CHAPTER SEVEN

7. Matrices As Transformations

y = 2x is a graph representative of a function, namely f(x) = 2x, i.e. the "doubling function". This function takes numbers and transforms them into twice their original value. Similarly the following are all functions transforming numbers into numbers.

$g(x) = x^2$, $h(x) = \sin x$, $j(x) = \log x$.

A matrix is a transformation (i.e. function) which changes vectors into vectors. For example, a matrix may change (**1,2,3**) to (**-3,4**).

It will also make perfectly good sense to think of a matrix as changing **points** to **points.** In fact this is the usual and probably most intuitive way of thinking of matrices. We say a matrix "maps" or "transforms" a set of points to a set of points. For example there exists a matrix which can map a circle into an ellipse or another matrix can map the whole of R^3 onto the set of points lying on y = 3x in R^3.

Thus a matrix is a function mapping from $R^n \longrightarrow R^m$.

A matrix is designated by a capital letter (say) A and is written thus $\begin{bmatrix} 2 & 1 & -3 \\ 6 & 2 & 1 \\ 4 & 8 & 5 \end{bmatrix}$.

It may have any number of rows or columns but this text will be restricted to discussion of matrices with at most three rows of columns. Each entry is called an element and for our text will always be a real number. An entry is identified by (say) a_{32} which means it is an element in matrix A in position row 3 and column 2. (i.e. 8 in the matrix above).

The dimensions of matrices are always referred to in the order Row before Column. For example, we say $\begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & -2 \end{bmatrix}$ is a "2 by 3" matrix meaning it has two rows and three columns.

The way in which a matrix 'operates' on a point (or vector) is different from that of elementary algebra.

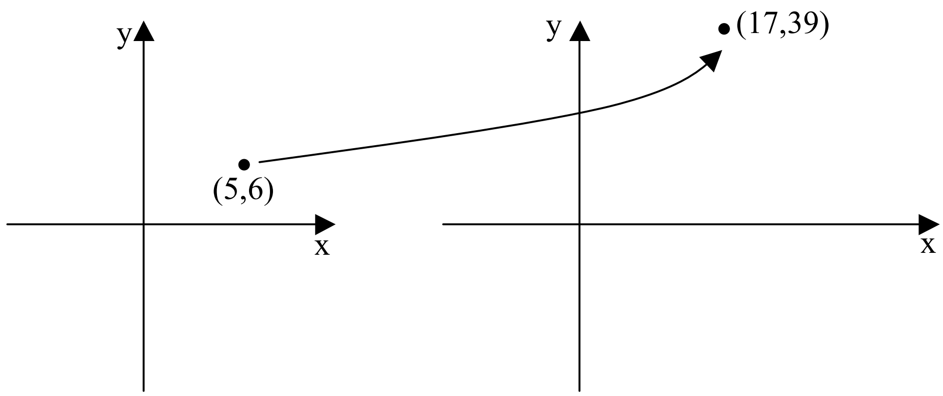
e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ operating on the vector (**5,6**) is written.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and the result (called the image) of this operation is obtained by

finding the dot product of the **first** row of the matrix with the vector, i.e. (**1,2**)•(**5,6**) and inserting that number in the **first** position of the image; similarly for the second row.

i.e. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\times 5 + 2\times 6 \\ 3\times 5 + 4\times 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$

This means that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ maps (5,6) to (17,39)



Whether you think of (5,6) and (17,39) as vectors or points does not matter essentially.

Example $\begin{bmatrix} -2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \end{bmatrix}$

In general $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix}$

It may seem trivial but an important idea is that

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e. the origin (or Zero Vector)

always remains as a fixed point for **any** matrix.

It is possible to multiply a matrix by a real number, as follows –

$$\text{e.g. } m\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ma & mb \\ mc & md \end{bmatrix}$$

i.e. every element is multiplied by the real number.

Also addition of two matrices of compatible dimensions is as one would expect.

$$\text{viz: } \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

In general A + B = C if $a_{ij} + b_{ij} = c_{ij}$ for all relevant i and j.

$$\text{i.e. } 3\begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 12 & 9 \end{bmatrix}$$

$$\text{and } \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 10 & 1 \end{bmatrix}$$

Exercise 7.1

1. Find a_{23} if A is $\begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{bmatrix}$
2. State the dimensions of A in question 1.
3. State the image of (0,0) for any matrix.
4. Find the image of (3,4) when "operated on" by $\begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix}$
5. Find the image of (a,b) when operated on by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   Hence deduce what function $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ represents.
6. Repeat Question 5 for $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
7. Find the image of (a,b) when operated by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Do the operation again. Try

   to guess what geometric operation the matrix represents.

8. i) Repeat Question 7 for $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   ii) Repeat Question 7 for $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Exercise 7.1 Answers

1. 6.
2. 3 x 3
3. (0,0)
4. (5,27)
5. (a,b). The identity function
6. (a,0). Projection onto x axis.
7. (-a,-b). Reflection about the origin.
8. i) (a,-b) Reflection about x axis.
   ii) (-b,a) Rotation of 90° anti-clockwise.

7.2 **Multiplication of Matrices**

Matrices may also multiply each other. Matrices are multiplied from left to right and in the multiplication of A•B = C, the dot product of the first row of A and the first column of B is entered in the first row **and** column of C. In general the dot product of the i^{th} row of A and the j^{th} column of B is entered in the position of the i^{th} row and j^{th} column of C, i.e. c_{ij}.

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\times5+2\times7 & 1\times6+2\times8 \\ 3\times5+4\times7 & 3\times6+4\times8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

If A•B = C then $\sum_{k=1}^{n} a_{ik} b_{kj} = c_{ij}$ where n = # of columns of A = # of rows of B (which have to be equal for multiplication to be possible).

Example

$$\begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 0 & 4 \end{bmatrix}$$

Multiplication of matrices is representative of the operation of one matrix followed by the operation of the other matrix.

For example, Exercise 7.1, in Question #7, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ represents a rotation of 180° about the origin. (Unless otherwise stated, rotation will always mean rotation about the origin).

If we repeat this operation we get $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a matrix which leaves points unchanged and is hence called the identity matrix.

It is denoted by I. This agrees with the notion of rotating 180° "followed by" rotating 180° since this clearly means rotating by 360°, i.e. "leaving unchanged".

It is important to understand that the order of operation matters. In the designation AB we multiply from left to right but from an operation point of view this means do **B** first followed by A because the point to be operated on appears immediately after B, thus AB$[\;]$. This also conforms with the idea of f(g(x)) meaning "do g first".

Example

A is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, B is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

A is a projection onto the x axis. (See Exercise 7.1, Question 6)

B is a rotation of 90° anticlockwise (a.c.) (See Exercise 7.1, 8ii)

Let point to be transformed be (3,4).

Now $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

and hence $AB\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

Note that this result is the same as B mapping (3,4) to (-4,3) and then A mapping (-4,3) to (-4,0).

i.e. (3,4) when rotated 90° a.c. first and then projected onto the x axis becomes (-4,0).

Check for yourself that the operations in reverse, i.e. (3,4) projected onto x axis and then rotated 90° (a.c.) becomes (0,3), i.e. a **different** point from (-4,0). Thinking of these operations geometrically make it clear.

Two matrices may only be multiplied if they are conformable, viz: A times B is possible fi the number of columns of A = number of rows of B.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ 26 & 6 \\ 0 & -1 \end{bmatrix}$ but $\begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 0 & 0 \end{bmatrix}$ is not defined.

In general if A is a m x n matrix and B is a n x p matrix then AB is a m x p matrix. It follows that 2 x 3 matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ maps (1,0,0) to (1,4). We say $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ is a function from $R^3 \longrightarrow R^2$. We call R^3 the **DOMAIN** and R^2 the **RANGE SPACE**.

Exercise 7.2

1. Multiply if possible –

   i) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

   ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

   iii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. If A is a 2 x 2 matrix such that A•A = I, does A = I? If so, prove it. If not, give two examples for A.
3. Write down the identity matrix mapping from R^3 to R^3.
4. Multiply $\begin{bmatrix} -1 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 2 & 4 \end{bmatrix}$
5. i) find the image of (**1,-2**) rotated 180°

   ii) Find the image (**1,-2**) rotated 90° a.c.
6. Try to find two matrices A and B such that AB = BA but A ≠ B neither A nor B = I. Do A and B have to be square matrices?
7. State the dimensions of C if AB = C and A is a 3 x 1 matrix and B is a 1 x 2 matrix.
8. State the Domain and Range Space of $\begin{bmatrix} -1 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix}$

9. i) Show that if A maps P and Q, distinct points in R^2, not collinear with the origin, to themselves then A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   ii) Try to find a matrix A, other than I, such that A maps both (1,1) and (2,2) to themselves.

Exercise 7.2 Answers

1. i) [14] ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ iii) not possible.

2. No. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 11 \\ 17 & 23 \end{bmatrix}$

5. i) (-1,2) ii)(2,1)

6. A = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Yes.

7. 3 x 2

8. Domain is R^3. Range Space is R^2.

9. ii) $\begin{bmatrix} a & 1-a \\ b & 1-b \end{bmatrix}$ for any a and b.

7.3 Simple Transformation in the Plane

There are many simple transformation in the plane and they are all common enough that each will be examined here.

They are:

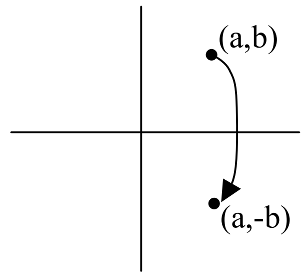
1. reflection
2. stretch
3. magnification
4. projection
5. shear
6. rotation
7. translation

Reflections

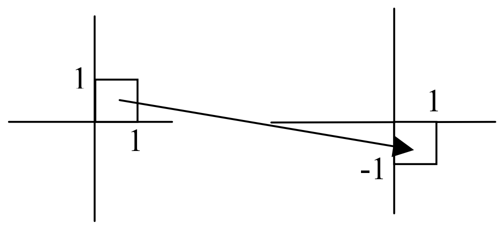
A reflection in the x axis

maps a point (a,b) to (a, -b)

and is represented by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

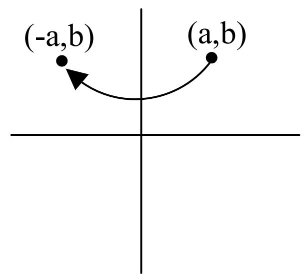


It maps the unit square
(i..e. square whose vertices
are (0,0) (1,0) (0,1) and (1,1))
to square whose vertices are
(0,0) (1,0) (0,-1) (1,-1)



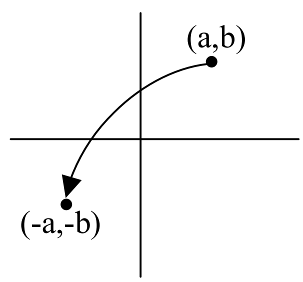
Similarly a reflection in the y axis

is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ mapping (a,b) to (-a,b).



A reflection in the origin is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

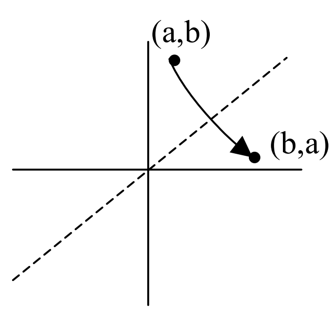
which is the same as a rotation of 180°



A reflection about the line y = x
means an interchange of first and
second co-ordinates

i.e. (a,b) is mapped to (b,a).

