# CHAPTER THREE

## 3.1 Cross Product

Given two vectors $\vec{a} = \overrightarrow{(a_1, a_2, a_3)}$ and $\vec{b} = \overrightarrow{(b_1, b_2, b_3)}$ in $R^3$, the cross product of $\vec{a}$ and $\vec{b}$ written $\vec{a} \times \vec{b}$ is defined to be:

$$\vec{a} \times \vec{b} = \overrightarrow{(a_2 b_3 - b_2 a_3, a_3 b_1 - a_1 b_3, a_1 b_2 - b_1 a_2)}$$

Note $\vec{a} \times \vec{b}$ called "$\vec{a}$ cross $\vec{b}$" is a **VECTOR** (unlike $\vec{a} \bullet \vec{b}$ which is a scalar).

Example

$$\overrightarrow{(1,2,3)} \times \overrightarrow{(4,5,6)} = \overrightarrow{(2\times 6 - 5\times 3, 3\times 4 - 1\times 6, 1\times 5 - 4\times 2)} = \overrightarrow{(-3,6,-3)}$$

Note that $\overrightarrow{(-3,6,-3)}$ is perpendicular to both $\overrightarrow{(1,2,3)}$ and $\overrightarrow{(4,5,6)}$. At first sight $\vec{a} \times \vec{b}$ seems an unwieldy formula but use of the following mnemonic will possibly help.

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ & & & & \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}$$

Each component of the cross product is obtained by considering the following square block, multiplying diagonally adding the $\searrow$ products and subtracting the $\nearrow$ products.

i.e. the first component is $\begin{array}{cc} a_2 & a_3 \\ b_2 & b_3 \end{array}$ i.e. $a_2 b_3 - b_2 a_3$

Theorem 3.1

$\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$

Proof

Let $\vec{a} = (\overrightarrow{a_1, a_2, a_3})$ and $\vec{b} = (\overrightarrow{b_1, b_2, b_3})$

$(\vec{a} \times \vec{b}) \bullet \vec{a} = (\overrightarrow{a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2}) \bullet (\overrightarrow{a_1, a_2, a_3})$

$= a_1a_2b_3 - a_1b_2a_3 + a_2a_3b_1 - a_2b_3a_1 + a_3a_1b_2 - a_3b_1a_2$

$= 0$

$\therefore$ By Theorem 2.1$(\vec{a} \times \vec{b})$ is perpendicular to $\vec{a}$.

Similarly $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

Theorem 3.2

$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \sin\theta$ where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$.

Proof

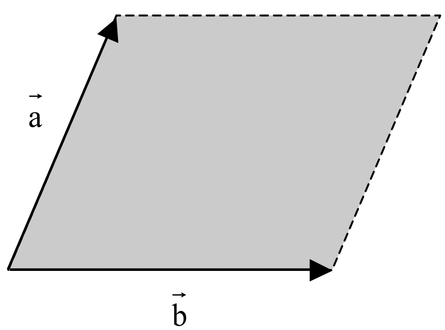
We will show $(\left|\vec{a}\right|\left|\vec{b}\right| \sin\theta)^2 = \left|\vec{a} \times \vec{b}\right|^2$. (note that there will be no difficulty here with negative signs since each of $\left|\vec{a}\right|, \left|\vec{b}\right|, \left|\vec{a} \times \vec{b}\right|$ and $\sin\theta$ is positive.)

$$
\begin{aligned}
(\left|\vec{a}\right|\left|\vec{b}\right| \sin\theta)^2 &= \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 \sin^2\theta \\
&= \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 (1 - \cos^2\theta) \\
&= \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 \left(1 - \frac{(\vec{a} \bullet \vec{b})^2}{\left|\vec{a}\right|^2 \left|\vec{b}\right|^2}\right) \quad \text{(Theorem 2.1)}
\end{aligned}
$$

$$= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \bullet \vec{b})^2$$
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$
$$= a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 - 2(a_1 b_1 a_2 b_2 + a_1 b_1 a_3 b_3 + a_2 b_2 a_3 b_3)$$
$$= (a_2^2 b_3^2 - 2a_2 b_2 a_3 b_3 + a_3^2 b_2^2) + (a_3^2 b_1^2 - 2a_1 b_1 a_3 b_3 + a_1^2 b_3^2) + (a_1^2 b_2^2 - 2a_1 b_1 a_2 b_2 + a_2^2 b_1^2)$$
$$= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$
$$= |\vec{a} \times \vec{b}|^2 \blacksquare$$

i.e. $\boxed{|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta}$

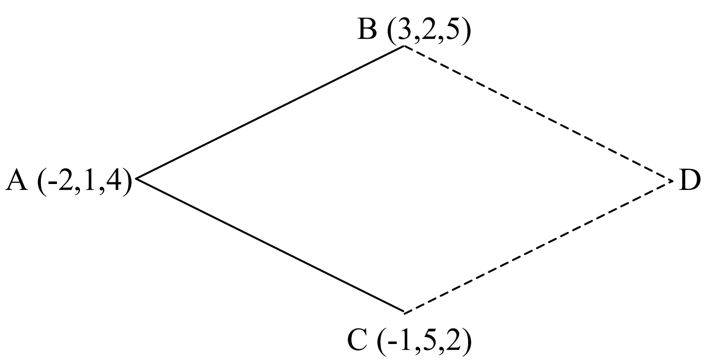
In itself, Theorem 3.2 is not over-important but it does give the very valuable corollary that $|\vec{a} \times \vec{b}|$ is the **<u>area of the parallelogram</u>** formed with $\vec{a}$ and $\vec{b}$ as two adjacent edges.




