## CHAPTER FIVE

## 5. Equations of Lines in $\mathbf{R}^{3}$

In this chapter it is going to be very important to distinguish clearly between points and vectors. Frequently in the past the distinction has only been a subtle one which has not borne heavily upon the concepts and one could almost think of the two ideas interchangeably. However, note that $(1,0,0),(0,1,0),(0,0,1)$ are coplanar, whereas $(\mathbf{1 , 0 , 0})$, $(\mathbf{0}, \mathbf{1}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{0}, \mathbf{1})$ most certainly are not.

Throughout this chapter we shall be dealing with points in the derived equations but will use vectors to derive those equations.

## Vector Equation of a Line in $\mathbf{R}^{\mathbf{3}}$

## Example

To find the equation of a line in $\mathrm{R}^{3}$ passing through the points $\mathrm{A}(1,2,3)$ and B(3,7,10)


Let P be an arbitrary point ( $\mathrm{x}, \mathrm{y}, \mathrm{z}$ ) on the line.
Note then that $\mathbf{A P}=\mathrm{t} \mathbf{A B}$ where t is some scalar.
But $\mathbf{O P}=\mathbf{O A}+\mathbf{A P}$
It therefore follows that $\mathbf{O P}=\mathbf{O A}+\mathrm{tAB}$

$$
\text { i.e. }(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1 , 2 , 3})+\mathrm{t}(\mathbf{2}, \mathbf{5}, 7)
$$

This is called the vector equation of a line. T is called a PARAMETER.
Note clearly that this equation means the set of all POINTS ( $\mathrm{x}, \mathrm{y}, \mathrm{z}$ ) which satisfy the condition that the VECTOR $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1 , 2 , 3})+t(\mathbf{2}, \mathbf{5}, \mathbf{7})$
i.e. the line passing through $(1,2,3)$ and $(3,7,10)$ is the set of all points in $\{(\mathrm{x}, \mathrm{y}, \mathrm{z}) \mid(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1}, \mathbf{2}, \mathbf{3})+\mathrm{t}(\mathbf{2}, \mathbf{5}, \mathbf{7})\}$.
AB i.e. $(\mathbf{2}, \mathbf{5}, \mathbf{7})$ in our example, is called the DIRECTION VECTOR of the line and can be thought of rather loosely as "the slope of the line".

## Parametric Equations For a Line in $\mathbf{R}^{\mathbf{3}}$

Continuing the previous example, we note that the vector equation for the line could be rewritten

$$
\left.\begin{array}{rl}
(\mathbf{x}, \mathbf{y}, \mathbf{z}) & =(\mathbf{1}+\mathbf{2 t}, \mathbf{2}+\mathbf{5 t}, \mathbf{3}+7 \mathrm{t}) \\
\text { i.e. } \mathrm{x} & =1+2 \mathrm{t} \\
\mathrm{y} & =2+5 \mathrm{t} \\
\mathrm{z} & =3+7 \mathrm{t}
\end{array}\right\}
$$

The set of three equations together forms what is called the PARAMETRIC
EQUATIONS for the line in $\mathrm{R}^{3}$.

## Symmetric Equations for a Line in $\mathbf{R}^{\mathbf{3}}$

If in the previous example we solve the equations in terms of t , we get

$$
\begin{aligned}
& t=\frac{x-1}{2} \\
& t=\frac{y-2}{5} \\
& t=\frac{z-3}{7}
\end{aligned}
$$

Rewriting these yields $\frac{x-1}{2}=\frac{y-2}{5}=\frac{z-3}{7}(=\mathrm{t})$
These are called SYMMETRIC EQUATIONS for a line in $\mathrm{R}^{3}$.

Note:
i) that the denominators in order form the components of the direction vector viz $(2,5,7)$ and the numerators tell us that the line passes through $(1,2,3)$. For example, symmetric equations for the line passing through $(-4,5,7)$ with direction vector $(3,-6,8)$ would be $\frac{x+4}{3}=\frac{y-5}{-6}=\frac{z-7}{8}$.
ii) that any multiple of the direction vector will serve as the denominators of the symmetric equations.
For example, $\frac{x+4}{5}=\frac{y-5}{-12}=\frac{z-7}{16}$ represents exactly the same line as the example quoted in note i)
iii) the co-ordinates of the point quoted are not unique. The co-ordinates of any point on the line will suffice. For example $\frac{x-2}{6}=\frac{y+7}{-12}=\frac{z-23}{16}$ also represents the same line as in note i) and ii) because $(2,-7,23)$ lies on the line. If the direction vector makes angles of $\alpha, \beta, \gamma$ with lines parallel to $\mathrm{x}, \mathrm{y}, \mathrm{z}$ axes respectively, then the direction vector may be written $(\cos \alpha, \cos \beta, \cos \gamma)$.