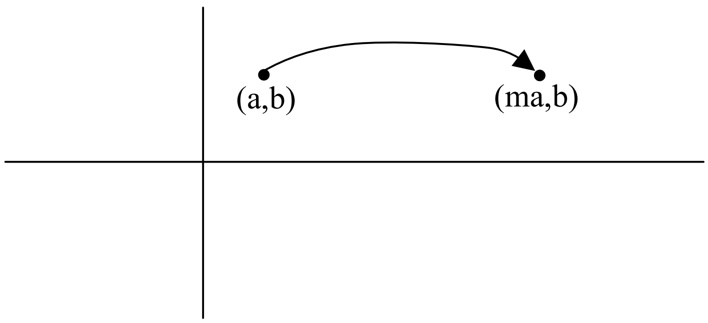
This is effected by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



The unit square is mapped to itself.

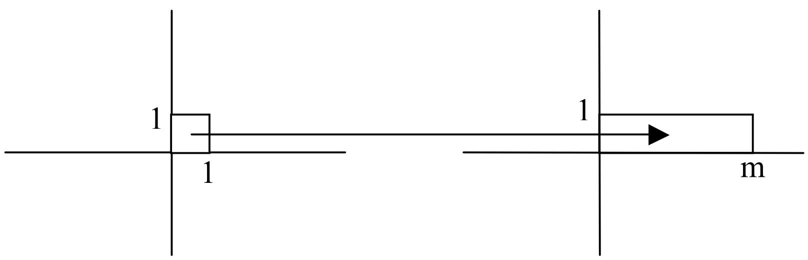
Stretch

A stretch in the x direction means (a,b) is mapped to (ma,b) where m is a constant.



The matrix representing this is $\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}$

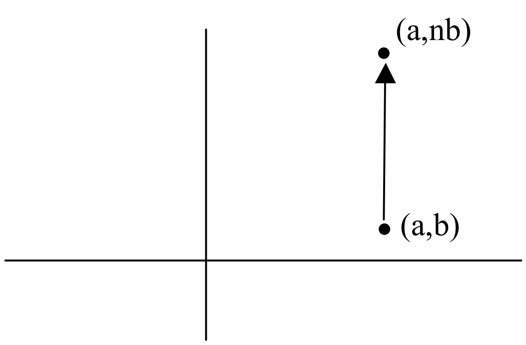
Note that $\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}$ would map the unit square as shown below.



Similarly a stretch in the y direction is:

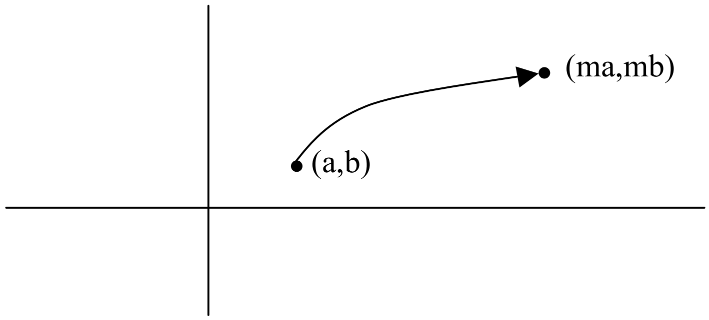
Where (a,b) is mapped to (a,nb).

The corresponding matrix for this mapping is $\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$

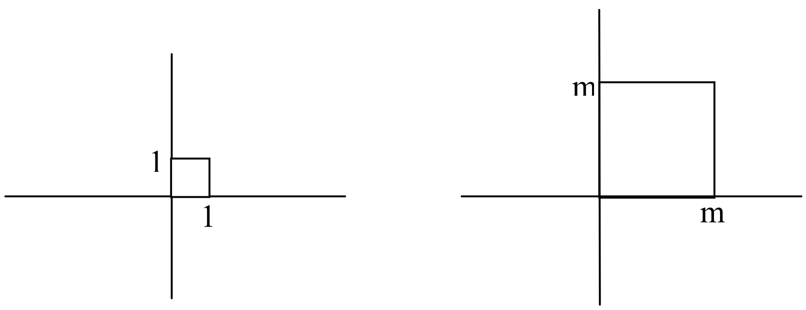


Magnification

A magnification is a stretch in the x and y direction by the same factor, i.e. (a,b) is mapped to (ma, mb)

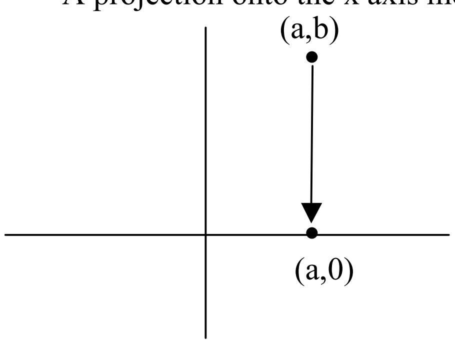


The matrix is $\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ and it changes the unit square as shown below: -



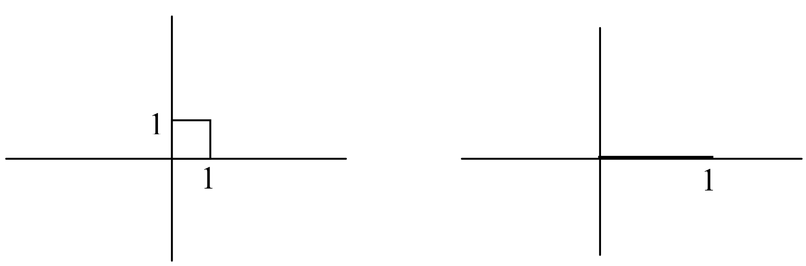
Projection

A projection onto the x axis means (a,b) is mapped to (a,0)

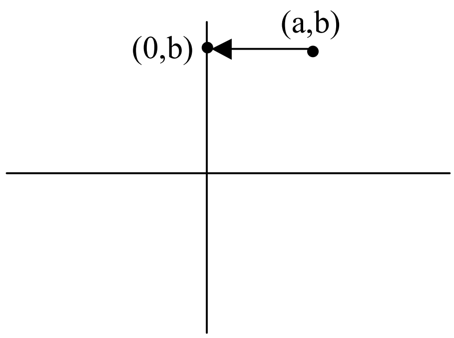


The matrix for this is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Note that this projection "squashes the unit square flat"



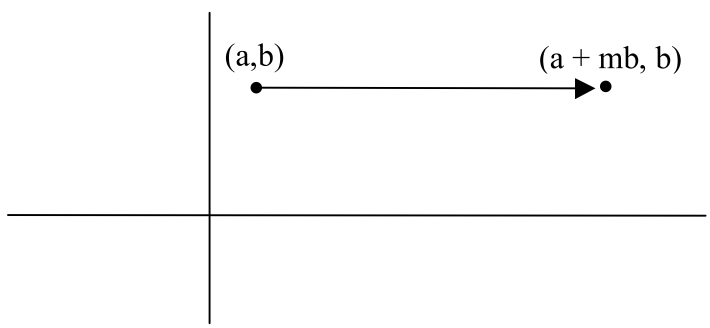
Similarly a projection onto the y axis means (a,b) is mapped to (0,b)



The matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and of course "squashes the unit square flat" onto the y-axis.

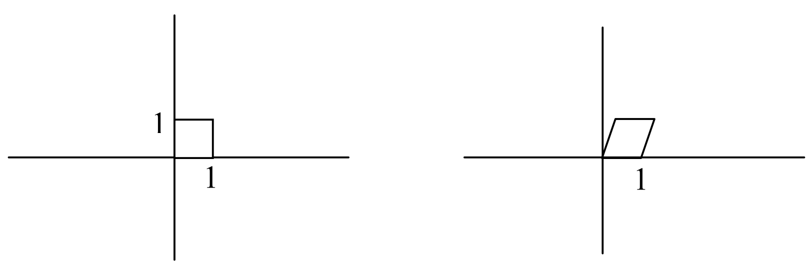
Shear

A shear in the x direction maps (a,b) to (a + mb, b)



The matrix for this is $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$

Note that a shear in the x direction maps the unit square as shown: -



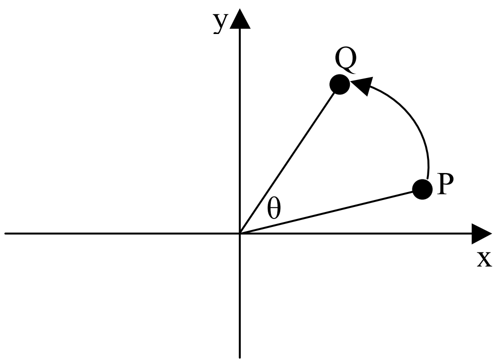
i.e. it distorts the shape but preserves the area. Similarly a shear in the y direction is

$\begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ where (a,b) is mapped to (a, ma + b).

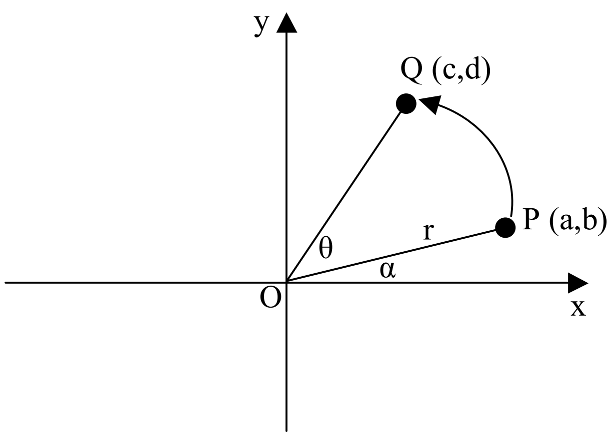
Rotation

To find the matrix representing a rotation of θ^o anti-clockwise.

A rotation, while easy to imagine visually, is a little more difficult to derive algebraically than the previous transformations. Hence we will develop it.



Let P be the point (a,b) and let the rotation of P through θ^o a.c. be Q. We wish to find co-ordinates of Q, i.e. (c,d).



Let S be any point on the x axis. Let $\angle$POS be α and OP = r.

