear). How many points of intersection are determined by n random lines?
4. Prove by induction that $\dfrac{1}{1\times 3} + \dfrac{1}{3\times 5} + \dfrac{1}{5\times 7} + \ldots \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$
5. Prove that the sum of the interior angles of a polygon is (n-2)180°.

6. Prove by induction or otherwise that $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}\ldots\binom{n}{n}=2^n$

7. Prove by induction that $n(n^2 + 5)$ is divisible by 6 for all positive integer values of n.

8. Prove by mathematical induction that the sum of the cubes of any three consecutive natural numbers is divisible by 9.

9. A motorist estimates that, by traveling along a main road at a certain steady speed, the probability that the next set of traffic lights will be green if the last set was green is $\frac{3}{4}$ and the probability that the next set of lights will be green if the last set was red is $\frac{1}{2}$. He sets out one day to test his theory. Prove that, if the first light he meets is green then the probability that the nth set of lights is green is

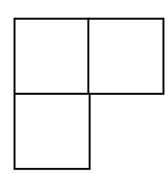
$$\frac{2}{3}+\frac{1}{3}\left(\frac{1}{4}\right)^{n-1}$$

10. Let $f(x) = \frac{x}{1-x}$. Define $f_2(x) = f(f(x))$, $f_3(x) = f(f(f(x)))$ etc for all integers. Prove by induction that $f_n(x) = \frac{x}{1-nx}$ for all positive integers n.

11. Show by Induction or otherwise that 3 is a factor of n3 - n for all positive integers n.

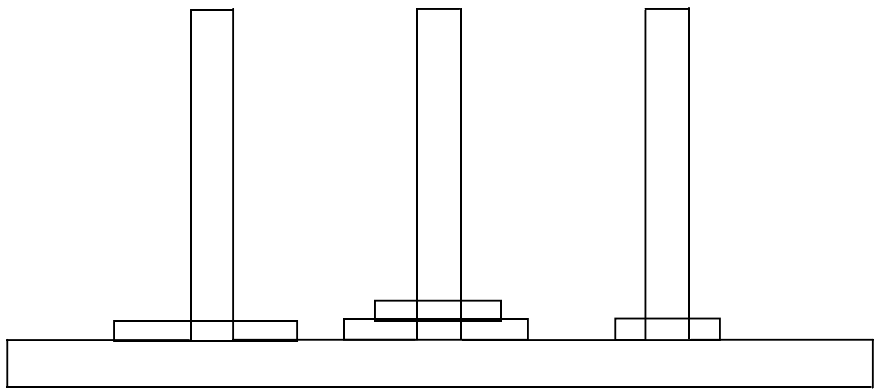
12. Prove $\sum_{i=r}^{n}\binom{i}{r}=\binom{n+1}{r+1}$ by Induction. Use Pascal's Theorem $\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$.

13. One square is deleted from a square checkerboard with 2^{2n} squares. Show that the remaining $2^{2n} - 1$ squares can always be tiled with shapes of the form which cover three squares.



14. The Towers of Hanoi is a classical ancient puzzle dating from, at least 1883. It consists of a board with three vertical pegs and a number of discs of a differing diameter each with a central hole which allows them to be threaded on to one of

the pegs. Initially the discs are all threaded on to one peg so that a disc with smaller diameter is always placed on top of a disc of larger diameter. The object of the puzzle is to transfer all the discs to another peg by a succession of moves. A move is defined as transferring a disc from one peg to another peg so that pegs with small diameter are always on top of discs with larger diameter (see diagram)



Use the method of induction to prove that if the puzzle has n discs then the minimum number of moves required to transfer all the discs from one peg to another is $2^n - 1$

15. Find the smallest value of n, $n \in N$, for which $10^n < n!$. Use the method of induction to prove that $n! > 10^n$ for all integers equal to and greater than this value of n.
16. Prove by induction that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ [Hint: you will need to use the product rule for differentiation]
17. If $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ find M^2, M^3, and M^4 and suggest a value for M^n. Use induction to test the validity of your suggestion.
18. Find the number of intersections when:

    a) 2 b) 3 c) 4 non parallel, non-trisecting lines are drawn in a plane. Suggest a formula for P_n, the number of points of intersection of n non-parallel, non-trisecting lines in the plane. Use the method of induction to test the validity of your conjecture.