Furthermore, if a point through which line passes is $\left(\mathrm{x}_{1}, \mathrm{y}_{1}, \mathrm{z}_{1}\right)$ then symmetric equations for the line would be

$$
\frac{\mathrm{x}-\mathrm{x}_{1}}{\cos \alpha}=\frac{\mathrm{y}-\mathrm{y}_{1}}{\cos \beta}=\frac{\mathrm{z}-\mathrm{z}_{1}}{\cos \gamma}
$$

## Example

Find vector, parametric and symmetric equations for a line passing through the point $(-4,2,1)$ parallel to the z axis.

A direction vector of the line is $(\mathbf{0}, \mathbf{0}, \mathbf{1})$
Therefore a vector equation is: -
$(\mathbf{x}, \mathbf{y}, \mathbf{z})=(-\mathbf{4}, \mathbf{2}, \mathbf{1})+\mathrm{t}(\mathbf{0}, \mathbf{0}, 1)$.
Also the parametric would be -

$$
\begin{aligned}
& x=-4 \\
& y=2 \\
& x=1+t
\end{aligned}
$$

Note that the symmetric equations cannot be written in the form of the previous example since there would be denominators of zero, i.e. $\alpha=\beta=90^{\circ}$.

Hence we write $\mathrm{x}=-4 ; \mathrm{y}=2 ; \frac{\mathrm{z}-1}{1}=\mathrm{t}$
Usually we simply write $\mathrm{x}=-4 ; \mathrm{y}=2$.

## Example

Find symmetric equations for a line passing through ( $1,2,3$ ), parallel to the xy plane and intersecting the z axis.


The line intersects the z axis at $(0,0,3)$ and hence a direction vector of the line can be thought of as $(\mathbf{1 , 2 , 0})$.
i.e. a vector equation for the line is -

$$
(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1 , 2 , 3})+\mathrm{t}(\mathbf{1 , 2 , 0})
$$

with resulting parametric equations -

$$
\begin{aligned}
& x=1+t \\
& y=2+2 t \\
& z=3
\end{aligned}
$$

Again we cannot write symmetric equations in the usual form and we write -

$$
\frac{x-1}{1}=\frac{y-2}{2} ; z=3
$$

## Example

Find an intersection point (if any) of -

$$
\frac{x-6}{2}=\frac{y-7}{2}=\frac{z-6}{3} \text { and } \frac{x-2}{1}=\frac{y-13}{-4}=\frac{z+7}{5}
$$

We use the parametric form of the two equations, viz.

$$
\begin{array}{ll}
x=6+2 t(1) & x=2+s(1)^{1} \\
y=7+2 t(2) & y=13-4 s(2)^{1} \\
z=6+3 t(3) & z=-7+5 s(3)^{1}
\end{array}
$$

Note that we use different parameters for two different lines.
Equating (1) with (1) ${ }^{1}$ and (2) with (2) ${ }^{1}$ leads to a solution that $\mathrm{s}=2$ and $\mathrm{t}=-1$.
Substituting these values of the parameters into (3) and (3) yields the fact that the $z$ values in both these equations are equal.
It therefore follows that $s=2, t=-1$ is the solution for the three pairs of equations, the lines do intersect and they intersect at $(4,5,3)$.
Note that it does not suffice to solve merely two pairs of equations since the values of the parameters would not necessarily satisfy the third pair of equations. In fact in general they will not, since usually two lines in $\mathrm{R}^{3}$ do not intersect (in contrast to the situation in $R^{2}$ ).

Furthermore, it is a bad mistake (but nevertheless a frequent one) to equate corresponding parts of the symmetric equations, vis; $\frac{x-6}{2}$ and $\frac{x-2}{1}$ since they are equal to different parameters.