Then a = r$\cos\alpha$ and b = r $\sin\alpha$

Since we are rotating about the origin OQ = r also.

i.e. $c = r\cos(\theta+\alpha)$

i.e. $c = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

i.e. $c = a\cos\theta - b\sin\theta$

Similarly, $d = r\sin(\theta+\alpha)$

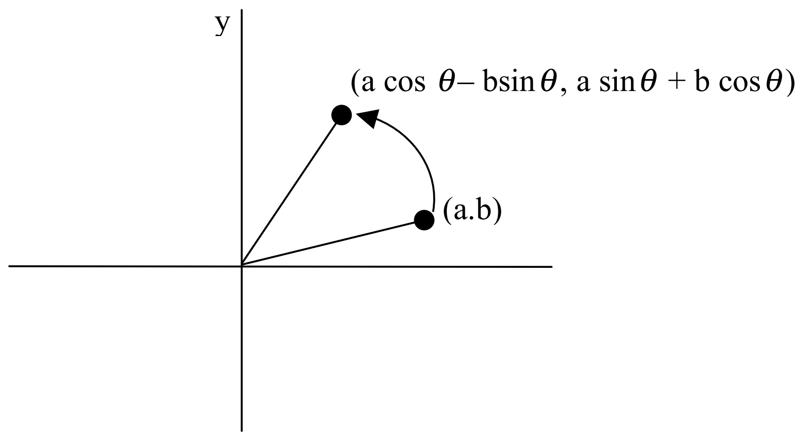
$= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$

$= a\sin\theta + b\cos\theta$

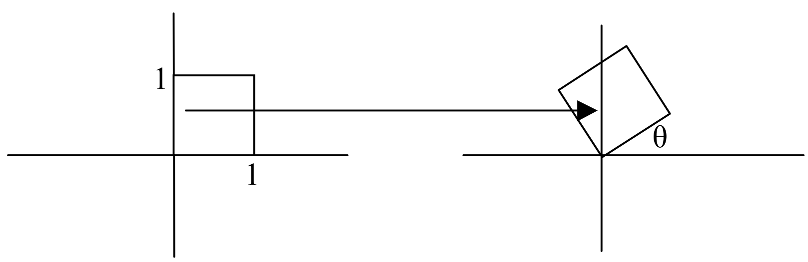
This means that (a,b) when rotated $\theta°$ a.c. becomes $(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$. For example (3,5) rotated 30° a.c. becomes (3cos30 – 5sin30, 3sin30 + 5cos30)

i.e. $\left(\dfrac{3\sqrt{3}-5}{2}, \dfrac{3+5\sqrt{3}}{2}\right)$.

hence the matrix representing a rotation of $\theta°$ a.c. is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

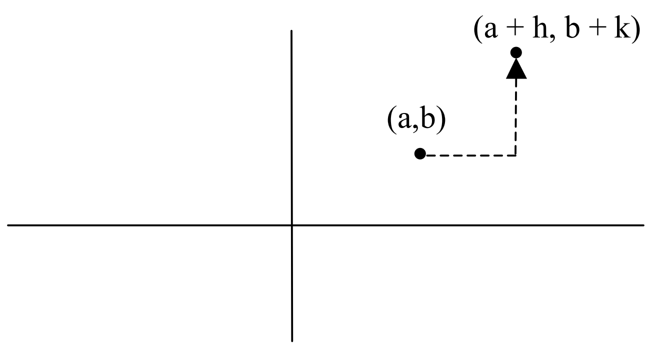


It will rotate the unit square as shown

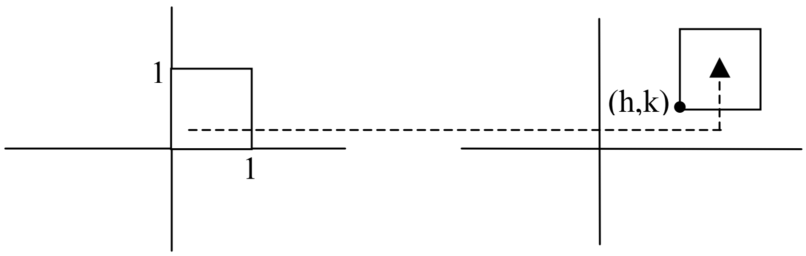


Translation

A translation moves each point (a,b) a number of units to the right and a number of units up. i.e. (a,b) is mapped to (a + h, b + k) where h and k are any real numbers.



It maps the unit square as shown



Unfortunately it is not possible to represent this transformation by a matrix. (Note that the origin is not a fixed point for this transformation).

Transformation	Mapping	Matrix Examples
Reflection	(a,b) (a,-b) (a,b) (-a,b)	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Stretch	(a,b) (ma, b)	$\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}$
Magnification	(a,b) (ma,mb)	$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$
Projection	(a,b) (a,0)	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Shear	(a,b) (a + mb, b)	$\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$
Rotation	(a,b) $(a \cos\theta - b\sin\theta, a \sin\theta + b \cos\theta)$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Translation	(a,b) (a + h, b + k)	None

Exercise 7.3

1. Find the image of (1,2) rotated 90° a.c.
2. Find the image of $(\sqrt{2}, \sqrt{2})$ rotated 45° a.c.
3. Describe the image of the unit square after a transformation by the matrix

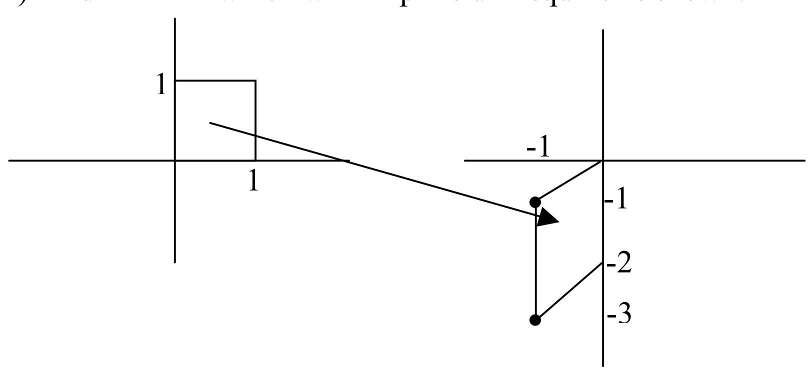
   i) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$
4. Find a matrix which will rotate 30° a.c. then reflect in the line y = x.

5. Identify the following: -

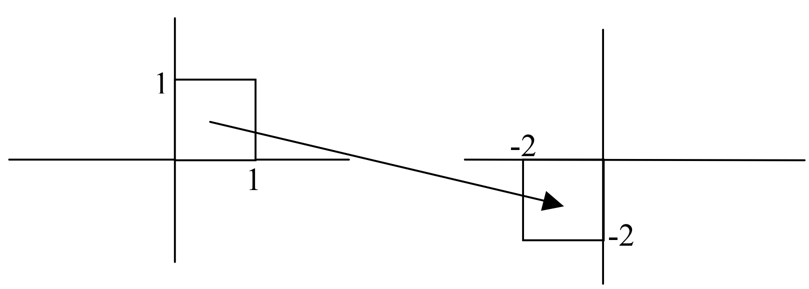
i) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ ii) $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ iii) $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ iv) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ v) $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

vi) $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ vii) $\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$ viii) $\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$ ix) $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ x) $\begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$

6. Find the matrix representing a rotation of 270° a.c.
7. Find the point (a,b) which maps to (-1,1) after the transformation represented by $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$.
8. Find the matrix representing a reflection about the line y = -x.
9. Make a chart headed Shape, Area and Distance and underneath each heading write down all the simple transformations which preserve the geometric property of the heading. For example a shear does not preserve shape and magnification does preserve shape but not area or distance. (Incidentally a distance-preserving transformation is called an ISOMETRY)
10. i) Find a matrix which will map the unit square as shown.



ii)



11. Find the image of the unit circle, i.e a circle centre (0,0) radius 1, when mapped by $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

12. i) Show that $\begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$ represents a stretch in the x direction by a factor of m followed by a stretch in the y direction by a factor of n.

    ii) Find the image of the unit circle when mapped by $\begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$.

Exercise 7.3 Answers

1. (-2,1)
2. (0,2)
3. i) A parallelogram with vertices at (0,0) (1,0) (3,1) (2,1).

   ii) 4 points lying on the line y = 2x.

   iii) A parallelogram with vertices at (0,0) (3,2) (7,5) (4,3).
4. $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
5. i) Identity ii) A reflection in the x axis. iii) A magnification by a factor of 2. iv) A reflection about the line y = x v) A projection onto the x axis. vi) A projection onto the y axis vii) A rotation of 60° a.c. viii) A shear in the y direction. ix) A shear in the x direction x) A reflection in y = x followed by a magnification by a factor of 2.
6. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
7. (-3,1)
8. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

9.

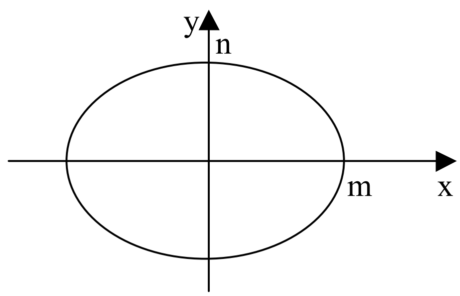
Shape	Area	Distance
Reflection Magnification Rotation Translation	Reflection Shear Rotation Translation	Reflection Rotation Translation

10. i) $\begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}$ or $\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}$

ii) $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ or $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

11. A circle centre (0,0) radius 2.

12. ii) An ellipse with centre (0,0)



7.4 Determinants

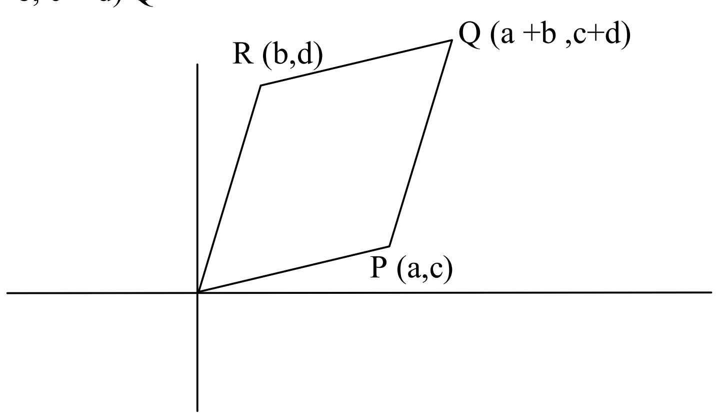
Consider the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when operating on the unit square.

Note $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ maps (1,0) to (a,c) P

(0,1) to (b,d) R

(1,1) to (a + b, c + d) Q

i.e. the unit square becomes



Note that OPQR is a parallelogram since OP is parallel to RQ and OR is parallel to PQ.

The area of OPQR = | (**a,c,o**) x (**b,d,o**) |

= |(**0,0,ad-bc**)|

= | ad – bc |

The expression ad – bc is a very important one and is called the **DETERMINANT** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Note that, as we have seen before, it is the product of numbers $\searrow$ minus the product of the numbers $\nearrow$.

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then its determinant is written det A or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Since the unit square has its area magnified by a factor of |ad – bc| it follows that: -

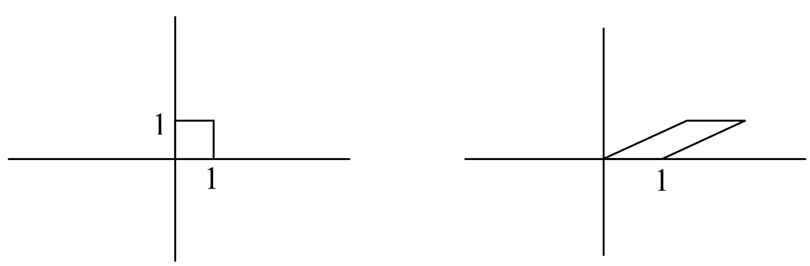
> If A is a 2 x 2 matrix, then |det A| represents the magnification of area of any plane figure when operated on by A

Examples

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation which clearly preserves the area of any figure. This is borne out by noting that its determinant = $\cos^2\theta + \sin^2\theta = 1$.