19. $P = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, a, b, $\in \Re$ and $a \neq 1$. Evaluate P^2, P^3, P^4 …. And suggest a formula for P^n in terms of a, b, and n. Test the validity of your formula using induction.
20. What is wrong with the following "proof"

    To prove $2^{n-1} = 1$ for all positive integers n.

    Firstly when n = 1, $2^{1-1} = 2^0 = 1$ $\therefore$ Statement is true when n = 1

    Secondly on the assumption $2^{k-1} = 1$ prove $2^k = 1$

    $$2^k = \frac{2^{k-1} \cdot 2^{k-1}}{2^{k-2}} = \frac{1 \cdot 1}{1} = 1$$
21. A sequence of integers $t_0, t_1, t_2, \ldots$ is defined by $t_0 = 0$, $t_1 = 1$, $t_2 = 4$, and $t_n = 3t_{n-1} - 3t_{n-2} + t_{n-3}$ for $n \geq 3$. Prove that $t_n = n^2$ for all n.
22. Suppose $t_n = 15t_{n-1} - 75t_{n-2} + 1253t_{n-3}$ for $n \geq 3$, and $t_0 = 1$, $t_1 = 5$, $t_2 = 35$. Prove $t_n = 5^n$ for all n.

Vector Proof

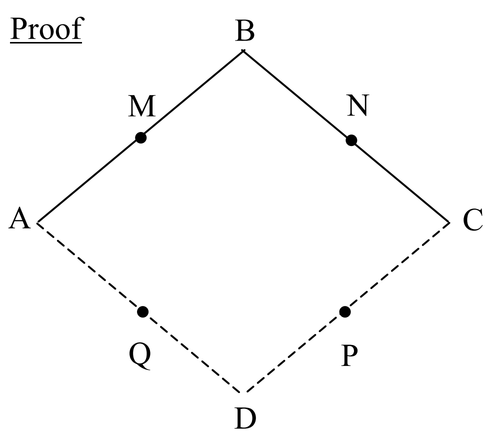
In geometry and occasionally elsewhere it is helpful to use vectors to prove theorems.

Example 15

Theorem

Given A,B,C,D are four points (not necessarily co-planar) forming a figure ABCD with M, N, P, Q as the mid-points of AB, BC, CD and DA respectively, prove that MNPQ is a parallelogram.

Proof



In ΔABC, $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC}$

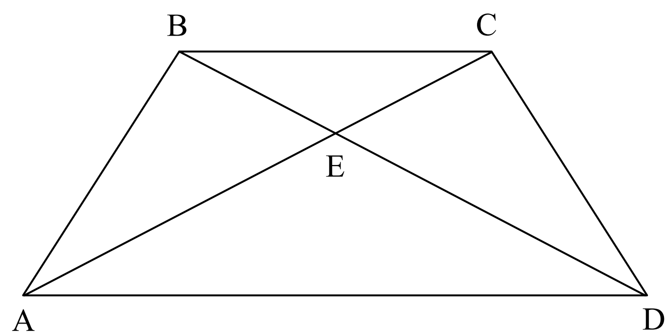
In ΔADC, $\overrightarrow{QP} = \frac{1}{2}\overrightarrow{AC}$

$\therefore \overrightarrow{MN} = \overrightarrow{QP}$

$\therefore$ MNPQ is a parallelogram.

Example 16

Prove that the diagonals AB and BD of trapezoid ABCE intersect each other in the same ratio at point E.

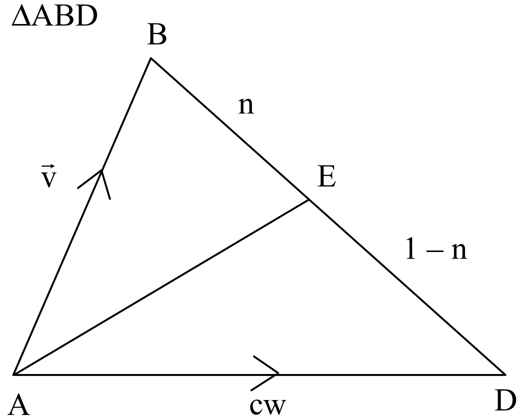


Let $\overrightarrow{AB} = \vec{v}$ and $\overrightarrow{BC} = \vec{w}$. Then $\overrightarrow{AD} = c\vec{w}$ for some scalar c.

