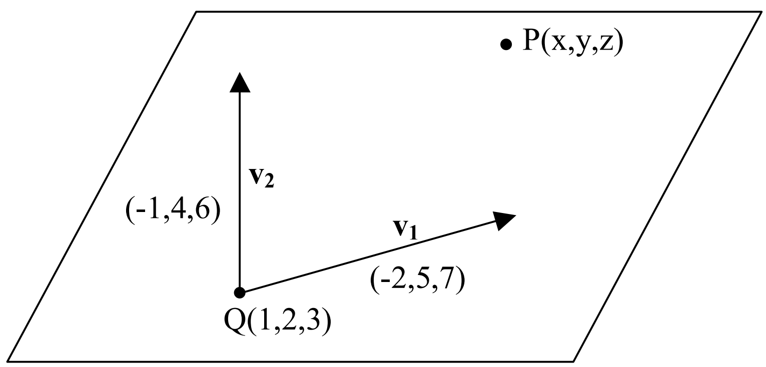
Chapter Six

Equations of Planes

6.1 The set of all linear combinations of two l.i. vectors forms a plane. If we also know a point in the plane then these are sufficient to determine the plane. The two l.i. vectors are called **direction vectors** of the plane.

Example

Find an equation of a plane containing point Q(1,2,3) and having direction vectors $\mathbf{v_1} = (\mathbf{-2,5,7})$ and $\mathbf{v_2} = (\mathbf{-1,4,6})$.



Let P(x,y,z) be an arbitrary point in the plane.

i.e. **QP**, $\mathbf{v_1}$. $\mathbf{v_2}$ are coplanar.

i.e. $\mathbf{AP}\bullet\mathbf{v_1} \times \mathbf{v_2} = 0$. Theorem 4.3

i.e. $(\mathbf{x-1,\ y-2,\ z-3})\bullet(\mathbf{-2,5,7}) \times (\mathbf{-1,4,6}) = 0$

i.e. $(\mathbf{x-1,\ y-2,\ z-3})\bullet(\mathbf{2,5,-3}) = 0$

i.e. $2x - 2 + 5y - 10 - 3z + 9 = 0$

i.e. $2x + 5y - 3z = 3$ is the equation of the plane.

Note: i) In general if $\mathbf{v_1}$ and $\mathbf{v_2}$ are direction vectors and (x_1,y_1,z_1) is a given point in the plane, then the equation of the plane is $(\mathbf{x-x_1,\ y - y_1,\ z - z_1})\bullet\mathbf{v_1}\times\mathbf{v_2} = 0$.

ii) $(\mathbf{v_1}\times\mathbf{v_2})$ is perpendicular to $\mathbf{v_1}$ and $\mathbf{v_2}$. Theorem 4.1. Therefore $(\mathbf{v_1}\times\mathbf{v_2})$ is perpendicular to the plane. We call this a **NORMAL** to the plane. i.e. **n**.

iii) Note that co-efficients of x,y,z in the plane's equation are the components of a normal to the plane.

Theorem 6.1

If $\mathbf{v_1}$ and $\mathbf{v_2}$ are two direction vectors of a plane and if $\mathbf{n} = \mathbf{v_1}\times\mathbf{v_2}$ then $\mathbf{n}$ is perpendicular to every vector in the plane.

Proof

Let $\mathbf{w}$ be an arbitrary vector in the plane.

i.e. $\mathbf{w} = s\mathbf{v_1} + t\mathbf{v_2}$ for some scalars s and t.

$$
\begin{aligned}
\text{Then } \mathbf{n}\bullet\mathbf{w} &= \mathbf{n}\bullet(s\mathbf{v_1} + t\mathbf{v_2}) \\
&= s(\mathbf{n}\bullet\mathbf{v_1}) + \mathbf{t}(\mathbf{n}\bullet\mathbf{v_2}) \\
&= s(0) + t(0) \qquad \text{since } \mathbf{n} \text{ is perpendicular to } \mathbf{v_1} \text{ and } \mathbf{v_2} \\
&= 0
\end{aligned}
$$

$\therefore$ $\mathbf{n}$ is perpendicular to $\mathbf{w}$.

Theorem 6.2

If $\mathbf{n} = (\mathbf{A},\mathbf{B},\mathbf{C})$ is a normal to a plane, then the equation of the plane may be written in the form

$$Ax + By + Cz = D.$$

Proof

Let $Q(q_1,q_2,q_3)$ be a given point in the plane and P be an arbitrary point (x,y,z) in the plane.

Then $\mathbf{QP}$ is perpendicular to $\mathbf{n}$.

i.e. $(\mathbf{x - q_1, y - q_2, z - q_3}) \bullet (\mathbf{A,B,C}) = 0$

i.e. $Ax - Aq_1 + By - Bq_2 + Cz - Cq_3 = 0$

i.e. $Ax + By + Cz = Aq_1 + Bq_2 + Cq_3$

i.e. equation is of the form $Ax + By + Cz = D$.

□

An interesting corollary of this theorem is the fact that if $\mathbf{n}$ is a normal to a plane and P is a point in the plane with position vector $\mathbf{p}$ then $\mathbf{n}\bullet\mathbf{p}$ is constant.

Example

Find an equation of a plane passing through E(1,2,3), F(2,5,4) and G(3,4,6).

Solution

A normal to the plane is **EF** $\times$ **FG**, i.e. (**1,3,1**)$\times$(**1,-1,2**).

i.e. (**7,-1,-4**)

$\therefore$ The equation is of the form 7x – y – 4z = D

Since E lies on the plane 7(1) – (2) – 4(3) = D

i.e. D = -7

The equation of the plane is 7x – y – 4z = -7.

The idea of the above example is to illustrate the value of knowing the components of a normal are the co-efficients of x,y,z in the equation. The example could equally well have been done by using **EP•EF**$\times$**FG** = 0 where P is an arbitrary point (x,y,z).