## Exercise 5.1

1. Find vector, parametric and symmetric equations for the following lines:
i) the line through $(1,2,3)$ and $(4,5,7)$
ii) the line through $(2,4,1)$ parallel to the $y$-axis.
iii) The line through $(0,1,4)$ parallel to $(\mathbf{2}, \mathbf{5}, \mathbf{- 1})$
iv) The line passing through the origin making angles of $60^{\circ}$ with both the $x$ and $y$ axes in $R^{3}$.
2. Show that the point $(4,1,2)$ lies on the line $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{2}, \mathbf{3}, \mathbf{1})+\mathrm{t}(\mathbf{2}, \mathbf{- 2}, \mathbf{1})$
3. Show that $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and $\frac{x-3}{4}=\frac{y-7}{8} ; z=-1$ intersect.
4. Find point of intersection of $\frac{x-1}{2}=\frac{y-1}{4}=\frac{z-1}{6}$ and $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1}, \mathbf{3}, \mathbf{1})+t(-1,2,-\mathbf{3})$
5. Find angle of intersection between -

$$
(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1}, \mathbf{1}, \mathbf{1})+\mathrm{t}(\mathbf{1}, \mathbf{0}, \mathbf{1}) \text { and } \mathrm{x}-1=\mathrm{y}-1 ; \mathbf{z}=1
$$

6. Find $m$ so that $\frac{\mathrm{x}-1}{2}=\frac{\mathrm{y}}{4}=\frac{\mathrm{z}}{3}$ and $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{0}, \mathbf{1}, \mathbf{0})+\mathrm{t}(\mathbf{1}, \mathbf{0}, \mathbf{m})$ intersect.
7. Find symmetric equations for a line parallel to $x=1+2 t, y=2+5 t, z=-t$ and passing through $(0,1,4)$.
8. Find equations of a line with direction vector perpendicular to both $(\mathbf{2}, \mathbf{- 1 , 3})$ and $(4,2,-1)$ and which passes through $(4,1,0)$
9. Is it possible for $\frac{x-3}{3 m}=\frac{y+1}{m+2}=\frac{z-3}{2}$ to be parallel to $\frac{x-1}{m+1}=\frac{y+5}{2}=\frac{z+2}{m-1}$ ?
10. Find angle of intersection between

$$
\begin{aligned}
& \mathrm{x}=1+\mathrm{s} \text { and }\{(\mathrm{x}, \mathrm{y}, \mathrm{z}) \mid \mathrm{x}=\mathrm{y} ; \mathrm{z}=1\} \\
& \mathrm{y}=1 \\
& \mathrm{z}=1+\mathrm{s}
\end{aligned}
$$

11. Find equations of a line passing through $(1,2,3)$ making an angle of $60^{\circ}$ with a line parallel to x axis and an angle of $45^{\circ}$ with a line parallel to y axis.

## Exercise 5.1 Answers

1. ii) $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{2}, \mathbf{4}, \mathbf{1})+\mathrm{t}(\mathbf{0}, \mathbf{1}, \mathbf{0})$

$$
\begin{aligned}
& \mathrm{x}=2 \\
& \mathrm{y}=4+\mathrm{t} \\
& \mathrm{z}=1 \\
& \mathrm{x}=2 ; \mathrm{z}=1 ; \mathrm{y}-4=\mathrm{t} .
\end{aligned}
$$

$$
\begin{aligned}
& \text { iii) }(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{0}, \mathbf{1}, \mathbf{4})+\mathrm{t}(\mathbf{2 , 5 , - 1}) \\
& \mathrm{x}=2 \mathrm{t} \\
& y=1+5 \mathrm{t} \\
& z=4-\mathrm{t} \\
& \frac{x}{2}=\frac{y-1}{5}=\frac{z-4}{-1} \\
& \text { iv) }(\mathbf{x}, \mathbf{y}, \mathbf{z})=\mathrm{t}(\mathbf{1}, \mathbf{1}, \pm \sqrt{\mathbf{2}}) \\
& \mathrm{x}=\mathrm{t} \\
& \mathrm{y}=\mathrm{t} \\
& z= \pm \sqrt{2} \mathrm{t} \\
& \mathrm{x}=\mathrm{y}=\frac{\mathrm{z}}{ \pm \sqrt{2}}=\mathrm{t}
\end{aligned}
$$