Let BE : ED = n:1 – n

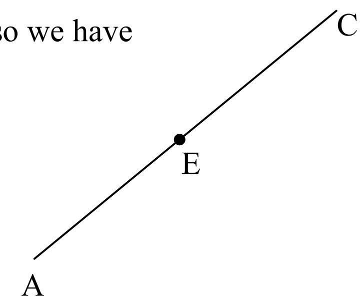
Let CE : EA = m : 1 – m

In ΔABD



$\overrightarrow{AE} = nc\vec{w} + (1-n)\vec{v}$ ①

Also we have



so that $\overrightarrow{AE} = (1-m)\overrightarrow{AC}$

$\therefore \overrightarrow{AE} = (1-m)(\overrightarrow{AB} + \overrightarrow{BC})$

$\overrightarrow{AE} = (1-m)(\vec{v} + \vec{w})$ ②

Comparing ① and ② leads to the equation

$(1-n)\,\vec{v} + nc\vec{w} = (1-m)\vec{v} + (1-m)\vec{w}$

But $\vec{v}, \vec{w}$ is an independent set of vectors and a linear combination of them is unique.

$\therefore 1-n = 1-m$ scalars of $\vec{v}$

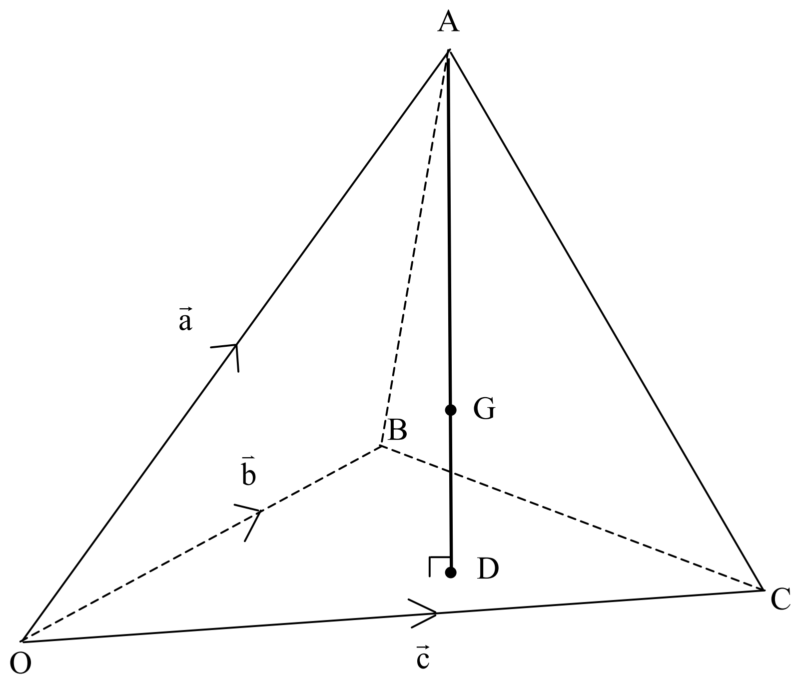
$\therefore v = m$

$\therefore$ AC and BD intersect each other in the same ratio.

Example 17

To prove that the altitudes of a tetrahedron do not, in general, intersect.

Proof (by contra-positive argument)



$\overrightarrow{OA} = \vec{a}$

Let $\overrightarrow{OB} = \vec{b}$

$\overrightarrow{OC} = \vec{c}$

Consider the altitude from A to the plane of triangle OBC.

Then $\overrightarrow{AD}$ is a normal of plane OBC. i.e. $\overrightarrow{AD} = s(\vec{b} \times \vec{c})$ for some scalar s.