Exercise 6.1

1. Find a vector perpendicular to the plane passing through the three points (1,-1,2), (2,0,-1) and (0,2,1).
2. Find the area of the triangle formed by the three points in Question 1.
3. Find a vector perpendicular to x + y + z = 6.
4. Find a vector perpendicular to x + y + z = 2.
5. Find the equation for a plane containing point (1,2,3), (3,5,7) and (5,9,-1).
6. Find equation of the plane passing through (1,2,3) having (**3,2,-1**) and (**-1,4,2**) as two direction vector.
7. Find the equations for the planes satisfying the following conditions: -

   i) passing through (6,3,0) and (-2,0,-3) and parallel to **i** + **j** + **k**.

   ii) parallel to both $\frac{x}{3} = \frac{y-2}{-1} = z+4$ and $\frac{x-6}{3} = \frac{y-6}{5} = \frac{z-1}{4}$ and passing through (2,0,-1).

   iii) passing through (1,2,-3) and perpendicular to the line joining (4,5,6) to (-1,3,2).

   iv) containing $x-1 = \frac{y-1}{2} = \frac{z-1}{3}$ and x = 1; y = z.

   v) whose nearest point to the origin is (2,3,4).

   vi) whose nearest point to (1,2,3) is (2,3,4).

8. Find C so that 2x + 3y + Cz = 8 is parallel to $\frac{x-1}{2} = \frac{y-2}{3} = z+1$

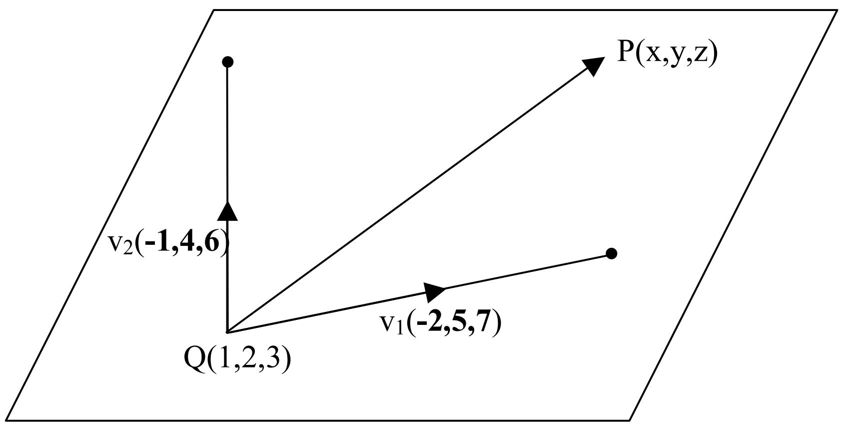
<u>Answers to Exercise 6.1</u>

1. **(2, 1, 1)** 2. $2\sqrt{6}$ 3. **(1, 1, 1)** 4. **(1, 1, 1)** 5. $20x-12y-z=-7$

6. $8x-5y+14z=40$ 7. (i) $y-z=3$ (ii) $x+y-2z=4$ (iii) $5x+2y+4z=-3$

(iv) $x+y-z=1$ (v) $2x+3y+4z=29$ (vi) $x+y+z=9$ 8. $C=-13$

6.2 **Vector and Parametric Equations for a Plane**



In the previous example, we could develop a vector equation for the plane as follows:

$\mathbf{QP} = s\mathbf{v_1} + t\mathbf{v_2}$

i.e. $\mathbf{OP} - \mathbf{OQ} = s(\mathbf{-2,5,7}) + t(\mathbf{-1,4,6})$

i.e. $(\mathbf{x,y,z}) - (\mathbf{1,2,3}) = s(\mathbf{-2,5,7}) + t(\mathbf{-1,4,6})$

i.e. $(\mathbf{x,y,z}) = (\mathbf{1,2,3}) + s(\mathbf{-2,5,7}) + t(\mathbf{-1,4,6})$.

This is a vector equation for the plane.

Similarly $\left.\begin{aligned} x &= 1 - 2s - t \\ y &= 2 + 5s + 4t \\ z &= 3 + 7s + 6t \end{aligned}\right\}$ are the parametric equations for the plane.

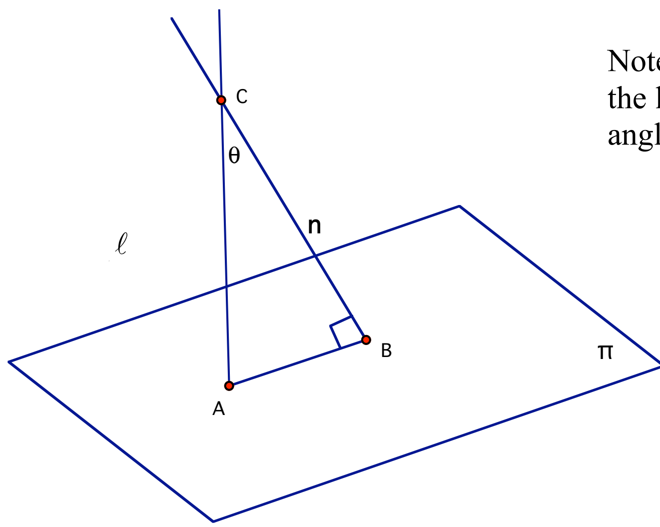
Eliminating s and t from these three equations would result in the equation $2x + 5y - 3z = 3$.
Note that the vector equation for a plane requires **two** parameters and two direction vectors because of the two dimensional nature of a plane.

Definitions

i) Two planes are parallel if their normals are parallel.