$\text{Area} = |\vec{a} \times \vec{b}|$

<u>Example</u>

Find area of $\Delta ABC$ where A is (-2,1,4), B is (3,2,5) and C is (-1,5,2)



$\overrightarrow{AB} = \overrightarrow{(5,1,1)}, \quad \overrightarrow{AC} = \overrightarrow{(1,4,-2)}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \overrightarrow{(-6,11,19)}$

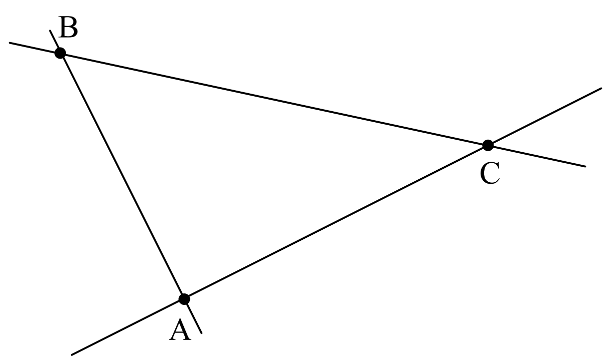
$$\begin{aligned}
\text{But Area of } \Delta ABC &= \frac{1}{2} \text{ area of parallelogram ABDC.} \\
&= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| \\
&= \frac{1}{2}\sqrt{(-6)^2 + (11)^2 + (19)^2} \\
&= \frac{1}{2}\sqrt{518} \\
&= 11.38 \text{ (approx.)}
\end{aligned}$$

Example

Find a vector perpendicular to both $\overrightarrow{(1,1,1)}$ and $\overrightarrow{(1,2,3)}$. Clearly $\overrightarrow{(1,1,1)} \times \overrightarrow{(1,2,3)}$ will suffice. (by Theorem 3.1)

Example

Find a vector perpendicular to the plane passing through the points A (1,-1,2), B(2,0,-1) and C (0,2,1)




$\overrightarrow{AB} = \overrightarrow{(1,1,-3)} \qquad \overrightarrow{AC} = \overrightarrow{(-1,3,-1)}$

Now note, since $\overrightarrow{AB} \times \overrightarrow{AC}$ will be perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then it will be a vector perpendicular to the plane containing $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\overrightarrow{AB} \times \overrightarrow{AC} = \overrightarrow{(8,4,4)}$ and hence $\overrightarrow{(8,4,4)}$ is a vector perpendicular to the plane. $(\overrightarrow{2,1,1})$ equally well serves as a vector perpendicular to the plane of course.

Exercise 3.1

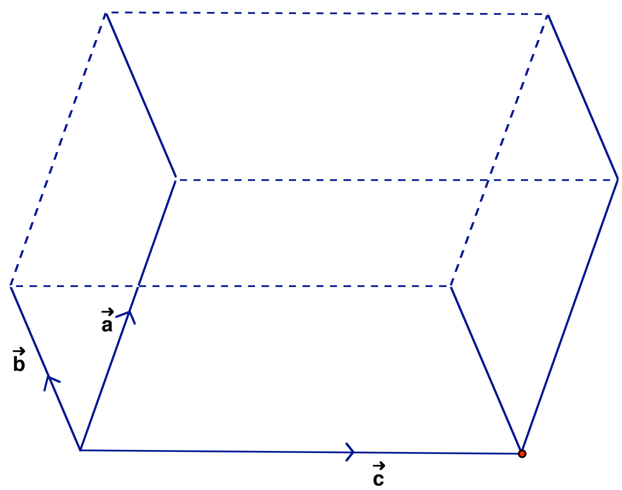
1. Find $\overrightarrow{(2,3,4)} \times \overrightarrow{(-1,2,5)}$
2. Find a vector perpendicular to both $\overrightarrow{(-1,3,2)}$ and $\overrightarrow{(3,2,1)}$.
3. Find area of parallelogram which has (0,0,0) (1,3,5) (-2,4,-1) and (-1,7,4) as its vertices.
4. Find area of $\Delta ABC$ where A is (1,2,3) B is (5,-2,1) and C is (7,0,6)
5. Show that $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$
6. Find a vector perpendicular to both $3\vec{i} - \vec{j}$ and $\vec{j} - 2\vec{k}$.
7. $\vec{a} \times \vec{i} = \overrightarrow{(0,1,2)}$ and $\vec{a} \bullet \vec{i} = 3$. Find $\vec{a}$.
8. Find angle between $\overrightarrow{(2,1,0)}$ and $\overrightarrow{(3,-1,0)}$.
9. (0,0,0) (1,2,3) and (1,1,n) are three vertices of a triangle whose area is $\frac{\sqrt{19}}{2}$. Find n.
10. True or False? (Assume $\vec{a}$ and $\vec{b}$ are non-zero vectors in the hypotheses.)
    a) $\vec{a} \times \vec{b} = \vec{0}$ implies $\vec{a}$ is a scalar multiple of $\vec{b}$.
    b) $\vec{a} \times \vec{b} = \vec{b}$ implies $|\vec{a}| = 1$.
    c) $\vec{a} \times \vec{b} = \vec{b}$ implies $\vec{b} = \vec{0}$
    d) $\vec{i} \times \vec{i} = \vec{k} \times \vec{k}$
    e) $\vec{a} \times \vec{i} = \vec{b} \times \vec{i}$ implies $\vec{a} \bullet \vec{i} = \vec{b} \bullet \vec{i}$
    f) $\vec{a} \times \vec{i} = \vec{b} \times \vec{i}$ implies $\vec{a} \bullet \vec{k} = \vec{b} \bullet \vec{k}$