Shear

$\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ Its determinant is again 1 and hence a shear preserves area. Note that a shear maps the unit square as below.



Note that the area of the image is 1 (it is a parallelogram with base length 1 and height 1)

Projection

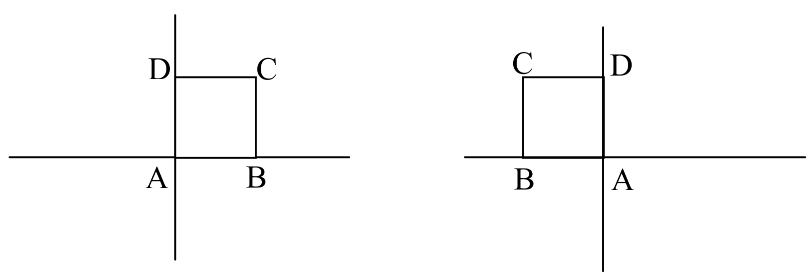
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Its determinant is zero and hence this means that the image of any plane figure mapped by a projection onto the x axis is zero.

This is clear, of course, by considering the geometric significance of a projection, i.e. it "squashes flat".

Reflection in $\mathbf{R^2}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Its determinant is -1 and since we consider absolute value of the determinant the reflection also preserves area. The image of any plane figure under a reflection is, of course, in fact a congruent plane figure.Note that a reflection **in $\mathbf{R^3}$** can have a determinant of **positive** 1.

The significance of the determinant being negative is one of **orientation**. This means that if a plane figure (say) labelled ABCD ANTICLOCKWISE then the image of ABCD after an operation whose determinant is negative will be ABCD CLOCK-WISE.



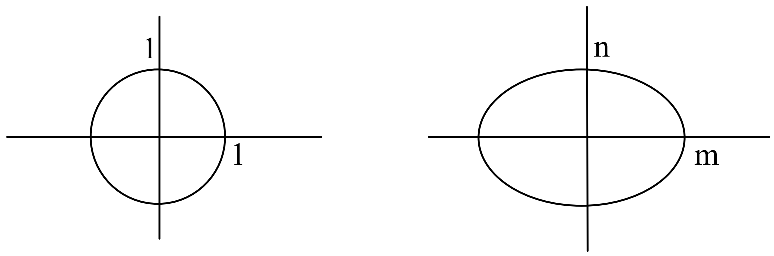
A full discussion of orientation is beyond the scope of this book but suffice it to say that, in R^2,

If det A > 0 then A preserves orientation

If det A < 0 then A reverses orientation.

Example

In Exercise 7.3 12 ii) you will have found that the image of a unit circle when operated on by $\begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$ is an ellipse.



The area of the unit circle is $\pi 1^2 = \pi$

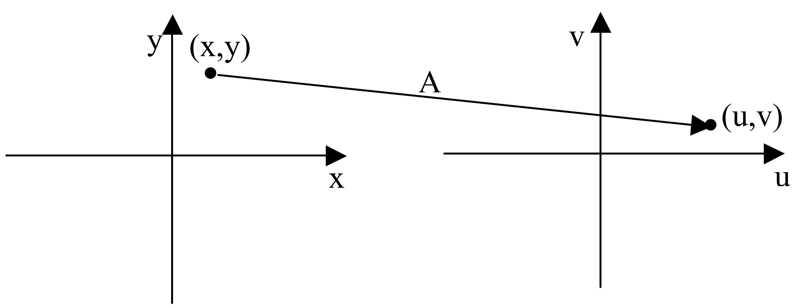
The determinant of $\begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$ is mn.

$\therefore$ The area of the ellipse is π times mn = πmn. (If you tried in Calculus class to find the area of an ellipse by integration methods you will appreciate the simplicity of the above all the more).

The inverse of a matrix

Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A maps (1,0) to (a,c) and (0,1) to (b,d).

In general A maps a point (x,y) to the point (u,v) where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

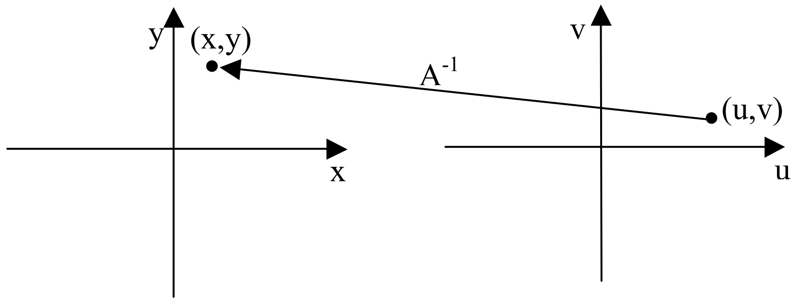


i.e. $ax + by = u$ (1)

$cx + dy = v$ (2)

The matrix which is capable of mapping all points back uniquely to the point from which each individually 'came' is called the inverse matrix, written A^{-1}.

For example, if A maps (x,y) to (u,v) then A^{-1} (u,v) is (x,y)



In particular A^{-1} will need to map (a,c) back to (1,0) and (b,d) back to (0,1). We need to solve equations (1) and (2) above for x and y.

Elementary algebra produces $x = \frac{du}{ad-bc} - \frac{bv}{ad-bc}$ and $y = \frac{-cu}{ad-bc} + \frac{av}{ad-bc}$

i.e. $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$ is A^{-1}.

Note that ad-bc appears here.

i.e. $\boxed{\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

Example

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Since A^{-1} is the operation which "cancels out" A, i.e. is the operation which brings all points back to their original position then it follows that A combined with A^{-1} is the identity matrix.

i.e. $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ maps (1,0) to (1,3) and $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ maps (1,3) back to (1,0).

Example

To find the inverse of a rotation of $\theta°$ a.c.

A rotation of $\theta°$ a.c. is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Its determinant is +1. Therefore, its inverse is

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

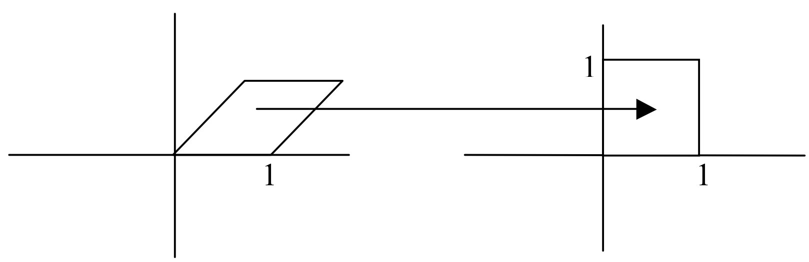
However, note that the inverse of a rotation of $\theta°$ a.c.is a rotation of $\theta°$ clockwise i.e. a rotation of $(-\theta)$ a.c. Note that since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, substitution of $-\theta$ in for the angle in the standard form of a rotation a.c. produces the required inverse

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Example

To find inverse of a shear $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Determinant is +1, i.e. its inverse is $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

Note that the inverse maps



Example

To find inverse of a projection $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The determinant = 0. This means that there is **no** inverse because the reciprocal of the determinant is not a real number. Note that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ maps both (1,1) and (1,2) to (1,0). Therefore if an inverse did exist tit would have to map (1,0) back to both (1,1) and (1,2) – clearly not possible.

Exercise 7.4

1. Find det $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

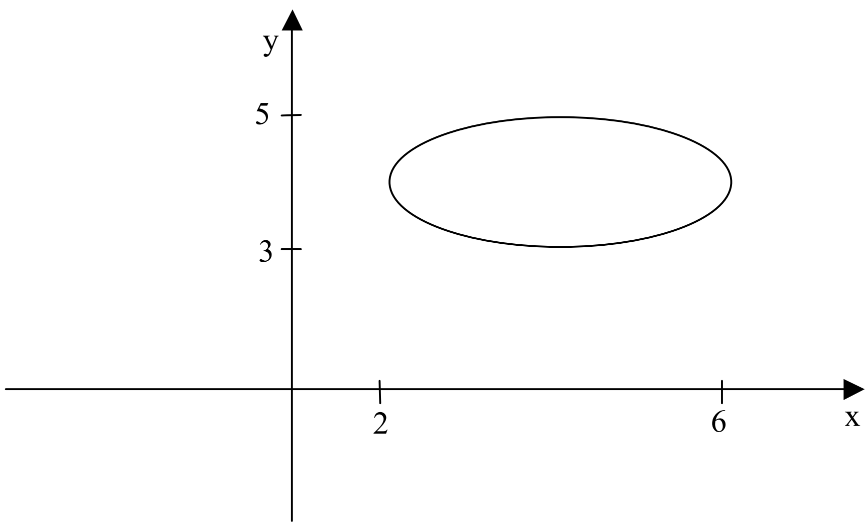
2. Find the matrix representing a rotation of 45° clockwise.

3. i) Let A = $\begin{bmatrix} -2 & 3 \\ 2 & 1 \end{bmatrix}$. Find A^{-1}. Check your solution by investigating whether

   $A^{-1}A = I$.

   ii) PQRS is a square in the domain of A, as in i), such that PQ = 3. Find the area of the image of PQRS under matrix A. Is the orientation preserved?

4.



   Find the area of ellipse illustrated.

5. If $A = A^{-1}$ does A = I?

6. Find $\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$

7. Find the inverse of magnification, reflection and stretch using the quoted formula and convince yourself that the inverse satisfy your notion of their geometric significance.

8. i) Find image(3,4) for matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   ii) Find image (1,5) for matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

iii) What do the results i) and ii) tell you about matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$?

9. Convince yourself that the determinant of a magnification satisfies your idea of its magnification of area from a geometric point of view.
10. Express a rotation of 60° a.c. as a matrix.
11. i) A, B are rotations in R^2. Does AB = BA?

    ii) A, B are reflections in R^2. Does AB = BA?

    iii) A is a rotation, B is a reflection. Does AB = BA?
12. A, B are both 2 x 2 matrices. Show that det(AB) = (det A)(det B).
13. Find a matrix which will map the unit square into a rhombus with one vertex at the origin, each side at length 1 and having an area equal to $\frac{1}{2}$.
14. Show that the matrix derived in Question 13 is not distance preserving despite the fact that each side of the rhombus in the image has length 1.
15. Does a 2 x 2 matrix always map two independent vectors to two independent vectors?
16. If $A^2 = I$ does det A = ± 1?
17. Find two matrices A,B so that AB = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but neither A nor B = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Does det A = det B = 0?
18. Is it possible to have a matrix A such that A maps all points in the Domain (R^2) to points lying on the line x + y = 1? If so, find the matrix.
19. Repeat Question 18 for the image points to lie on x + y = 0
20. A is a function $R^2 \rightarrow R^2$ such that A(x,y) = (x+y, x-y). Write A as a matrix
21. Is it possible for the determinant of a rotation/reflection to be zero?
22. Is it possible for a 2 x 2 matrix to map the unit square into parallelogram EFGH where E is (1,1), F is (2,5), G is (4,7), H is (3,3)?
23. Explain why a rotation about the point (1,1) cannot be represented by a matrix.
24. Can a matrix represent a reflection about the line y = x + 1? about y = 2?
25. Find a 2 x 2 matrix such that it maps each point on the line x + y = 1 to a point on the line x + y = 2.