Assumption Let G be the point on $\overrightarrow{AD}$ which is the common point of intersection of the four altitudes.

$$\begin{aligned}\text{Then } \overrightarrow{OG} &= \overrightarrow{OA} + \overrightarrow{AG} \\ &= \overrightarrow{OA} + r\overrightarrow{AD} \quad \text{for some scalar r} \\ &= \vec{a} + r(s(\vec{b} \times \vec{c})) \\ &= \vec{a} + rs(\vec{b} \times \vec{c}) \quad (1)\end{aligned}$$

mutatis mutandis

$$\overrightarrow{OG} = \overrightarrow{OB} + \overrightarrow{BG} = \vec{b} + mn(\vec{a} \times \vec{c}) \quad (2)$$

Comparing (1) and (2) we have

$$\vec{a} + rs(\vec{b} \times \vec{c}) = \vec{b} + mn(\vec{a} \times \vec{c})$$

$$\therefore \vec{a} - \vec{b} = mn(\vec{a} \times \vec{c}) - rs(\vec{b} \times \vec{c})$$

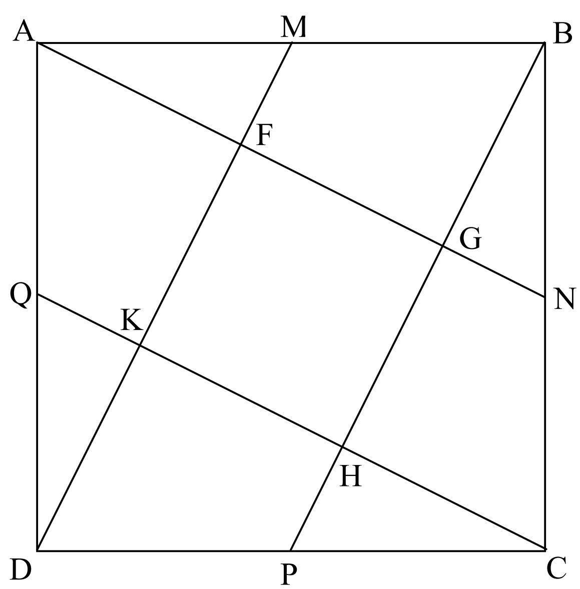
$$= (mn\vec{a} - rs\vec{b}) \times \vec{c}$$

Now note that this last statement is a contradiction since it implies that $\vec{a} - \vec{b}$ is perpendicular to $\vec{c}$ i.e. it implies that BA is perpendicular to OC which in general is not true.

Hence the assumption is false and the altitudes do not intersect.

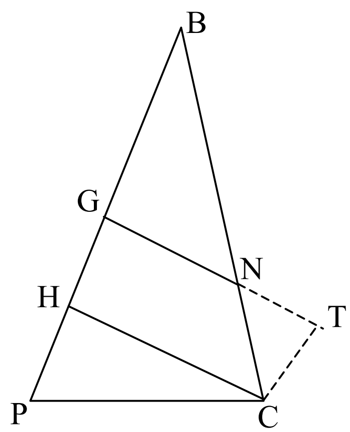
Example 18

Given square ABCD, each side length 1. Let M,N,P,Q be mid-points of AB, BC, CD, DA respectively. Join AN, BP, CQ, DM forming a quadrilateral (as shown) FGHK. Find area of FGHK.



An algebraic solution exists as follows

Consider



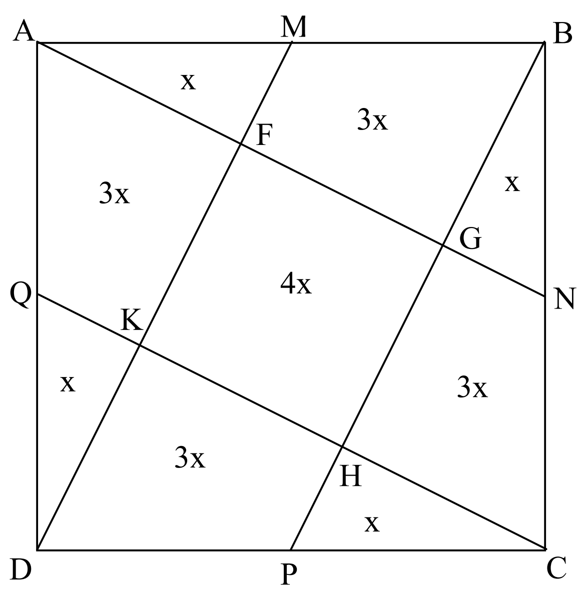
Extend GN and draw CT parallel to PB.

It is clear that $\Delta GBN \approx TCN$ and have same area.