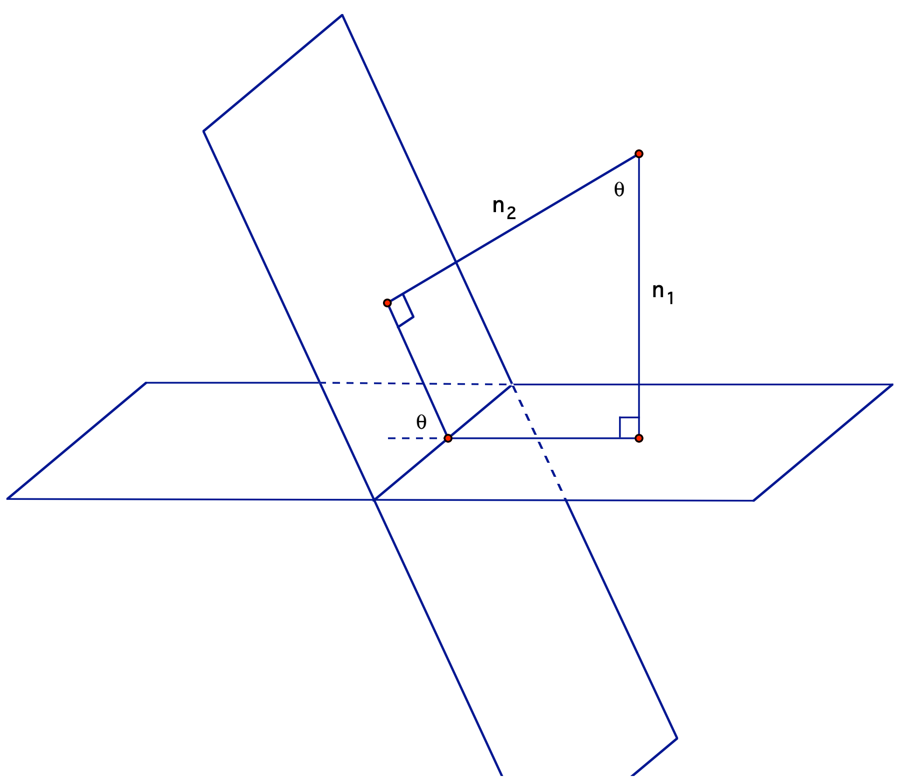
ii) Two planes are perpendicular if their normals are perpendicular

iii) The angle between a line and a plane is defined to be the complement of the angle which the line makes with a normal to the plane, i.e. the angle between line ℓ and plane π is $90 - \theta$.



Note: the angle between the line and the plane is angle CAB.

iv) The acute angle, θ, of intersection of two planes is defined to be the acute angle between their normals.



Exercise 6.2

1. Find the angle of intersection of $x = \frac{y}{2} = \frac{z}{3}$ and $x + y + z = 0$.
2. Show that $x = t$, $y = 4 + 2t$, $z = 5 + 2t$ is parallel to the plane $4x - y - z = 6$.
3. Find the point on $x + 2y + 3z = 21$ nearest to the origin
4. Find the equation of the plane which bisects at right angles the line joining the points (4,6,10) and (8,6,8)
5. Find the equation of the plane containing $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}$

   and $x - 4 = \frac{y - 5}{2} = \frac{z-6}{3}$.
6. Find C so that $2x + 3y + Cz = 8$ is perpendicular to $2x + 3y + z = 8$.
7. Find D so that $2x + 3y + Cz = 8$ is parallel to $Cx + Dy + 8z = 8$.
8. Find the angle between $x + y + 4z = 6$ and $2x - y + 2z = 3$.
9. Can all planes in R^3 be represented by equations of the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$?
10. Let p be the plane $9x - 2y - 24z = 17$.

    Let π be the plane $2x - 3y + z = 4$.

    Let ℓ be the line $x = 2y + 1 = 3z + 2$

    i) Show that p contains ℓ

    ii) Show that p is perpendicular to π

    iii) Find the equation of the plane perpendicular to both p and π and which passes through the origin.

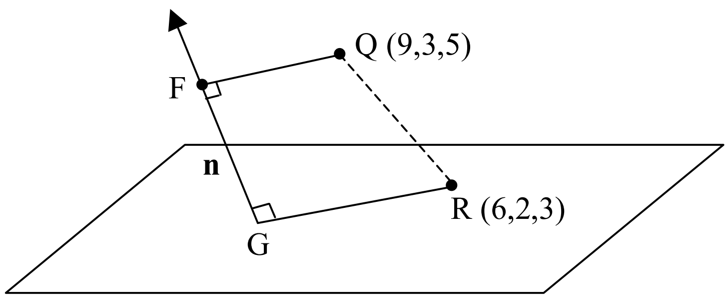
Exercise 6.2 Answers

1. 67.8°
3. $((1\frac{1}{2}, 3, 4\frac{1}{2})$
4. $2x - z = 3$
5. $x - 2y + z = 0$
6. C = -13
7. $D = \pm\ 6$.

8. 45°
9. No. e.g. $x + y + z = 0$
10. iii) $74x + 57y + 23z = 0$.

6.3 **Distance Between a Point and a Plane**

To find distance from Q (9,3,5) to the plane $2x + 3y + 4z = 30$.



Let R be **any** convenient point on $2x + 3y + 4z = 30$. We will let R be (6,2,3) in this case.

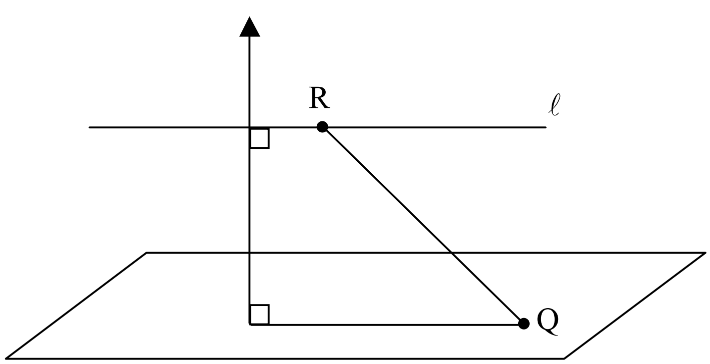
> Then the distance of Q from the plane is the length of the projection of **RQ** onto a normal of the plane.

i.e. in our example the distance of Q from the plane is the distance FG (namely the length of the projection of **RQ** onto **n**). See diagram.

$$\text{i.e. distance } = \frac{|\mathbf{RQ} \bullet \mathbf{n}|}{|\mathbf{n}|} = \frac{|(\mathbf{3,1,2}) \bullet (\mathbf{2,3,4})|}{|(\mathbf{2,3,4})|} = \frac{17}{\sqrt{29}} = 3.157(\text{approx})$$

> i.e. distance of a point Q from a plane $= \dfrac{|\mathbf{RQ} \bullet \mathbf{n}|}{|\mathbf{n}|}$
> where R is any point on the plane with normal **n**

To find distance between a plane and a parallel line.