26. Find two matrices of the form $\begin{bmatrix} m & n \\ n & m \end{bmatrix}$, other than $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ satisfying the equation $A^2 = A$. Such a matrix is called IDEMPOTENT.

27. Find a 2 x 2 matrix A other than $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ such that A^2 = zero matrix. If A^n = zero matrix, A is said to be NILPOTENT of order n.

Exercise 7.4 Answers

1. 11

2. $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

3. i) $\begin{bmatrix} -\frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ ii) 72. No.

4. 2π

5. Not necessarily

6. $\begin{bmatrix} \frac{-2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix}$

8. i) (11,22) ii) (11,22)iii) It is not a 1-1 function and has no inverse. 0.

10. $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

11. i) Yes ii) No, e.g. A is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, B is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ iii) no.

13. Any matrix of the form $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$ where b = $\frac{1}{2}$ or -$\frac{1}{2}$ and a = $\frac{\sqrt{3}}{2}$ or -$\frac{\sqrt{3}}{2}$.

15. No (If det = 0, the two image vectors are dependent)

16. Yes.

17. $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} B = \begin{bmatrix} 2 & 3 \\ -1 & -\frac{3}{2} \end{bmatrix}$. Yes.

18. No, since this "matrix" does not preserve the origin.

19. Yes. Any matrix of the form $\begin{bmatrix} m & n \\ -m & -n \end{bmatrix}$ e.g. $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

21. No, since reflections and rotations preserve area and have determinants -1 and +1 respectively.

22. No, since origin is not preserved.

23. No, since origin is not preserved

24. No. No, since origin is not preserved.

25. Any matrix of the form $\begin{bmatrix} a & b \\ 2-a & 2-b \end{bmatrix}$ e.g. $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$ where $a \neq b$

26. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

27. Any matrix of the form $\begin{bmatrix} a & \frac{-a^2}{b} \\ b & -a \end{bmatrix}$ e.g. $\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$

7.5 Properties of Matrices

Matrix multiplication is associated, ie.. (A B)C = A(B C).

It is left to the reader to verify this easy but time-consuming exercise. It follows that ABC is well-defined without brackets.

$\underbrace{A\ A\ A\ A}_{n \text{ times}}$ ----- A is hence written A^n.

Example

If $A^4 = I$ then $\det(A^4) = \det I$.

Since $\det(AB) = (\det A)(\det B)$ it follows that $\det(A^4) = (\det A)^4 = \det I = 1$

i.e. $\det A = \pm 1$.

A very important property of matrices is that

$$\boxed{\text{if det A = 0 then } A^{-1} \text{ does not exist}}$$

Such a matrix A is called **SINGULAR**.

From the derivation of inverses, viz: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ it is clear that det A = 0 would negate the existence of A^{-1}.

For example the inverse of $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ does not exist because its determinant is zero.

Note that for the matrix above (3,6) is mapped to (0,0). If an inverse did exist it would have to map (0,0) to (3,6) which is not possible for any matrix.

To find inverse of AB

Let the inverse of AB be X

Then X(AB) = I

Post multiply by B^{-1}

i.e. $X(AB)B^{-1} = IB^{-1}$

i.e. $XA(BB^{-1}) = IB^{-1}$ Associative Property

i.e. $XA = B^{-1}$

Post multiply by A^{-1}

i.e. $XAA^{-1} = B^{-1}A^{-1}$

i.e. $X = B^{-1}A^{-1}$

i.e. $\underline{\underline{(A\,B)^{-1} = B^{-1}A^{-1}}}$ **note order**

If A^{-1} or B^{-1} does not exist then $(AB)^{-1}$ does not exist.

Transpose

Interchanging the rows and columns of a matrix produces the TRANSPOSE of a matrix.

e.g. if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then A transpose = $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. This is written A^T.

Solving Equations

Consider $2x + 3y = 6$ (1)

$4x + 5y = 8$ (2)

Note that these two equations could be written in the form $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ (*)

From a matrix point of view, solving for x and y is equivalent to finding the point (x,y) which maps to (6,8) under matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

Let A = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$.

Pre-multiply both sides of equation (*) by A^{-1} producing $A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

i.e. $I\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

i.e. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

i.e. the solution is x = - 3, y = 4.

It is not suggested that this is a superior technique for solving these simple equations – merely an illustration of one of the very many uses to which matrices can be put.

If we try to solve $\begin{matrix} x + 2y = 1 \\ 2x + 4y = 3 \end{matrix}$ using matrices, i.e. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ it is not possible to do so because $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has determinant zero and hence has no inverse. Note that the two equations do not have a solution since geometrically they represent two parallel non-intersecting lines in R^2. Furthermore $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ maps no point to (1,3)

Compound Transformations in R^2

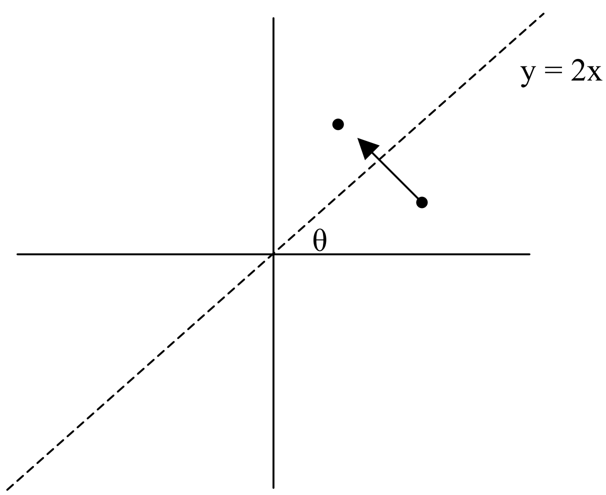
By compound transformation we mean a sequence of simple transformations. For example, the matrix representing the compound transformation of rotation 30° a.c., followed by reflection in the line y = x, followed by a magnification by a factor of 2 is

$$\underset{\substack{\uparrow\\ \text{Magnification}}}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \text{ times } \underset{\substack{\uparrow\\ \text{Reflection}}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \text{ times } \underset{\substack{\uparrow\\ \text{Rotation}}}{\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

Note the order of the matrices.

Example

To find the matrix representing a reflection about the line y = 2x.



The idea is to

1. rotate $\theta°$ clockwise
2. reflect about the x axis.
3. rotate $\theta°$ a.c.

This series of operations, 1), 2), 3) will produce the compound transformation of a reflection about y = 2x. In diagram, $\tan\theta = 2$, $\therefore \cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$

Therefore a rotation of $\theta°$ clockwise is $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

The reflection about the x axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The rotation of $\theta°$ a.c. is $\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$\therefore$ A reflection about the line $y = 2x$ is $\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

which is $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$.

This means, for example, that the image of (2,3) when reflected about y = 2x is

$$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

i.e. $\left(\frac{6}{5}, \frac{17}{5}\right)$.

Note that the determinant of the reflection is -1. This is true for reflections in R^2 since they preserve area but change orientation.

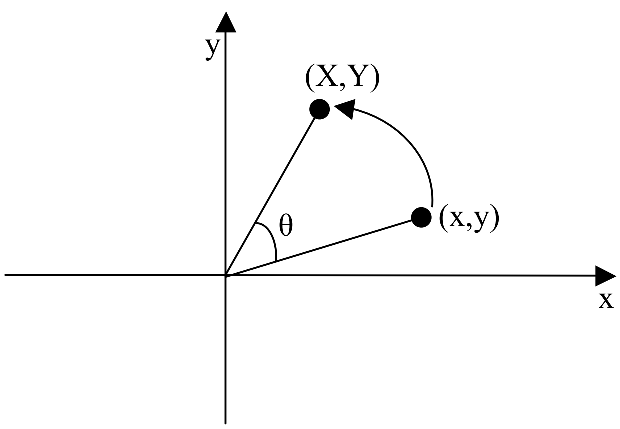
Rotation of conic sections

$y = x^2$ is well known as a parabola whose axis of symmetry is the y-axis. If we rotate the parabola 45° about the origin (anti-clockwise) its equation may be written in the form

$(x + y)^2 = \sqrt{2}(y - x)$.

This is clearly not an easy 'formula' to digest or remember. However, when the hyperbola $\frac{x^2}{2} - \frac{y^2}{2} = 1$ is rotated 45° anti-clockwise about the origin its equation can be written $y = \frac{1}{x}$. One might guess this from their graphs but clearly some work has to be done.

In general, to find an equation for any graph when rotated $\theta°$ (a.c.) about the origin can be effected as follows



Let (x,y) be an arbitrary point on our original graph and let (X,Y) be its new co-ordinates when rotated $\theta°$ a.c.. Then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

Pre-multiplying by the inverse of the rotation we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

i.e. $x = X\cos\theta + Y\sin\theta$ (1)

$y = -X\sin\theta + Y\cos\theta$ (2)

To find the new equation for the graph when rotated we simply substitute for x and y as given above in (1) and (2) and simplify.

Example 1

To find an equation of the image x + 2y = 2 when rotated 90° (a.c.) about the origin.

$x = X\cos 90 + Y\sin 90$ i.e. $x = Y$

$y = -X\cos 90 + Y\cos 90$ $y = -X$

Substituting into the original equation we get $Y - 2X = 2$ as the new equation for the rotated graph.

Example 2

To find an equation of xy + x = 1 when rotated 60° (a.c.) about the origin.

$x = X\cos 60 + Y\sin 60$ i.e. $x = X\frac{1}{2} + Y\frac{\sqrt{3}}{2}$

$y = -X\sin 60 + Y\cos 60$ $y = -X\frac{\sqrt{3}}{2} + Y\frac{1}{2}$

Substituting for x and y yields

$(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y)(-\frac{\sqrt{3}}{2}X + \frac{1}{2}Y) + \frac{1}{2}X + \frac{\sqrt{3}}{2}Y = 1$ which when simplified becomes

$3X^2 + 2\sqrt{3}XY - 3Y^2 - 2\sqrt{3}X - 6Y + 4\sqrt{3} = 0$

Note that this method will work for all relations, not simply graphs which are symmetric about the origin.

Example 3

To find equation of $y = x^3$ when rotated 45° (a.c.) about the origin.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

i.e. $x = \frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y$ and $y = -\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y$

Substituting these values for x and y into $y = x^3$ yields

$-\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y = (\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y)^3$ which when simplified becomes

$2(Y - X) = (X + Y)^3$

Usually equations of conic sections whose lines of symmetry are not horizontal or vertical are "nasty" and contain an xy term. For example, xy = 1, which is not "nasty", represents a hyperbola whose axis of symmetry is y = x.

In general, given

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ then the graph is

a) an ellipse if $B^2 - 4AC$ is negative
b) a parabola if $B^2 - 4AC$ is zero
c) a hyperbola if $B^2 - 4AC$ is positive

Incidentally it is not a coincidence that this relates to the $b^2 - 4ac$ in the quadratic formula.
Sometimes we may be given a graph whose equation is written in a form containing an xy term and we wish to rotate the graph so that its equation no longer has an xy term. This will mean the graph is in a so-called 'standard position'.

Given $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ it is easy to show by substituting for x and y in its rotated form that the xy term will disappear when the graph is rotated $\theta°$ a.c. when

$\tan 2\theta = \frac{B}{C - A}$

It is left as an exercise for the student to confirm this.