3. They intersect at $(-1,-1 .-1)$
4. $\left(1 \frac{1}{2}, 2,2 \frac{1}{2}\right)$
5. $60^{\circ}$
6. $\mathrm{m}=\frac{1}{2}$
7. $\frac{x}{2}=\frac{y-1}{5}=\frac{z-4}{-1}$
8. $\frac{\mathrm{x}-4}{-5}=\frac{\mathrm{y}-1}{14}=\frac{\mathrm{z}}{8}$
9. Yes, if $\mathrm{m}=2$.
10. $60^{\circ}$
11. $\mathrm{x}-1=\frac{\mathrm{y}-2}{\sqrt{2}}=\mathrm{z}-3$

### 5.2 Example

To find the vector equation of a line intersecting the line $\ell, \quad \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ at right angles and passing through point $\mathrm{A}(0,1,4)$.


Note it does not suffice to find a line perpendicular to $\ell$ passing through $(0,1,4)$ since this line might not intersect $\ell$. The solution depends upon finding a point B on $\ell$ so that $\mathbf{A B}$ is perpendicular to $\ell$.

The parametric equations for $\ell$ are:
$x=1+2 t, y=2+3 t, z=3+4 t$, therefore $B$ has co-ordinates $(1+2 t, 2+3 t, 3+4 t)$.
i.e. $\mathbf{A B}=(\mathbf{1}+\mathbf{2 t}, \mathbf{1}+\mathbf{3 t}, \mathbf{- 1 + 4 t})$

Since $\mathbf{A B}$ is perpendicular to $\ell$ then $\mathbf{A B}$ is perpendicular to the direction vector of $\ell$ i.e., $(\mathbf{2 , 3 , 4})$
$\therefore(1+2 t, 1+3 t,-1+4 t) \bullet(2,3,4)=0$
i.e. $1+29 t=0$
i.e. $t=-\frac{1}{29}$
$\therefore \mathbf{A B}=\left(\frac{27}{29}, \frac{26}{29},-\frac{33}{29}\right)$
i.e. $\left(\frac{\mathbf{2 7}}{\mathbf{2 9}}, \frac{\mathbf{2 6}}{\mathbf{2 9}},-\frac{\mathbf{3 3}}{\mathbf{2 9}}\right)$ is a direction vector of the required line.

More simply we can say (27,26,-33).
An alternative method for solving for the example 5.2 is as follows.


The direction of the line $A B$ which is $\perp$ to $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ where $B$ is on the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is

$$
\begin{aligned}
(\mathbf{A C} \times \mathbf{C D}) \times \mathbf{C D} & \text { i.e. }((\mathbf{1}, \mathbf{1},-\mathbf{1}) \times(\mathbf{2}, \mathbf{3}, 4)) \times(\mathbf{2}, \mathbf{3}, 4) \\
& =(\mathbf{7}, \mathbf{6},-\mathbf{1}) \times(2,3,4) \\
& =(-27,-26,33)
\end{aligned}
$$

$\therefore$ Equation of the line AB is $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{0}, \mathbf{1}, \mathbf{4})+\mathrm{t}(\mathbf{- 2 7}, \mathbf{- 2 6}, \mathbf{3 3})$

$$
\text { or } \frac{x}{-27}=\frac{y-1}{-26}=\frac{z-4}{33} .
$$

## Example

Find the distance from $\mathrm{A}(1,2,3)$ to the line $\ell, \quad \frac{\mathrm{x}-1}{2}=\frac{\mathrm{y}-1}{3}=\frac{\mathrm{z}-1}{2}$


## Solution



Let B and C be two distinct points on $\ell$.
We will let B be $(1,1,1)$ and C be $(3,4,3)$.
Now Area $\triangle \mathrm{ABC}=\frac{1}{2}|\mathbf{A B} \times \mathbf{B C}|$

$$
\text { Therefore, } \begin{aligned}
& \frac{1}{2}|\mathbf{B C}||\mathbf{A D}|=\frac{1}{2}|\mathbf{A B} \times \mathbf{B C}| \\
&|\mathbf{A D}|=\frac{|\mathbf{A B} \times \mathbf{B C}|}{|\mathbf{B C}|} \\
&=\frac{|(\mathbf{0}, \mathbf{- 1},=\mathbf{2}) \times(\mathbf{2}, \mathbf{3}, \mathbf{2})|}{|(\mathbf{2}, \mathbf{3}, \mathbf{2})|} \\
&=\frac{|(\mathbf{4}, \mathbf{- 4 2})|}{|(\mathbf{2}, \mathbf{3}, \mathbf{2})|} \\
&=\frac{6}{\sqrt{17}}
\end{aligned}
$$