We merely let R and Q be a point on the line ℓ and the plane respectively and the distance between the line and plane is then $\dfrac{|\mathbf{RQ} \bullet \mathbf{n}|}{|\mathbf{n}|}$

Example

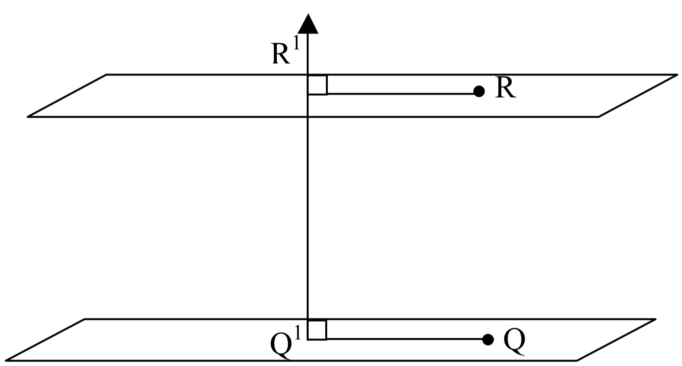
To find distance between $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-2}{4}$ and $x - 6y + 4z = 8$.

Let R be (1,-1. 2) and Q be (0,0,2). i.e. $\mathbf{RQ} = (\mathbf{-1,1,0})$

Distance = $\dfrac{|\mathbf{RQ} \bullet \mathbf{n}|}{|\mathbf{n}|} = \dfrac{|(\mathbf{-1,1,0}) \bullet (\mathbf{1,-6,4})|}{|(\mathbf{1,-6,4})|} = \dfrac{7}{\sqrt{53}}$ = .96 approx.

N.B. Be careful to check that the line is parallel to the plane!

To find distance between two parallel planes



Again let R and Q be a point on each plane and the distance is then again $\dfrac{|\mathbf{RQ} \bullet \mathbf{n}|}{|\mathbf{n}|}$

As before, the projection of **RQ** onto the common normal is R^1Q^1 in diagram. This is the distance between the planes.

Example

Find the distance between $2x + y + z = 3$ and $2x + y + z = 6$. Let R be (0,0,3) and Q be (0,0,6), i.e. $\mathbf{RQ} = (\mathbf{0,0,3})$.

Distance = $\dfrac{|(\mathbf{0,0,3}) \bullet (\mathbf{2,1,1})|}{|(\mathbf{2,1,1})|} = \dfrac{3}{\sqrt{6}}$ = 1.22 approx.

Note, in general, distance between $Ax + By + Cz = D$ and $Ax + By + Cz = D^1$ is

$$\frac{|D - D^1|}{\sqrt{A^2 + B^2 + C^2}}$$

Exercise 6.3

1. Find distance between (0,1,1) and (1,2,3)
2. Find distance from (1,1,1) to $x + 2y + 3z = 4$.
3. Find distance from (3,0,3) to $2x - y + z = 8$.
4. Find distance $(\mathbf{x,y,z}) = (\mathbf{1,1,1}) + t(\mathbf{1,1,-1})$ to $x + 2y + 3z = 4$.
5. Find distance between $x + 2y + 3z = -4$ and $x + 2y + 3z = 6$.
6. Find distance from $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ to $x + y + z = 0$.
7. Find m so that $x = 3 + t$, $y = 4 + 2t$, $z = 5 + mt$ is –
   i) parallel to $4x - y - z = 6$.
   ii) Find distance between the line and the plane.
   iii) Find equation for a plane perpendicular to the plane $4x - y - z = 6$ but parallel to the line and $5\sqrt{2}$ units from the line.
8. Show that the formula derived for distance of a point (x_1, y_1, z_1) from a plane $Ax + By + Cz = D$ in Section 6.3 is the same as $\left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$

Exercise 6.3 Answers

1. $\sqrt{6}$

2. $\frac{\sqrt{14}}{7}$

3. $\frac{1}{\sqrt{6}}$

4. $\frac{\sqrt{14}}{7}$

5. $\frac{10}{\sqrt{14}}$

6. Distance is zero. (line is not parallel to plane)

7. i) $m = 2$ ii) $\frac{1}{\sqrt{2}}$ iii) $y - z = 9$ or $y - z = -11$

6.4 **Intersection of plane and a non-parallel line.**

To find the intersection of a plane and a non-parallel line we substitute the parametric form for the line into the equation of the plane and solve for the parameter.

Example

To find intersection of $\frac{x-1}{2} = \frac{y-2}{3} = z+1$ and x + 2y – z = 13.

Solution

Parametric form for the line is :
$$\left.\begin{aligned} x &= 2t + 1 \\ y &= 3t + 2 \\ z &= t - 1 \end{aligned}\right\}$$

and substituting into the equation for the plane we get –

(2t + 1) + 2(3t + 2) – (t – 1) = 13

i.e. 7t = 7 and t = 1

The point of intersection is hence (3,5,0)

Intersection of two non-parallel planes

Two non-parallel planes will intersect in a line.

One method of finding symmetric equations of the line of intersection would be to eliminate one of the variables from the two equations for the planes and then insert a parameter for one of the two remaining variables and then solve for each variable in terms of that parameter. The **recommended method** however, is as follows.

Example

To find intersection of 2x + y + 3z = 7 and x – y + z = 5.