Question

Given $x^2 + 2\sqrt{3}\,xy - y^2 = 1$ a) state which conic section it is, and b) find the angle of rotation needed to eliminate the xy term, and c) state an equation for the rotated conic section.

Answer

a) A = 1, B = $2\sqrt{3}$ and C = -1

therefore $B^2 - 4AC = 16$ which is positive, therefore the graph is a hyperbola

b) $\tan 2\theta = \frac{B}{C - A} = \frac{2\sqrt{3}}{-1-1} = -\sqrt{3}$ therefore $\tan 2\theta = -\sqrt{3}$ and hence $2\theta = 120°$ and hence

$\theta = 60°$. Therefore the conic needs to be rotated 60° (a.c.)

c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 60 & \sin 60 \\ -\sin 60 & \cos 60 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

Therefore $x = \frac{1}{2}X + \frac{\sqrt{3}}{2}Y$ and $y = -\frac{\sqrt{3}}{2}X + \frac{1}{2}Y$

Substituting into $x^2 + 2\sqrt{3}\,xy - y^2 = 1$ yields, when simplified, $2Y^2 - 2X^2 = 1$ which is a hyperbola as shown in a)

Exercise 7.5

1. True of False?
   - i) $(A\,B)^{-1} = A^{-1}B^{-1}$
   - ii) $\det A = 0 \Rightarrow A$ has no inverse.
   - iii) $A \bullet B = A \bullet C \Rightarrow B = C$
   - iv) $A \bullet B = A \bullet C \Rightarrow B = C$ or A^{-1} does not exist.
   - v) $A^3 = I \Rightarrow \det A = 1$.
   - vi) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ preserves area.
   - vii) $\det(AB) = \det A \det B$
   - viii) The image of the unit square transformed by a matrix is a parallelogram
   - ix) The determinant of a reflection in R^2 is negative
   - x) $AB = BA$ if A,B are rotations
   - xi) A shear preserves area
   - xii) $A(BC) = (AB) = C$
   - xiii) $AB =$ zero matrix $\Rightarrow$ A or B = zero matrix
   - xiv) $A = A^{-1} \Rightarrow A = I$
   - xv) Determinant of a rotation is +1
   - xvi) $A^2 = A \Rightarrow A = I$ or zero matrix
   - xvii) $\det(ABC) = (\det A)(\det B)(\det C)$
   - xviii) $(\det A)(\det B) = (\det B)(\det A)$

xix) $(A\,A)^{-1} = A^{-1}\,A^{-1}$

xx) $\begin{bmatrix} -.6 & -.8 \\ .8 & -.6 \end{bmatrix}$ is a rotation in R^2.

xxi) $(A^{-1})^{-1} = A$

xxii) $[A^2]^{-1} = [A^{-1}]^2$

xxiii) $A^2 = I$ has exactly two solutions for A.

xxiv) $AB = BA \Rightarrow A,B$ are rotations.

xxv) $A^{-1} = B^{-1} \Rightarrow A = B$.

xxvi) $A^2 = A \Rightarrow \det A = 0$ or 1.

xxvii) A is a reflection $\Rightarrow A^2 = I$.

xxviii) $A = B^{-1} \Rightarrow \det A = \dfrac{1}{\det B}$.

xxix) $\det(P^{-1}AP) = \det A$

2. Let A be a 2 x 2 matrix. If $\det A = \det(A^{-1})$ what can you deduce about A?
3. Find a matrix representing a rotation of 60° a.c. followed by a reflection in the line y= x.
4. Find the point (a,b) mapping to (1,2) for matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
5. find the point (a,b) mapping to (1,2) for matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
6. i) Show how you would use matrices to solve $x + 2y = 3$, $4x - 2y = 1$.

   ii) Show how you would use matrices to solve $4x + 3y = 1$, $5x + 4y = 1$
7. It is **not** true that ($AB = AC \Rightarrow B = C$) (Fact).

   i.e. 'Cancellation' law for matrices is **not** true.

   Find the mistake

$$AB = AC$$
$$\therefore A^{-1}(AB) = A^{-1}(AC)$$
$$\therefore (A^{-1}A)B = (A^{-1}A)C$$
$$\therefore IB = IC$$
$$\therefore B = C$$

8. Find the matrix representing a projection onto y = x.
9. Find the matrix representing a reflection about 3y = 4x.
10. Find the matrix representing a reflection about $y = \sqrt{3}x$
11. Show that the matrix representing a reflection about the line $y = (\tan\alpha)x$ is
$\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$.
12. Write down $(ABC)^{-1}$ as the product of inverses.
13. Find a matrix A such that $A^3 = I$ but $A \neq I$ (No calculation required).
14. Convince yourself that
    i) $(AB)^T = B^TA^T$
    ii) $(A^T)^{-1} = (A^{-1})^T$.
15. Find the matrix which is capable of mapping (1,1) to (2,2) and mapping (1, -1) to (1,-1) i.e. (1,-1) is a fixed point. Is the matrix unique?
16. Explain why $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not a projection onto y = x despite the fact that this matrix map all points onto y = x.

17. State the type of conic section for each of following
    a) $3x^2 + 6xy + 3y^2 - 2x - 7y = 8$
    b) $3x^2 + 6xy - 3y^2 - 2x - 7y = 8$
    c) $3x^2 + 2xy + 3y^2 - 2x - 7y = 8$
18. Find the image of $y = x^2$ when rotated 60° (a.c.)
19. Given $xy + x = 1$
    a) State what type of conic section it is
    b) Name the angle through which the graph needs to be rotated to eliminate the xy term.
    c) Find an equation of the rotated conic in b)
20. Given $x^2 - 2\sqrt{3}\,xy + 3y - 2\sqrt{3}\,x + 2y = 0$
    a) State what type of conic section it is.
    b) Name the angle through which the graph needs to be rotated to eliminate the xy term.

c) Find an equation of the rotated conic.

21. Find an equation for y = 2x + 1 after it is rotated 90° (a.c.)

Exercise 7.5 Answers

1. i) False ii) True iii) False iv) True v) True vi) True
vi) True viii) True ix) True x) True xi) True xii) True
xiii) False xiv) False, e.g. A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ xv) True xvi) False, e.g. A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
xvii) True xviii) True xix) True xx) True xxi) True xxii) True
xxiii) False, e.g. A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and many more. xxiv) False
xxv) True xxvi) True xxvii) True xxviii) True xxix) True

2. det A = ± 1 i.e. A preserves area

3. $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$ 4. $(0, \frac{1}{2})$ 5. Any point lying on x + 2y = 1.

7. A does not necessarily have an inverse, e.g. det A = 0 or A may not be square.

8. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ 9. $\begin{bmatrix} -\frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{bmatrix}$ 10. $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ 12. $(C^{-1})(B^{-1})(A^{-1})$.

13. A could be a rotation of 120° a.c., i.e. $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

15. $\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ 17. a) parabola b) hyperbola c) ellipse

18. $-2\sqrt{3}X + 2Y = X^2 + 3Y^2 + 2\sqrt{3}XY$

19. a) hyperbola b) 45° c) $Y^2 + \sqrt{2}\,Y = 2 + X^2 - \sqrt{2}\,X$

20. a) parabola b) 60° (a.c.) c) $y = x^2$ 21. $y = -\frac{1}{2}x - \frac{1}{2}$

7.6 3 x 3 Matrices

The determinant of a 3 x 3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + d\begin{vmatrix} h & i \\ b & c \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$

$= a(ei - hf) + d(hc - bi) + g(bf - ec)$

$= aei + dhc + gbf - ahf - dbi - gec$

A useful mnemonic for obtaining this is as follows –

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

Add the products $\searrow$ and subtract the products $\nearrow$

Example:

$$\det\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 3 \\ 5 & 1 & 2 \end{bmatrix} = 1x4x2 + 3x3x5 + 1x2x1 - 5x4x1 - 1x3x1 - 2x2x3$$

$= 8 + 45 + 2 - 20 - 3 - 12 = 20$

However, note that the determinant of the matrix

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is (**a,d,g**) • (**b,e,h**) x (**c,f,i**)

i.e. the triple scalar product of the three columns of the matrix considering each column as a vector. If a matrix is not square it does not have a determinant. The determinant may also be found by expanding along a row, e.g. in above example

determinant = $a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} g & i \\ d & f \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

or as a triple scalar product (**a,b,c**) •(**d,e,f**) x (**g,h,i**)

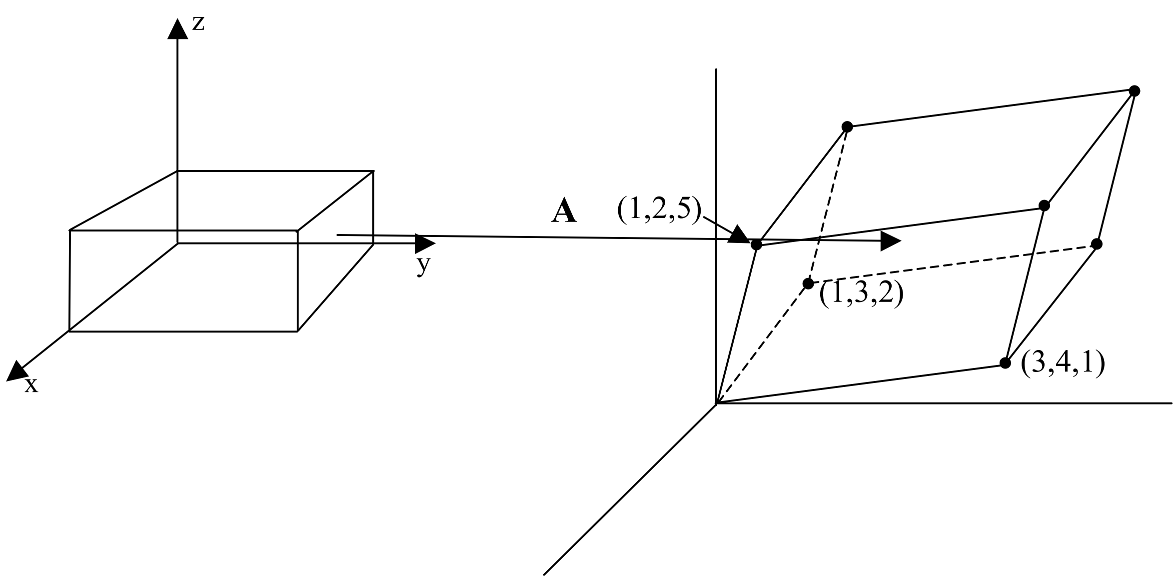
Consider the mapping of the unit cube (i.e. a cube with vertices at (0,0,0) (1,0,0) (0,1,0) (0,0,1)) by matrix A = $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 3 \\ 5 & 1 & 2 \end{bmatrix}$.

note that A maps (**1,0,0**) → (**1,2,5**)

(**0,1,0**) → (**3,4,1**)

(**0,0,1**) → (**1,3,2**)

i.e. A maps **i** to the vector whose components are in the first column of A.
Similarly the mapping of **j** and **k** are in the second and third columns of A. This is true for all matrices and will take on greater significance later.



In fact A maps the unit cube to a parallelepiped three adjacent edges of which are (**1,2,5**), (**3,4,1**) and (**1,3,2**). It is left to the reader to verify this.