The distance from A to line $\ell$ is $\frac{6}{\sqrt{17}}$
Note that the solution does not depend upon the choice of B and C, merely that they lie on line $\ell$.
In general, two lines in $\mathrm{R}^{3}$ do not intersect regardless of whether they are parallel or not.
For example -

$$
\begin{aligned}
(\mathbf{x}, \mathbf{y}, \mathbf{z}) & =(\mathbf{1 , 2 , 2})+\mathrm{t}(\mathbf{2}, \mathbf{3}, \mathbf{2}) \\
\text { and }(\mathbf{x}, \mathbf{y}, \mathbf{z}) & =(\mathbf{3}, \mathbf{4}, \mathbf{5})+\mathrm{s}(\mathbf{4}, \mathbf{5}, \mathbf{6})
\end{aligned}
$$

do not intersect, because if

$$
\left.\left.\begin{array}{l}
x=1+2 t \\
y=2+3 t \\
z=2+2 t
\end{array}\right\} \text { and } \quad \begin{array}{l}
x=3+4 s \\
y=4+5 s \\
z=5+6 s
\end{array}\right\}
$$

are equated, no solution for $s$ and $t$ satisfies all three pairs of equations.
Furthermore, the lines are not parallel because they have non-parallel direction vectors $(\mathbf{2}, 3,2)$ and $(4,5,6)$.
Two non-parallel, non-intersecting lines in $\mathrm{R}^{3}$ are called SKEW lines.

## Example

To find the shortest distance between two skew lines.

$$
x-1=\frac{y-1}{2}=\frac{z-1}{3}(1) \text { and } \frac{x}{3}=\frac{y-2}{2}=z(2)
$$



A standard technique would be to find the unique points $A$ and $B$ on the lines (1) and (2) so that $\mathbf{A B}$ was perpendicular to both the direction vectors of lines (1) and (2) and then
find $|\mathbf{A B}|$. This would solve the problem but it is a lengthy and awkward one and a recommended solution is as follows:
i) Find the cross product of the direction vectors (1) and (2)
ii) Choose points C and D, one on (1), one on (2)
iii) Find the length of the projection of $\mathbf{C D}$ on the vector found in i) - (a multiple of $\mathbf{A B}$ )
(This would be the required distance)

## Solution

i) Direction vector of line (1) is $(\mathbf{1 , 2 , 3})$

Direction vector of line (2) is $(\mathbf{3}, \mathbf{2}, \mathbf{1})$
Their cross product is $(-\mathbf{4}, 8,-4)$, i.e. a vector perpendicular to both lines.
ii) Let C be $(1,1,1)$ and D be $(0,2,0)$ These choices are completely random remember, except that C is on line (1) and D is on line (2).
i.e. $\mathbf{C D}=(\mathbf{- 1 , 1 , - 1})$
iii) The required distance is then $\frac{|C D \bullet(-4,8,-4)|}{|(-4,8,-4)|}$

$$
=\frac{|\mathbf{C}(-\mathbf{1}, \mathbf{1},-\mathbf{1}) \bullet(-\mathbf{4 , 8},-\mathbf{4})|}{|(-4,8,-\mathbf{4})|}=\frac{16}{\sqrt{96}}=\frac{4}{\sqrt{6}}=1.63 \text { (approx.) }
$$

In general, to find the distance between skew lines -

$$
(\mathbf{x}, \mathbf{y}, \mathbf{z})=\mathbf{p}+\mathrm{tv}(1) \text { and }(\mathbf{x}, \mathbf{y}, \mathbf{z})=\mathbf{q}+\mathbf{s w}(2)
$$

all we need to recognise is that $\mathbf{v} \times \mathbf{w}$ is a common perpendicular to (1) and (2) and $\mathbf{p}-\mathbf{q}$ is a vector joining two points on (1) and (2).
The distance is $\frac{|(\mathbf{p}-\mathbf{q}) \bullet(\mathbf{v} \times \mathbf{w})|}{|(\mathbf{v} \times \mathbf{w})|}$
This is not a formula to memorise necessarily but one which is very instructive and worth comprehending.