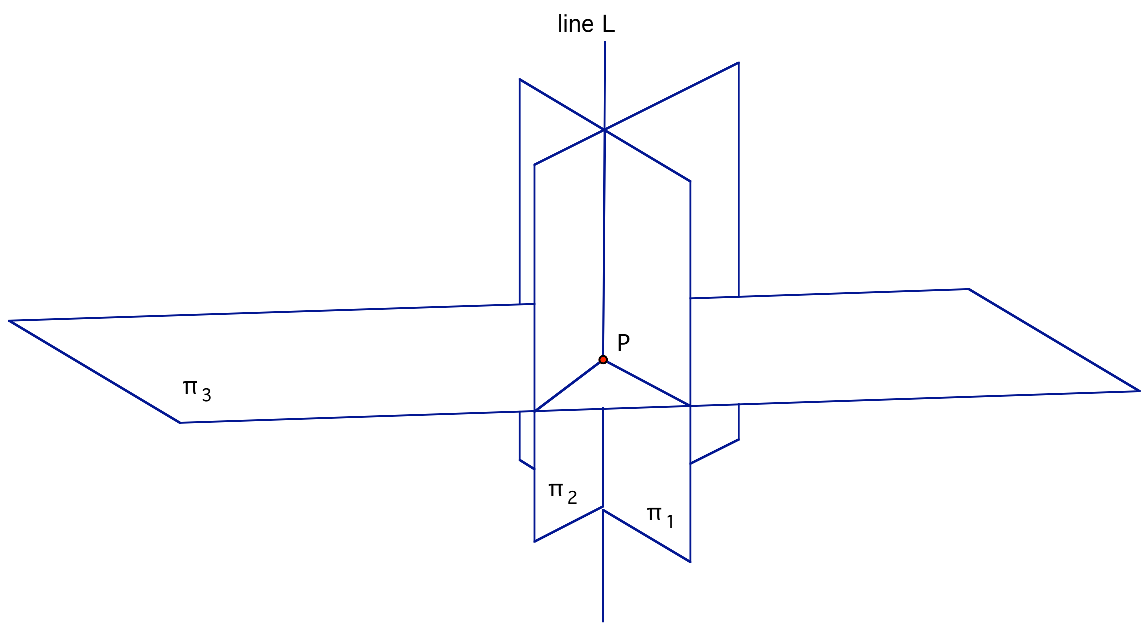
Note that the cross product of the normals of the two planes will be a vector contained in both planes and hence a direction vector of the line of intersection. For example **(2,1,3)** × **(1,-1,1)** = **(4,1,-3)** is a direction vector of the line of intersection.

We then find, by any method, a single point common to both planes, e.g. letting x = 0 and solving for y and z yields the point (0,-2,3). This is then a point on the line of intersection.

i.e. $\frac{x}{4} = \frac{y+2}{1} = \frac{z-3}{-3}$ is the line of intersection.

Intersection of Three Planes

Many situations can occur when three planes intersect. The most common is a single point of intersection as shown –



The point of intersection is P.

Let the planes be π_1, π_2, π_3 with normals $\mathbf{n_1}$, $\mathbf{n_2}$ and $\mathbf{n_3}$. The intersection of π_1 and π_2 will give a line whose direction vector is $\mathbf{n_1} \times \mathbf{n_2}$. This is line L in the diagram. If this line of intersection L is **not** parallel to π_3 then clearly it will intersect π_3 in a point (as P in diagram) and hence intersection of the three planes will be a point.

Interpreting the foregoing algebraically gives the following.

$\mathbf{n_1} \times \mathbf{n_2}$ is not parallel to π_3

means $\mathbf{n_1} \times \mathbf{n_2}$ is not perpendicular to $\mathbf{n_3}$.

and hence $(\mathbf{n_1} \times \mathbf{n_2}) \cdot \mathbf{n_3} \neq 0$.

This guarantees the intersection is a point.

i.e.

$$\boxed{\mathbf{n_1} \times \mathbf{n_2} \cdot \mathbf{n_3} \neq 0 \Rightarrow \text{unique intersection of three}}$$

Compare this result with Theorem 4.3.

Three planes satisfying the above condition are called linearly independent because their normals are linearly independent.

Three planes which are dependent, i.e. where $\mathbf{n_1} \times \mathbf{n_2} \bullet \mathbf{n_3} = 0$ can intersect in many ways as we now see.

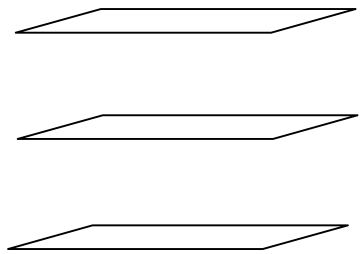
Example

To find intersection of

$x + y + z = 1$

$x + y + z = 2$

$x + y + z = 3$



Three parallel planes with clearly no intersection

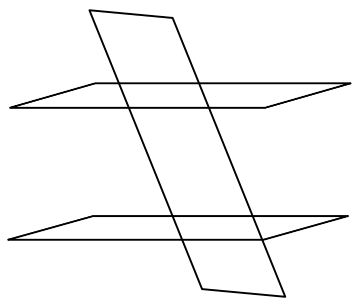
$\mathbf{n_1} \times \mathbf{n_2} \bullet \mathbf{n_3} = 0$

Example

$x + y + z = 1$ (1)

$x + y + z = 2$ (2)

$x + y + z = 3$ (3)



Here we have two parallel lines with a third non-parallel plane. Intersection is again the empty set. $\mathbf{n_1} \times \mathbf{n_2} \bullet \mathbf{n_3} = 0$.