Remember that the volume of the parallelepiped is:
|(**1,2,5**)•(**3,4,1**) x (**1,3,2**)| = 20. Since the volume of the unit cube is 1, it follows that the magnification of the volume is |(**1,2,5**)•(**3,4,1**) x (**1,3,2**)|i.e. det A. (in this case, 20).

In general,

$$\boxed{|\det 3\text{x}3 \text{ matrix } A| = \text{magnification of volume}}$$

Compare with the corresponding statement for 2x2 matrices.

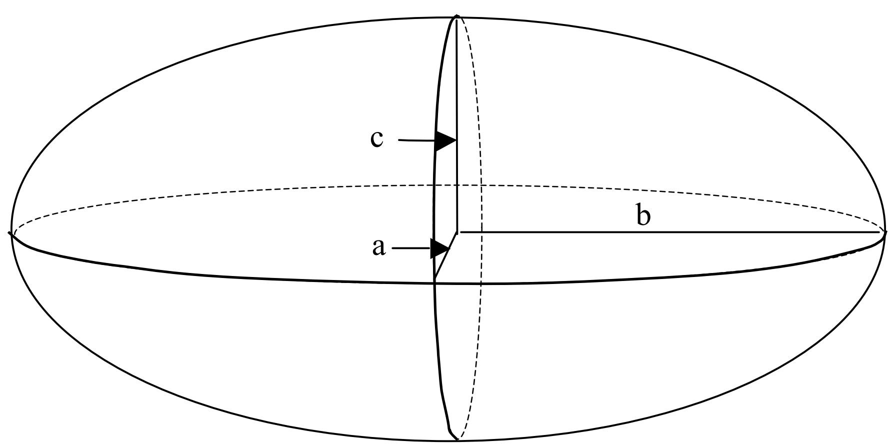
Example

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ has determinant zero.

It therefore follows that the image of the unit cube by this matrix has zero volume, i.e. is a plane, line, etc. In fact it is a plane.

Example

Note that $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is a three-dimensional stretching matrix, stretching by factors of a,b,c in the x,y,z directions respectively, e.g. a unit sphere (centre (0,0,0), radius 1) would be mapped by this matrix to an ellipsoid with "radii" a,b,c, as shown.



The volume of the unit sphere is $\frac{4}{3}\pi 1^3$ which is $\frac{4}{3}\pi$.

Determinant of matrix is abc.

$\therefore$ Volume of ellipsoid is $\frac{4}{3}\pi\,\text{abc}$.

Inverse of a 3x3 matrix

We will show how to get the inverse of a 3x3 matrix by example. To find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix} = A$.

Steps

1. Find determinant, i.e. 40+84+96-105-48-64 = 3.
2. Replace each element by the determinant of the reduced matrix formed by deleting both the row and column in which that element is.

   i.e. $\begin{bmatrix} -8 & -10 & -3 \\ -8 & -13 & -6 \\ -3 & -6 & -3 \end{bmatrix}$

3. Interchange rows and columns, i.e. transpose

   $$\begin{bmatrix} -8 & -8 & -3 \\ -10 & -13 & -6 \\ -3 & -6 & -3 \end{bmatrix}$$

4. Multiply each element by $(-1)^{i+j}$ where i and j represent the number of the row and column of each element.

   $$\begin{bmatrix} -8 & 8 & -3 \\ 10 & -13 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

5. Divide each element by the determinant.

   $$\begin{bmatrix} -\frac{8}{3} & \frac{8}{3} & -1 \\ \frac{10}{3} & -\frac{10}{3} & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

   This is the inverse of A.

Note that $\begin{bmatrix} -\frac{8}{3} & \frac{8}{3} & -1 \\ \frac{10}{3} & -\frac{10}{3} & 2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

which is the identity matrix mapping from $R^3 \rightarrow R^3$.

Note also that A(**1,0,0**) = (**1,4,7**) and A^{-1}(**1,4,7**) = (**1,0,0**). In general, of course, if A(**v**) = **w** then A^{-1}(**w**) = **v**.

Remember also in the steps 1 – 5 that if in step 1) the determinant is zero, then no inverse exists.

Example

To find the inverse of $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 4 \\ 5 & 3 & -1 \end{bmatrix}$

The determinant is zero and hence no inverse exists. Note however how this relates to the existence of a solution to (say)

$$x + y - z = 2 \quad (1)$$
$$2x - y + 4z = 3 \quad (2)$$
$$5x + 3y - z = 8 \quad (3)$$

This system of equations can be written in matrix form. $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 4 \\ 5 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$

Since there is no inverse there is no unique solution to the system of equations. If we think of (1), (2), (3) geometrically as planes we also know that no unique solution exists by the work of Chapter 6. (see section 6. 4) because if $\mathbf{n_1}$, $\mathbf{n_2}$, $\mathbf{n_3}$ are normals to the planes (1), (2), (3) then $\mathbf{n_1} \bullet \mathbf{n_2} \times \mathbf{n_3} = 0$.

i.e. (**1,1,-1**) • (**2,-1,4**) x (**5,3,1**) can be thought of as the determinant of the matrix or as the triple scalar product of the normals an idea closely relating the algebra and geometry.

To solve - $x + 2y + 3z = 12$

$2x - y - 3z = -7$

$3x + 3y + z = 7$

This set of equations can be rewritten $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -7 \\ -7 \end{bmatrix}$

The inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ is $\begin{bmatrix} \frac{8}{13} & \frac{7}{13} & -\frac{3}{13} \\ -\frac{11}{13} & -\frac{8}{13} & \frac{9}{13} \\ \frac{9}{13} & \frac{3}{13} & -\frac{5}{13} \end{bmatrix}$

Therefore $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{13} & \frac{7}{13} & -\frac{3}{13} \\ -\frac{11}{13} & -\frac{8}{13} & \frac{9}{13} \\ \frac{9}{13} & \frac{3}{13} & -\frac{5}{13} \end{bmatrix}\begin{bmatrix} 12 \\ -7 \\ -7 \end{bmatrix}$

i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ Hence x = 2, y = -1 and z = 4.

Remember that the transpose of a matrix A is the matrix with rows and columns interchanged, e.g. if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

If the transpose of a matrix A is also its inverse, then we say A is **ORTHOGONGAL.**

i.e. A is orthogonal if $A^T = A^{-1}$.

Orthogonal matrices are important because all distance preserving 2x2 matrices are orthogonal and vice versa. For example, all reflections and rotations in R^2 are orthogonal, e.g. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has its inverse and transpose as $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

In fact, every reflection in R^2 is its own inverse (thinking of the geometric significance of a reflection confirms this) and hence every reflection in R^3 is its own transpose.

Note that an orthogonal matrix has column vectors which are **mutually perpendicular**

unit vectors. e.g. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$ is an orthogonal matrix. This is true because

$\{(\boldsymbol{\frac{\sqrt{3}}{2},0,\frac{1}{2}}),(\boldsymbol{-\frac{1}{2},0,\frac{\sqrt{3}}{2}}),(\mathbf{0,1,0})\}$ is an orthonormal set of vectors. Hence its inverse is its

transpose. i.e. $\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \end{bmatrix}$

An orthogonal matrix in effect maps {**i**,**j**,**k**} to a set of vectors which are in 'similar positional relationship' to one another.

A matrix which is equal to its transpose is called **SYMMETRIC**.

e.g. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 6 \\ -3 & 6 & 5 \end{bmatrix}$

i.e. a matrix A is symmetric if $A = A^T$ i.e. if $a_{ij} = a_{ji}$ for all i,j. It is by definition a square matrix.

A square matrix whose elements are all zero except for those on the main diagonal is

called **DIAGONAL** e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a diagonal matrix.

In certain special cases it is possible to find inverse using the following techniques. For

example, to find the inverse of matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ we note –

A(**i**) = **j** since first column of A is $\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}$

Similarly A(**j**) = **k** and A(**k**) = **i**.

$\therefore A^{-1}$ is a matrix such that $A^{-1}(\mathbf{i}) = \mathbf{k}$ (1)

$A^{-1}(\mathbf{j}) = \mathbf{i}$ (2)

$A^{-1}(\mathbf{k}) = \mathbf{j}$ (3)

(1) means that the first column of A^{-1} is $\begin{matrix}0\\0\\1\end{matrix}$

(2) means that the first column of A^{-1} is $\begin{matrix}1\\0\\0\end{matrix}$

(3) means that the first column of A^{-1} is $\begin{matrix}0\\1\\0\end{matrix}$

i.e. $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Note that $A^{-1} = A^T$, i.e. A is orthogonal. (In fact, A is a 120° rotation about x = y = z.)

A mapping which completely 'fills up' the Range Space is called ONTO. This means that the Range = Range Space. i.e. for any **w** in the Range there is a **v** in the Domain mapping to **w**.

Similar ideas maybe used to derive matrices representing transformations in R^3 which at first seem complex.

Example

To find the matrix representing a reflection in the plane x + y + z = 0.

The image of **i** reflected in x + y + z = **0** is $\left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$.

The image of **j** reflected in x + y + z = **0** is $\left(\frac{-2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$.

The image of **k** reflected in x + y + z = **0** is $\left(\frac{-2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

The matrix representing this transformation is $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

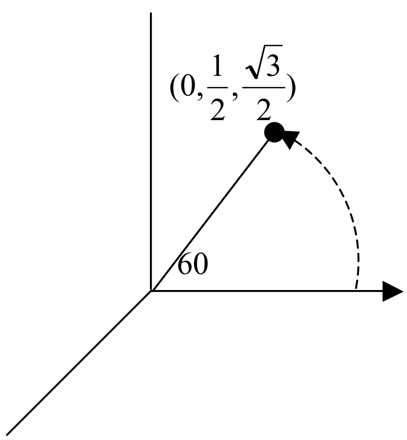
Note that its determinant is -1 (it is a reflection) and that its columns are the images of **i**, **j**, **k**.

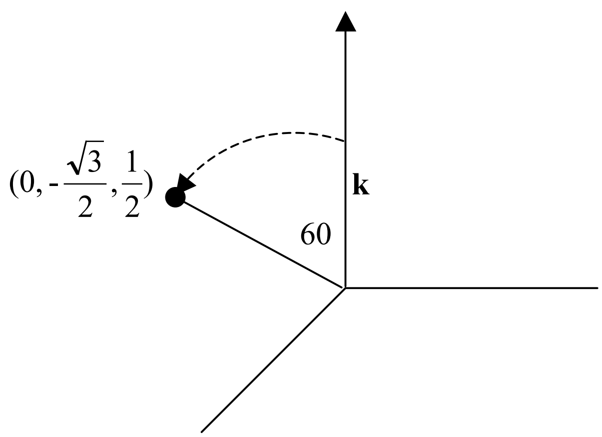
Example

To find the matrix representing a rotation of 60° a.c. about the x axis in R^3.

Let A be the required matrix. Then A(**i**) = **i**.