It is also clear that Area GTCH = 4 times area ΔGBN

Let Area $\Delta GBN = x$ then Area BNCH = 3x

Continuing, by similarity, the diagram can be labeled in terms of area as follows



By Similarity it is clear that FGHK is a square

Area ABCD = 20x = 1

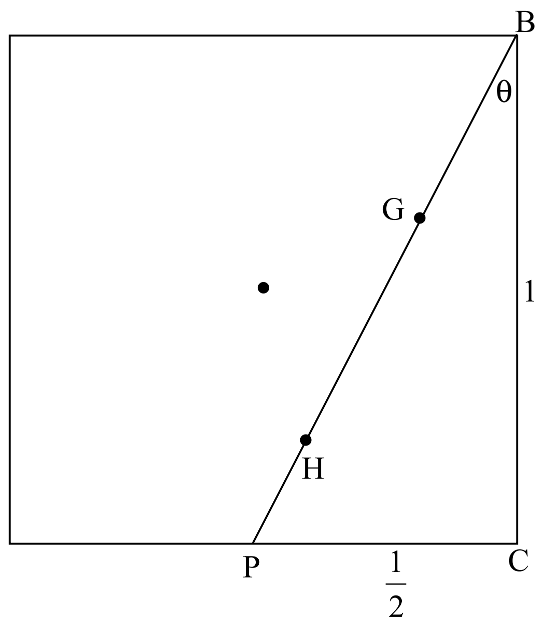
Area FGHK = 4x

$\therefore$ Area FGHK = $\frac{1}{5}$

Certain assumptions have been made in this solution but they are surely quite clear.

However, consider this Linear Algebra solution

Let center of square be 0 (0,0) and let $\angle$ PBC = θ



P is $(0, \frac{1}{2})$

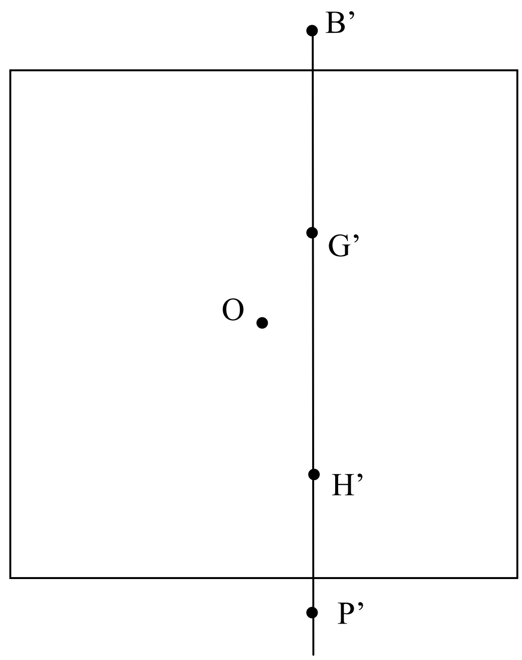
Let's rotate about O, $\theta°$ anti-clockwise

The matrix is $\begin{bmatrix} \text{cosè} & \text{- sinè} \\ \text{sinè} & \text{cosè} \end{bmatrix}$

But $\tan\theta = \frac{1}{2}$ $\therefore$ $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$

$\therefore$ Point P is moved to $\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} = (\frac{1}{2\sqrt{5}}, -\frac{1}{\sqrt{5}})$

This means B'G'H'P' is vertical but most importantly



The x co-ordinate of H' and G' is $\frac{1}{2\sqrt{5}}$

$\therefore$ The rotated square F'G'H'K' has dimensions $\frac{1}{\sqrt{5}}$ by $\frac{1}{\sqrt{5}}$ and hence its area is $\frac{1}{5}$.

Remember that rotations preserve area.

Example 19

To sovle (say) $x + y + 2z = 5$

$$x + 2y + z = 5$$

$$2x + y + z = 5$$

it is "clear" that , because of symmetry of the equations, the roles played by x, y and z are the same. Therefore, since there is a unique solution ($\mathbf{n_1} \bullet \mathbf{n_2} \times \mathbf{n_3} \neq 0$) then that solution must occur when $x = y = z$ i.e. $4x = 5$ and $x = y = z = \frac{5}{4}$.

Question:

Investigate whether it is true that if $x + y = 14$ and $x^2 + y^2 = 100$ then $x^2 + y^3 = y^2 + x^3$.

At first sight it seems that because the two equations are symmetric in x and y that $x = y$ and hence the "proof" is trivial but note that the system of equations

$$\begin{aligned} x + y &= 14 \\ x^2 + y^2 &= 100 \end{aligned}$$

does not have a unique solution and hence x is not necessarily equal to y.