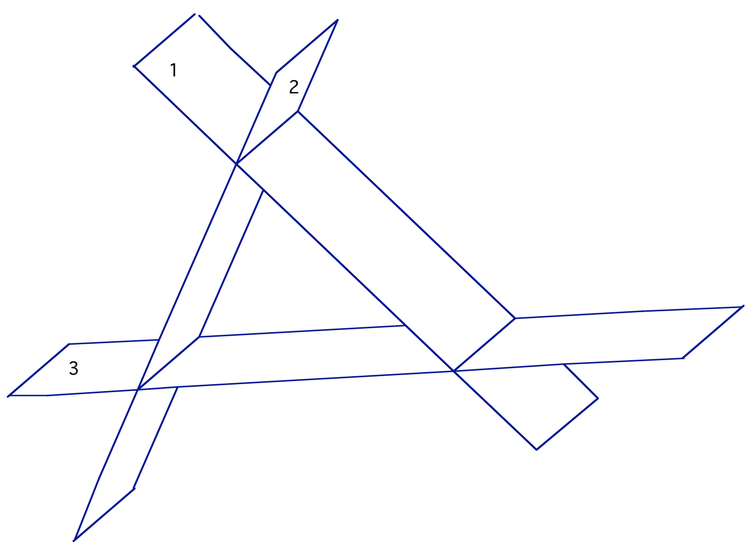
Example

$x + y + z = 0$ (1)

$x + 2y + 3z = 0$ (2)

$2x + 3y + 4z = 2$ (3)

The intersection of plane 1 and plane 2 is a line $\left(x = \frac{y}{-2} = z\right)$ which is parallel to, but not intersecting, plane 3. The intersection of the three planes is again the empty set $\varnothing$.
Note: $n_1 \times n_2 \bullet n_3 = 0$

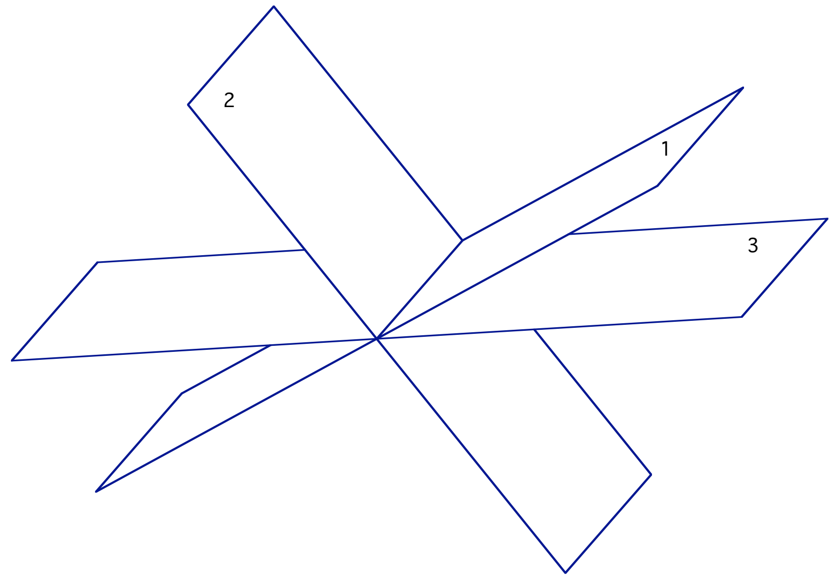


Example

x + y + z = 0   (1)

x + 2y + 3z = 0   (2)

2x + 3y + 4z = 0 (3)



Here, the intersection of planes (1) and (2) is again a line $x = \frac{y}{-2} = z$ as in the previous example, but the whole line of intersection is contained within plane (3) and so the intersection of the three planes is the line $x = \frac{y}{-2} = z$. Again, $\mathbf{n_1} \times \mathbf{n_2} \bullet \mathbf{n_3} = 0$.

To find the intersection point of three linearly independent planes

Example

x + y + z = 0 (1)

x + 2y + 3z = 0 (2)

2x + 3y + 5z = 3 (3)

We could solve these three equations using simultaneous equations techniques of elementary algebra. An alternative, but not necessarily recommended method, is as follows.

We test $\mathbf{n_1} \times \mathbf{n_2} \bullet \mathbf{n_3}$ and find it is not zero (in fact it is $(\mathbf{1,-2,1}) \bullet (\mathbf{2,3,5}) = 1$) and hence we know that the intersection is a common point of intersection of planes (1) and (2). We hence deduce that the line of intersection of planes (1) and (2) is $x = \frac{y}{-2} = z$.

Parametrically this is represented:

x = t, y = -2t, z = t.

Substituting in equation (3) yields

2t + 3(-2t) + 5t = 3

i.e. point of intersection is (3,-6,3).

Exercise 6.4

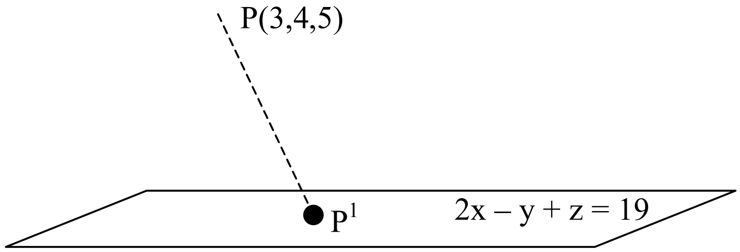
1. Find intersection of x + y + z = 1, x + 3y + 5z = 5 and x + 5y + 9z = 9.
2. Do x + y + z = 1, x + y + 2z = 1, 2x + 2y + 3z = 3 intersect? If so, where?
3. Find distance between x + 2y + 3z = 4 and $\frac{x-1}{2} = \frac{y-1}{-1}; z = 1$
4. Find the acute angle of intersection of x + y + 4z = 6 and 2x – y + 2z = 3.
5. Find the intersection of 4x + 2y + 6z = 14 and x – y + z = 5.
6. Find the scalar equation of a plane passing through (-1,1,2), (2,3,2) and (3,0,4).
7. Find the equation of the plane containing the point (1,2,-3) and perpendicular to the line $\frac{x-4}{2} = \frac{y+3}{3} = \frac{z}{-4}$
8. Find the vector equation of a plane containing the points (2,-1,2) and (3,0,1) and perpendicular to the plane 2x – 3y + z = 2.
9. Find intersection of x + y + z = 3, 2x – y + 3z = 4 and x + 2y – z = 2.
10. Find the equation of the plane containing the line $\frac{x+1}{2} = y = \frac{z-4}{-3}$ and which passes through the origin.
11. Find the number of solutions of the equations x – 2y + 3z = D, x + y – 2z = 8, Ax – 3y + 4z = 10 if
    - i) $A \neq 3$ and D = 1
    - ii) A = 3 and D = 1
    - iii) A = 3 and D = 2