## Example

To find the reflection of point $P(10,3,5)$ in the line $\ell, \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$


## Solution

Let $\mathrm{P}^{1}$ be the reflected point and let $\mathrm{PP}^{1}$ intersect line $\ell$ at R . Since R is on the line, $R$ has co-ordinates $(2 t+1,3 t+2,4 t+3)$.
i.e. $\mathbf{P R}=(\mathbf{2 t}-\mathbf{9}, \mathbf{3 t} \mathbf{- 1}, \mathbf{4 t} \mathbf{- 2})$

Note that $\mathbf{P R}$ is perpendicular to $\ell$
$\therefore(\mathbf{2 t}-\mathbf{9}, \mathbf{3 t}-\mathbf{1}, \mathbf{4 t} \mathbf{- 2}) \cdot(\mathbf{2}, \mathbf{3}, 4)=\mathbf{0}$
i.e. $4 \mathrm{t}-18+9 \mathrm{t}-3+16 \mathrm{t}-8=0$
$\mathrm{t}=1$
$\therefore \mathrm{R}$ is $(3,5,7)$
$\therefore P^{\prime}$ is $(-4,5,9)$

## Exercise 5.2

1. Find the point of intersection of $\frac{x-1}{4}=\frac{y-2}{5}=\frac{z-3}{6}$ and $\frac{x-4}{1}=\frac{y-5}{2}=\frac{z-6}{3}$.
2. Find the symmetric equations for a line parallel to $x=1-2 t, y=3+t, z=4$ passing through ( $-1,2,0$ ).
3. A line passing through $(-1,0,4)$ makes an angle of $60^{\circ}$ with a line parallel to the $x$ axis and an angle of $45^{\circ}$ with a line parallel to z axis. Find the parametric equations for the line.
4. Find the reflection of $(-1,3,5)$ in the line $\frac{x-2}{1}=\frac{y-1}{2}=z$
5. Find the distance between $x=y=z$ and $x=y=z-1$ (Distance is not 1 ).
6. Find the distance from $(1,2,3)$ to the line $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{2}, \mathbf{3}, \mathbf{4})+\mathrm{t}(\mathbf{1}, \mathbf{- 1 , 3})$
7. Find the equations of the line intersecting $\frac{x-1}{2}=\frac{y-2}{3}=z+1$ at right angles and passing through $(1,3,4)$.
8. Find the distance between the skew lines $x=y ; z=0$ and $x=0 ; y=z+1$.
9. Find the equation of the line intersecting -

$$
\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-1}{3} \text { at right angles and passing through }(4,4,-3) .
$$

10. Find the distance from $(1,2,3)$ to $x=\frac{y}{2}=\frac{z}{4}$.
11. Find the distance between $x-6=y-3=z-1$ and $\frac{x-4}{3}=\frac{y+2}{3} ; z=4$
12. Find the distance from the point Q with the position vector $\mathbf{q}$ to the line $(\mathbf{x}, \mathbf{y}, \mathbf{z})=$ p+tv.
13. Find the distance between $\frac{x-1}{2}=\frac{y-2}{3} ; z=3$ and $\frac{x+1}{2}=\frac{y+2}{3} ; z=4$.
14. Find the line passing through $(1,2,3)$ which intersects both $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and $\frac{x-3}{4}=\frac{y-4}{5}=\frac{z-5}{5}$.
15. Find the shortest distance between the line passing through ( $1,0,-1$ ) and $(-1,1,0)$ and the line passing through $(3,1,-1)$ and $(4,5,-2)$.