In fact the solutions are (6,8) and (8,6). The two solutions are however symmetric in x and y.

It follows that $x^2 + y^3$ is not equal to $y^2 + x^3$ and hence the proof fails because clearly for example when $x = 6$ and $y = 8$, $6^2 + 8^2$ is not equal to $8^2 + 6^3$.

Exercise 11.4

All these proofs can be done using vector methods.

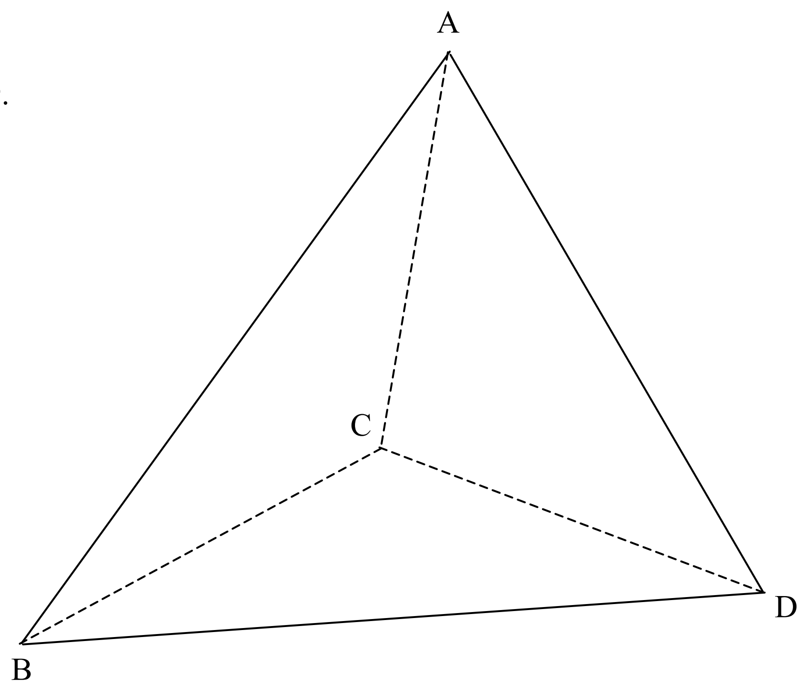
1. Prove that the angle in a semi-circle is 90°.
2. Prove that in ÄABC $\dfrac{(\overrightarrow{AB}\times\overrightarrow{AC})\times\overrightarrow{BC}}{\left|\overrightarrow{BC}\right|^2}$ is the vector $\overrightarrow{AD}$ where AD is an altitude of

   ÄABC
3. Prove the Cosine Rule using vectors
4. Prove that the diagonals of a parallelogram bisect each other.
5. Prove that if $\overrightarrow{AB} = \overrightarrow{DC}$ then $\overrightarrow{AD} = \overrightarrow{BC}$
6. Given A,B,C,D are 4 coplanar but not collinear points with M,N as mid-points of

   AC and BD respectively, prove $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD})$
7. Prove that the diagonals of a parallelepiped bisect each other at a common point. Remember that a parallelepiped is a figure whose 3 pairs of opposite faces are parallel parallelograms.

8. Given tetrahedron ABCD such that M,N,P,Q,R,S are the mid-points of edges AC, BC, CD, BD, AD, AB respectively prove that MQ, NR and SP bisect each other at a common point.

9.



ABCD is a regular tetrahedron i.e. each of its faces is an equilateral triangle.

Prove $\overrightarrow{AB} \bullet \overrightarrow{CD} = 0$

10. Prove that a parallelogram whose diagonals are perpendicular is a rhombus.

11. ABCD is a regular tetrahedron. E is the mid-point of AD and F is the mid-point of BC. Prove EF is perpendicular to AD and EF is perpendicular to BC.

12. In ÄOPQ let $\overrightarrow{OP} = \vec{x}$ and $\overrightarrow{OQ} = \vec{y}$.

Show that $\frac{|\vec{x}|}{|\vec{x}| + |\vec{y}|}\vec{y} + \frac{|\vec{y}|}{|\vec{x}| + |\vec{y}|}\vec{x}$ is the angle bisector of $\angle POQ$