Exercise 6.4 Answers

1. **(x,y,z)** = (**-1,2,0**) + t(**1,-2,1**)
2. No.
3. $\frac{2}{\sqrt{14}}$
4. 45°
5. $\frac{x}{4} = \frac{y+2}{1} = \frac{z-3}{-3}$
6. 4x – 6y – 11z = -32

7. $2x + 3y - 4z = 20$.
8. $(\mathbf{x,y,z}) = (\mathbf{2,-1,2}) + s(\mathbf{1,1,-1}) + t(\mathbf{2,-3,1})$
9. (1,1,1)
10. $4x - 5y + z = 0$
11. i) one ii) infinite iii) zero.

To find the projection of point P(3,4,5) onto plane 2x – y + z = 19.



Let projection of P onto the plane be P^1, thus PP^1 will be a multiple of the normal $(\mathbf{2,-1,1})$ to the plane

Let P^1 be (p_1, p_2, p_3)

Then $2p_1 - p_2 + p_3 = 19$ (1) since P^1 is on the plane.

And $(\mathbf{p_1 - 3, p_2 - 4, p_3 - 5}) = m(\mathbf{2, -1, 1})$ (2) since $PP^1 = m\mathbf{n}$

i.e. $p_1 = 2m + 3$, $p_2 = - m + 4$, $p_3 = m + 5$.

Substituting into (1) yields

$4m + 6 + m - 4 + m + 5 = 19$

i.e. $m = 2$ and hence $p_1 = 7$, $p_2 = 2$, $p_3 = 7$

i.e. $P^1 = (7,2,7)$

To find the reflection of point P (3,4,5) in the plane 2x - y + z = 19

Let the reflection of point P in the plane R. Using the method of the last example and noting that P^1 is the mid point of PR it is easy to deduce that R is (11,0,9). Alternatively we could say $(\mathbf{2,-1,1})$ is a normal to the plane and so the line through P perpendicular to the plane is $\frac{x-3}{2} = \frac{y-4}{-1} = \frac{z-5}{1}$. A general point on the line is $(2s + 3, -s + 4, s + 5)$.

Since P^1 is on the plane, co-ordinates of P^1 have values occurring when

$2(2s + 3) - (-s + 4) + (s+5) = 19$. Solving this yields $s = 2$.

$\therefore$ Co-ordinates of R have values occurring when s = 4, i.e. R is (11,0,9)

Exercise 6.5

1. Find reflection of (-2,1,3) in the plane $x - y + z = 3$.
2. Does the line $x = y = z$ intersect the plane $z = 1$? Explain.
   Does $x = y = z = w$ in R^4 intersect $x = y = z$ in R^4?
3. Find the projection of point (-2,1,3) onto the plane $2x + y - z = 5$.
4. Show (1,0,0), (0,2,0), (0,0,3), $(\frac{2}{5},\frac{4}{5},\frac{3}{5})$ are coplanar.
5. Find reflection of (1,0,1) in the plane $2x + y - z = 4$.
6. Find length of projection (**2,3,5**) onto the plane $x + z = 8$.
7. Find projection of point (2,3,5) onto the plane $x + z = 8$
8. Prove (1,0,0), (0,1,0), (0,0,1), (m, n, 1-m-n) are coplanar points.
9. Tetrahedron OABC has vertex O at the origin. A is (-1,0,1). B is (0,-1,1). $\angle AOC = \angle BOC = 60°$. $OC^2 = OA^2 + OB^2$. find the co-ordinates of the two possible positions of C and show $OC \perp AB$.
10. Find the maximum value of $x + 2y + z$ where x,y,z are restricted to satisfy $x^2 + y^2 + z^2 = 24$.
11. Find an equation of the plane tangent to the sphere $x^2 + y^2 + z^2 = 169$ at the point (3,4,12)

Exercise. 6.5 Answers

1. (0,-1,5)
2. yes
3. $(\frac{10}{6},\frac{17}{6},\frac{7}{6})$
5. (3,1,0)
6. $\frac{3\sqrt{6}}{2}$
7. $(2\frac{1}{2},3,5\frac{1}{2})$
9. $(-\sqrt{2},\sqrt{2},0)$
10. 12
11. $3x + 4y + 12z = 169$

Practice Test on Chapter 6

1. Find the shortest distance between x = y = z and x – 2y + z = 4
2. Which plane is nearer the origin, x + 2y + 3z = $\sqrt{2}$ or x + 2y + 4z = $\sqrt{3}$
3. Find an equation for the plane containing the point (1,1,2) (2,1,1) and (1,2,0)
4. Find an equation for the plane parallel to x + 2y + 2z = 3 and which is 4 units from it, farther from the origin.
5. Find a point which is equidistant from x + y + z = 1 and x + y + z = 3
6. Find an equation of the plane which is perpendicular to the plane x = z, which passes through the point (1, $\sqrt{2}$, 3) and whose normal makes an angle of 60° with the x-axis.
7. Find the values of m and n so that the intersection of the three planes
   x + y + z = 0 is a line. State equations for the line.
   x + 2y + 2z = 1
   2x – y + mz = n
8. Find the point on the plane x + 2y + 2z = 18 which is nearest to the point (1,2,2).