## Harder Questions

16. Show that: $\mathbf{i})(\mathbf{1 , 2 , 3}),(4,-2,6)$ and $(7,-6,9)$ are coplanar vectors.
ii) Show however that -

$$
\begin{aligned}
& (\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1}, \mathbf{0}, \mathbf{1})+\mathrm{r}(\mathbf{1}, \mathbf{2}, \mathbf{3}) \\
& (\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1 , 0 , 1})+\mathrm{s}(\mathbf{4},-\mathbf{2}, \mathbf{6}) \\
& (\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{2}, \mathbf{0}, \mathbf{2})+\mathrm{t}(\mathbf{7}, \mathbf{- 6 , 9})
\end{aligned}
$$

are not coplanar lines. Explain why not in view of i). Find a number y to replace 0 as the middle component in $(\mathbf{2 , 0 , 2})$ in the third equation so that the lines would be coplanar.
17. Find the area of the triangle formed by the intersection of $x=\frac{y}{2}=\frac{z}{3}$ and

$$
x-1=y-2=z-3 \text { and } \frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{4}
$$

18. Find the point of intersection of the medians of the triangle with vertices $(1,0,0)$, $(0,2,0)(0,0,3)$.
19. Find a vector equations of the line through the origin that intersects both

$$
x-1=\frac{y-2}{-2}=\frac{z-3}{-1} \text { and } \frac{x+1}{3}=\frac{y+4}{2}=\frac{z-7}{-1}
$$

20. Find a point $A$ on line (1) and point $B$ on line (2) so that $\mathbf{A B}$ is perpendicular to both line (1) and line (2).

$$
\begin{aligned}
& \frac{x-3}{2}=\frac{y-5}{3}=\frac{z+2}{-5}(1) \\
& \frac{x-9}{3}=\frac{y+10}{-7}=\frac{z-13}{4}(2)
\end{aligned}
$$

Find the distance of AB
21. Find $m$ so that $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{1 , 0 , 1})+r(\mathbf{1 , 2 , 3})$

$$
\begin{aligned}
(\mathbf{x}, \mathbf{y}, \mathbf{z}) & =(\mathbf{1}, \mathbf{0}, \mathbf{1})+\mathrm{s}(\mathbf{4},-\mathbf{2}, \mathbf{6}) \\
\text { and }(\mathbf{x}, \mathbf{y}, \mathbf{z}) & =(\mathbf{m}, \mathbf{m}, \mathbf{m})+\mathrm{t}(7,-\mathbf{6}, \mathbf{9})
\end{aligned}
$$

are coplanar lines (see Question 16)
22. Investigate how many lines pass through a given point and intersect both of two given skew lines in $\mathrm{R}^{3}$.
23. Find the distance between $x=y=2 z$ and $2 x=y=z-1$.

## Exercise 5.2 Answers

1. $(5,7,9)$
2. $\frac{\mathrm{x}+1}{-2}=\mathrm{y}-2 ; \mathrm{z}=0$
3. $\mathrm{x}=-1+\mathrm{t}$
$\mathrm{y}=\mathrm{t}$
$\mathrm{z}=4+\sqrt{2} \mathrm{t}$
4. $\frac{\sqrt{6}}{3}$
5. $\sqrt{\frac{24}{11}}$
6. $\frac{\mathrm{x}-1}{8}=\frac{\mathrm{y}-3}{5}=\frac{\mathrm{z}-4}{-31}$
7. $(7,3,-3)$
8. $\frac{1}{\sqrt{3}}$
9. $\frac{11}{\sqrt{107}}$
10. $\frac{x-4}{3}=\frac{y-4}{3}=\frac{z+3}{-4}$
11. $\mathrm{y}=-\frac{4}{3}$
12. $\frac{\sqrt{5}}{\sqrt{21}}$
13. $\frac{\sqrt{6}}{2}$
14. $\frac{3}{\sqrt{2}}$
15. $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$
16. $\frac{|(\mathbf{q}-\mathbf{p}) \times \mathbf{v}|}{|v|}$
17. $(\mathbf{x}, \mathbf{y}, \mathbf{z})=\mathrm{t}(\mathbf{1 , 0 , 1})$
18. $\frac{\sqrt{221}}{13}$
19. A is $(1,2,3) \mathrm{B}$ is $(3,4,5) \mathrm{AB}=$ $2 \sqrt{3}$
20. $\mathrm{x}-1=\mathrm{y}-2=\mathrm{z}-3$
21. $\mathrm{m}=\frac{4}{7}$
22. A unique line if the point and line (1) lie in a plane not parallel to line 2 ; otherwise there is no line.
23. $\frac{2}{\sqrt{17}}$