A(**j**) = $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{\sqrt{\mathbf{3}}}{\mathbf{2}})$ A(**k**) = $(\mathbf{0}, -\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$ See diagrams





A maps **j** to $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{\sqrt{\mathbf{3}}}{\mathbf{2}})$

A maps **k** to $(\mathbf{0}, -\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$

A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Cramer's Rule

Given equations

$$ax + by + cz = m$$
$$dx + ey + fz = n$$
$$gx + hy + iz = p$$

Then $x = \dfrac{\begin{vmatrix} m & b & c \\ n & e & f \\ p & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $\quad y = \dfrac{\begin{vmatrix} a & m & c \\ d & n & f \\ g & p & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$ and $z = \dfrac{\begin{vmatrix} a & b & m \\ d & e & n \\ g & h & p \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$

Note that x = (the determinant of co-efficients matrix with **x** co-efficient column replaced by constant term column) divided by determinant of co-efficients matrix. Similarly for y and z.

Example

To solve
$$3x + 2y + 5z = 18$$
$$2x - 3y - 2z = -3$$
$$4x + 4y - 3z = 6$$

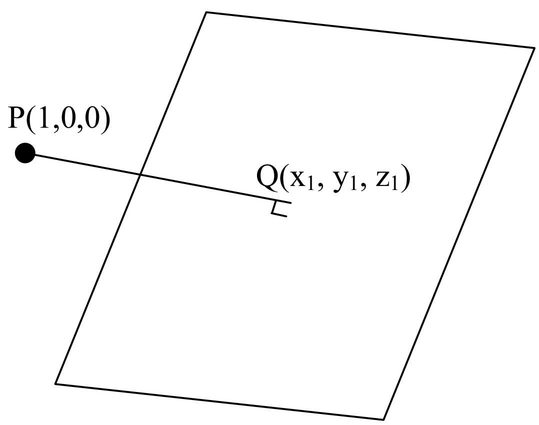
$$x = \frac{\begin{vmatrix} 18 & 2 & 5 \\ -3 & -3 & -2 \\ 6 & 4 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 5 \\ 2 & -3 & -2 \\ 4 & 4 & -3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} 3 & 18 & 5 \\ 2 & -3 & -2 \\ 4 & 6 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 5 \\ 2 & -3 & -2 \\ 4 & 4 & -3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 3 & 2 & 18 \\ 2 & -3 & -3 \\ 4 & 4 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 5 \\ 2 & -3 & -2 \\ 4 & 4 & -3 \end{vmatrix}}$$

i.e. $x = \dfrac{294}{147} = 2$ $\qquad y = \dfrac{147}{147} = 1$ $\qquad z = \dfrac{294}{147} = 2$

To find the matrix representing a projection onto Ax + By + Cz = 0

It suffices to find the images of (1,0,0). (0,1,0). (0,0,1) by this mapping since they will be the columns of the matrix.

To find the image of (1,0,0) projected onto Ax + By + Cz = 0



Let the projection of P(1,0,0) onto $Ax + By + Cz = 0$ be $Q(x_1, y_1, z_1)$

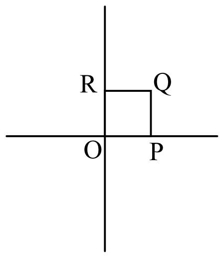
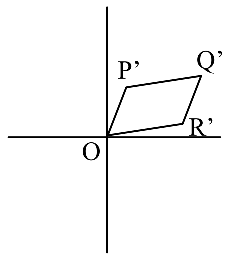
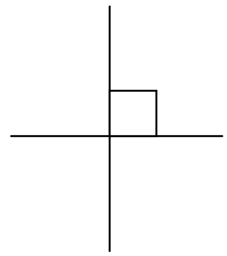
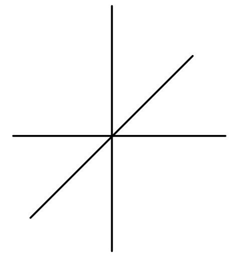
Then $Ax_1 + By_1 + Cz_1 = 0$ (1) since Q is on the plane and $(x_1 - 1, y_1, z_1) = m\,(A,B,C)$ (2) since PQ is a normal to the plane.

Solving (1) and (2) yields

$$(x_1, y_1, z_1) = \frac{1}{A^2 + B^2 + C^2}\begin{bmatrix} B^2 + C^2 & -AB & -AC \\ -AB & A^2 + C^2 & -BC \\ -AC & -BC & A^2 + B^2 \end{bmatrix}$$

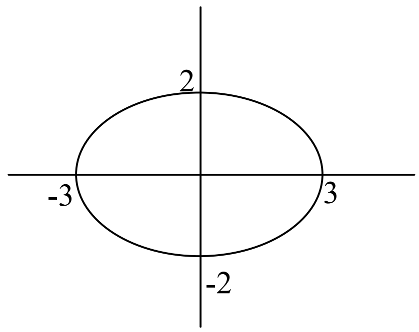
e.g. Projection onto $x + y + z = 0$ is $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Summary

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
Det A = -2	Det B = 0
 Area 1 Counterclockwise  Area 2 Clockwise	 Area 1  No Area
A^{-1} exists and is $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$	B^{-1} does not exist
A is not a zero divisor i.e. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has only the trivial solution	B **is** a zero divisor $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has **many** solutions for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ e.g. $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$
A has independent row vectors	B has dependent row vectors
A has independent column vectors	B has dependent column vectors
Given any point (a,b) in R^2 $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has a **unique** solution	$B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ does **not** have a unique solution (x,y) e.g. $B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has **no** solution. $B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ has many.
A is 1-1	B is not 1-1
A is onto	B is not onto.
AC = AD means C = D for all matrices C and D.	BC = BD does not mean C = D

Chapter 7 Review Exercise

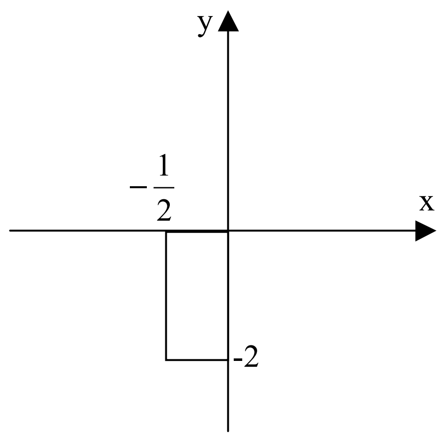
1. Find the image of the unit square after a transformation $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$.
2. A is a function such that A(x,y) = (x + y, x – y). Write A as a matrix.
3. Find the area of ellipse drawn.



4. Is it possible to write the function T(x,y) = (2x, y + 2) as a matrix?
5. Is $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ a singular matrix?
6. Find two matrices A and B such that AB = zero matrix but neither A nor B is the zero matrix and BA $\neq$ zero matrix.
7. Find $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}^{-1}$
8. Solve $\begin{aligned} 3x + 2y &= 1 \\ 5x + 4y &= 1 \end{aligned}$ using matrices,
9. Write down the matrix A where A(x,y,z) = (2x + z, y)
10. Does $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ always map squares to parallelograms?
11. Multiply $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
12. Write $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ as a product of two simple transformations in R^2.
13. Show that $(AB)^T = B^T A^T$
14. Find a matrix representing a reflection about 2y = x.

15. A is a rotation of $\theta°$ a.c. and B is a reflection about y = x. AB = BA. Find θ.

16. Find a matrix which will map the unit square to



17. Find a matrix mapping the unit square to ABCD where A is (0,0), B is (2,1), C is (3,1). What are the co-ordinates of D?

18. Find the image of (1,2) rotated 30° a.c.

19. Find a matrix which maps (1,1) to (1,1) but which maps (2,-1) to (3,1). Find another point left unmoved by this transformation.

20. Describe each of the following: -

i) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ v) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

21. Given an example of a matrix which is its own inverse.

22. Find the point (a,b) which maps to (1,2) after the transformation $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

23. Find **all** the points mappings to (1,2) after the transformation $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

24. Find the mistake.

Let $A^2 = A$

$\therefore A \bullet A = A$

$\therefore A^{-1}(A \bullet A) = A^{-1}A$

$\therefore IA = I$

$\therefore A = I.$

25. Write down a matrix which will map (1,2) to (1,2,3)

26. Find the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$.

27. i) Find the image of the unit cube after transformation by $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

    ii) Find the volume of the image of the unit cube for this matrix.

28. i) Does a 3x3 matrix with determinant = 0 have an inverse?

    ii) Can a reflection about $x + y + z = 1$ be represented as a matrix?

    iii) Can a projection onto $x + y + z = 1$ be represented as a matrix?

29. i) Find a matrix A representing a 90° rotation a.c. about the x axis **in $\mathbf{R}^3$**

    ii) Find the inverse of matrix found in i)

30. Find the image of (1,1) rotated 90° about the point (3,4).

31. Find the image of (1,1,1) reflected about the point (3,4,5)

32. Find the image of (1,1) reflected about $x + y = 1$.

33. Find the inverse of $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

34. Show that the line $y = -\ x$ is left unchanged by the matrix $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

35. Find inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

36. What is wrong with this?

    Let B be a translation such that $B(x,y) = (x + 1, y + 2)$.

    Then B(**i**) = (**2,2**) and B(**j**) = (**1,3**)

    $\therefore$ Matrix representation for B is $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

37. Solve $x + 2y + z = 4$ by Cramer's Rule

$$2x - y + 2z = 3$$

$$x - 3y + 2z = 0$$

38. Find the matrix representing a reflection in the plane x – y + z = 0.

39. Find the matrix representing a rotation of 90° a.c. about the line x = y = z.

40. Find the inverse of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

41. Is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ distance – preserving?

42. Find the matrix representing a projection onto x – y – z = 0.

43. If $A^2 = A$ does $A = A^T$?

44. Find a function which represents a rotation of 90 anti-clockwise about the point (1,1).

Chapter 7 Review Answers

1. A rhombus, whose vertices are (0,0), (-1,-1) (0,-2) (1,-1).

2. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ 3. 6π 4. No. 5. Yes.

6. $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 12 & -9 \\ -4 & 3 \end{bmatrix}$ 7. $\begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$ 8. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 10. Yes. 11. $\begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$ 12. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ 15. 180° 16. $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix}$ or $\begin{bmatrix} 0 & -\frac{1}{2} \\ -2 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ D is (1,0) 18. $(\frac{\sqrt{3}}{2} - 1, \frac{1}{2} + \sqrt{3})$

19. $\begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ (m,m) (for any value of m)

20. i) Projection onto y axis. ii) A shear in the x direction

iii) A stretch in the x direction iv) A reflection about the line y = x

v) A reflection about the origin

21. Any reflection e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 22. $(0, \frac{1}{2})$ 23. Any point lying on $x + 2y = 1$.

24. A^{-1} may not exist 25. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ 26. 3.

27. i) A parallelepiped with vertices (0,0,0) (1,2,2) (2,1,3) (1,2,1) (3,3,5) (2,4,3) (3,3,4) and (4,5,6) ii) 3. 28. i) no ii) No iii) No

29. i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ Note A is orthogonal since $A^T = A^{-1}$.

30. (6,2). 31. (5,7,9) 32. (0,0) 33. $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$ 35. $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

36. B is not a linear translation. 37. (1,1,1)

38. $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ 39. $\begin{bmatrix} \frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} \\ \frac{1+\sqrt{3}}{3} & \frac{1}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}$ 40. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

41. No, since it is not orthogonal, i.e. $A^T \neq A^{-1}$. 42. $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

43. No, e.g. $\begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix}$ 44. $f(x,y) = (2 - y, x)$