Answers

1. 1.63 (approx) 2. They are equidistant from the origin.3. x + 2y + z = 5.
4. x + 2y + 2z = 15 5. Any point on the plane x + y + z = 2 will suffice.
6. x + $\sqrt{2}$ y + z = 6 or x - $\sqrt{2}$ y + z = 2 7. m – 01, n = -3. Equations for the line are
x = -1; y + z = 1 8. (2,4,4)

Practice Test 2 on Chapter 6

1. Find the shortest distance between x = y = z and x – 2y + z = $\sqrt{6}$
2. Find the angle of intersection of x – y + z = 3 and 2x + y + z = 3
3. Find the reflection of the point (1,2,3) in the plane x + y + z = 9
4. Describe in geometric terms, in vector terms or in the form of equation(s), the set of all points 1 unit from 2x – 2y + z = 0 and 1 unit from x + 2y + 2z = 0
5. Find an equation of the plane containing x = y =z and x = 2y = 2z

6. Let **n** be a vector. Describe the set of all points (x,y,z) so that **(x,y,z)•n** = 3. Describe the set of all points (x,y,z) so that **(x,y,z)**×**n** = **(-5,1,1)**
7. A square piece of paper ABCD of side length 10 cms is folded along the diagonal BD so that the planes of the triangles ABD and CBD are perpendicular. Find the shortest distance between the edges AB and CD.

Answers

1. 1
2. 61.87°
3. (3,4,5)
4. **(x,y,z) = (1,0,1) +** t**(2,1,-2)**
   (x,y,z) = (-1,0,-1) + t**(2,1,-2)**
   (x,y,z) = (1,-1,-1) + t**(2,1,-2)**
   (x,y,z) = (1,2,-1) + t**(2,1,-2)**
5. y = z
6. a) a plane b) a line
7. $\frac{10\sqrt{2}}{3}$

Summary Vector Test on Chapters 1 – 6

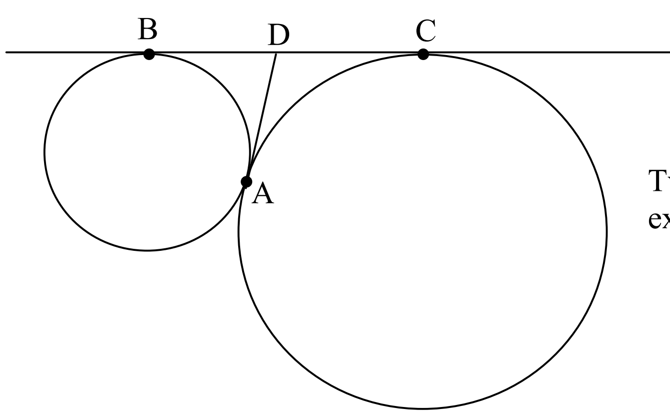
1. Find equations of the plane containing $x - 1 = \frac{y-1}{2} = \frac{z-1}{3}$ and x = 2; y = z – 1.
2. Find m so that 2x + 3y + mz = 8 is parallel to $\frac{x-1}{2} = \frac{y-2}{m} = z+1$
3. Find the distance between the two lines x = y – 1; z = 0 and $x - 1 = \frac{y}{2}$; z = 1.
4. Find the distance from (1,1,1) to x + y + z = 1
5. Find equation of the plane containing (1,1,1), (1,-1,3) and (2,1,0).
6. Find the equation of the line through (1,5,1) intersecting the lines x =1; $\frac{y-2}{2} = z-2$ and $\frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{4}$
7. Find the reflection of (1,1,1) in the line $x = \frac{y}{2} = \frac{z}{3}$.

8. $\{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{i + 3j} + m\mathbf{k}\}$ is a dependent set of vectors. Find the value(s) of m.
9. Find the intersection of $x + y + z = 3$, $2x - y + z = 2$ and $3x - y + z = 3$.
10. find the length of projection of $(\mathbf{-2,2,3})$ on $x + y - z = 4$.
11. Find the equation of the plane through the origin parallel to the lines $(\mathbf{x,y,z}) = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} + s(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $(\mathbf{x,y,z}) = 4\mathbf{i} - 5\mathbf{j} - 8\mathbf{k} + t(3\mathbf{i} + 7\mathbf{j} - 6\mathbf{k})$. Show that one of the lines lies in the plane and find the distance of the other line from the plane.
12. Show that A,B,C are collinear iff $\mathbf{AC} \times \mathbf{AB} = \mathbf{0}$. Are (1,2,-3), (3,1,0), (-3,4, -9) collinear?
13. A,B,C are three non-collinear points with position vectors **a**,**b**,**c** respectively.
    i) Write down the position vector of point D so that ABCD is a parallelogram
    ii) If A,B,C, are (1,0,3), (2,2,3), (2,4,6) respectively find the area of parallelogram ABCD.
    iii) Find volume of pyramid ABCDV where V is the point (5,1,11)
14. $(\mathbf{x,y,z}) = (\mathbf{2,1,1}) + r(\mathbf{1,0,1})$

    $(\mathbf{x,y,z}) = (\mathbf{3,-1,2}) + s(\mathbf{0,1,0})$

    $(\mathbf{x,y,z}) = (\mathbf{m,m,1}) + t(\mathbf{1,m,1})$

    are coplanar lines. Find m and an equation for the plane.
15.



Two circles are tangent externally at A.

B-D-C is a common external tangent and AD is a common internal tangent as shown in diagram. Prove by a vector method that $\angle BAC = 90°$.

16 .**v** makes an angle of 30° with the x-axis and an angel of 70° with the y-axis.

i. Can **v** be a vector in R^2?

ii. If **v** is in R^3, find the angle which **v** makes with the z-axis.

Answers to Summary Vector Test

1. $x + y - z = 1$
2. $m = -1$
3. 1
4. $\frac{2}{\sqrt{3}}$
5. $x + y + z = 3$
6. $x = 1;\ \frac{y}{5} = \frac{z+1}{2}$
7. $(-\frac{1}{7}, \frac{5}{7}, \frac{11}{7})$
8. $m = 4$
9. (1,1,1)
10. $\sqrt{14}$
11. $2x + z = 0,\ \sqrt{5}$
12. Yes
13. i) $\mathbf{a} - \mathbf{b} + \mathbf{c}$ ii) 7 iii) $12\frac{1}{3}$
14. $m = 2,\ x - z = 1.$
16. i) no

    ii) 68.6°