## Chapter Test on Chapter 5

1. Write equations for the line passing through the points $(1,2,3)$ and $(4,6,8)$
2. Find the intersection point of the lines $\frac{x-1}{2}=\frac{y-2}{3} ; z=3$ and $x=\frac{y}{2}=\frac{z+1}{4}$
3. Find the angle of intersection of the lines $x=y ; z=2$ and $x=3 ; y=z+1$
4. Find the shortest distance from the point $(1,1,1)$ to the line $(\mathbf{x}, \mathbf{y}, \mathbf{z})=(\mathbf{2}, \mathbf{0}, \mathbf{4})$ $+\mathrm{t}(\mathbf{1},-\mathbf{2}, \mathbf{1})$
5. Find the shortest distance between the lines $x=y=z$ and $2 x=y=3 z+1$.
6. Find the reflection of the point $(0,1,1)$ in the line $x=1 ; y=\frac{z+1}{2}$
7. Find the two possible values for m so that $\mathrm{x}=\mathrm{y}=\frac{\mathrm{z}}{\mathrm{m}}$ intersects the line $\frac{x}{m}=m y=\frac{z-1}{2}$ and find the two possible points of intersection.

## Chapter Test on Chapter 5 Answers

1. $\frac{x-1}{3}=\frac{y-2}{4}=\frac{z-3}{5}$
2. $(1,2,3)$
3. $\theta=60^{\circ}$
4. $\sqrt{5}$
5. . 196
6. $(2,1,1)$
7. $\mathrm{m}=1$ or -1
points of intersection are $(1,1,-1)$ and $(-1,-1,-1)$

## Chapter Test 2 on Chapter 5

1. Write equations for the line passing through the points $(1,2,3)$ and $(4,5,6)$
2. Find the shortest distance from the point $(1,2,3)$ to the line $x=y=z$.
3. Find where $\mathrm{x}=\mathrm{y}=\mathrm{z}$ intersects $\frac{\mathrm{x}-3}{2}=\frac{\mathrm{y}-1}{3}=\frac{2 \mathrm{z}}{7}$
4. Find the shortest distance between $x=y=z$ and $x=2 y ; z=3$
5. Find the point $A$ on the line $x=y=2-z$ and the point $B$ on the line $(2-5 t, 3+t, 4+t)$ so that $A B$ is perpendicular to both of these lines. Find the magnitude of the vector AB and hence deduce the shortest distance between these two lines.
6. $x^{2}+y^{2}+z^{2}=r^{2}$ is a sphere of radius $r$ in $R^{3}$ whose centre is $(0,0,0)$. Find the value of $r$ so that the line $x=y ; z=4$ is tangent to this sphere.

## Chapter Test 2 on Chapter 5 Answers

1. $\mathrm{x}-1=\mathrm{y}-2=\mathrm{z}-3$
2. $\sqrt{2}$
3. $(7,7,7)$
4. 1.225
5. A is $(1,1,1) \mathrm{B}$ is $(2,3,4)|\mathbf{A B}|=\sqrt{14}$ which is the shortest distance
6. radius is 4

## Chapter Test 3 on Chapter 5

1. State equations for the line passing through the point $(2,3,4)$ parallel to the $x$-axis.
2. Find the reflection of the point $(6,0,1)$ in the line $x-1=\frac{y-1}{2}=\frac{z-2}{3}$
3. Find the shortest distance between the two lines $x=y-1 ; z=0$ and

$$
x-1=\frac{y}{2} ; z=1
$$

4. Consider the two skew lines $x=y=z$ and $2 x=y=z-2$. $P$ is a point on line 1 and $Q$ is a point on line $2 . \mathrm{M}$ is the point $(2,3,4)$. Is it possible for $M$ to be the midpoint of $P Q$ ? If so, state the co-ordinates of $P$ and $Q$. If not, change the third coordinate of point $M$ so that $M$ is the mid-point of $P Q$.
5. $x=2 y+1=3 z+m$ intersects $2 x-1=y+m=z+1$. Find the value(s) of $m$.

## Chapter Test 3 on Chapter 5 Answers

1. $\mathrm{y}=3 ; \mathrm{z}=4$
2. $(-4,2,3)$
3. 1
4. Yes P is $(2,2,2)$ and Q is $(2,4,6)$
5. $\mathrm{m}=1$