Exercise 3.1 Answers

| | | | | | |
|---|---|---|---|---|---|
| 1. $\overrightarrow{(7,-14,7)}$ | 2. $\overrightarrow{(1,-7,11)}$ | 3. $\sqrt{710}$ | 4. $4\sqrt{17}$ | | |
| 6. $\overrightarrow{(2,6,3)}$ | 7. $\overrightarrow{(3,-2,1)}$ | 8. 45° | 9. n = 0 or 3.6 | | |
| 10. a) True | b) False | c) True | d) True | e) False | f) True |

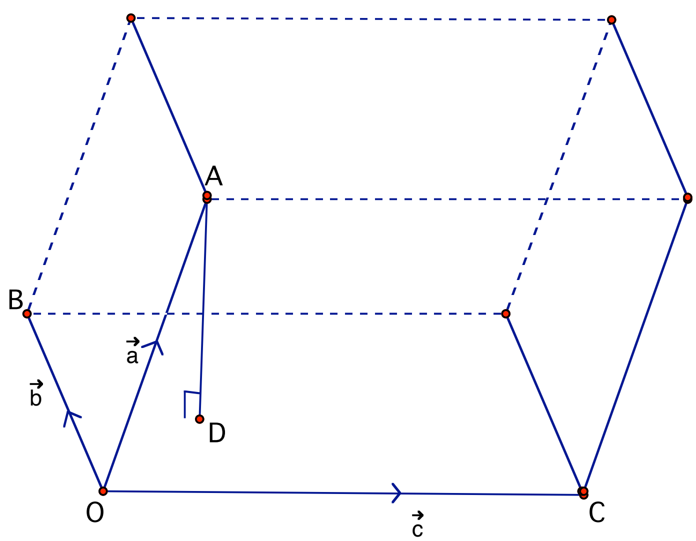
Theorem 3.3

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors considered as 3 adjacent edges of a parallelepiped as shown in diagram.




Then the volume of the parallelepiped is $\left|\vec{a} \bullet (\vec{b} \times \vec{c})\right|$

Proof




Let D be a point in the base of the parallelepiped such that $\overrightarrow{AD}$ is a vector perpendicular to the plane in which $\vec{b}$ and $\vec{c}$ lie. Then $\vec{a} \bullet (\vec{b} \times \vec{c}) = \left|\vec{a}\right|\left|(\vec{b} \times \vec{c})\right| \cos\theta$ (Theorem 2.1) where $\theta$ is the angle between $\vec{a}$ and $(\vec{b} \times \vec{c})$. i.e. $\theta = \angle OAD$ in the diagram.

But note $|\vec{a}|\cos\theta = |\overrightarrow{AD}|$ = height of parallelepiped where $|(\vec{b}\times\vec{c})|$ is the area of the base.

(Theorem 3.2 corollary).

$\therefore \vec{a}\bullet(\vec{b}\times\vec{c})$ = area of base times height of parallelepiped.

= volume of parallelepiped.

The absolute value symbol about $\vec{a}\bullet(\vec{b}\times\vec{c})$ in the statement of the theorem caters for the possibly that $\theta$ is obtuse. □

This is an extremely important theorem for its corollary.

$$\boxed{\vec{a},\vec{b},\vec{c} \text{ are coplanar iff } \vec{a}\bullet(\vec{b}\times\vec{c}) = 0}$$

This follows immediately from the fact that if $\vec{a}\bullet(\vec{b}\times\vec{c}) = 0$ then $\left|\vec{a}\bullet(\vec{b}\times\vec{c})\right| = 0$ and hence the volume of the corresponding parallelepiped is zero, i.e. the parallelepiped is "squashed flat" i.e. is merely a plane. i.e. $\vec{a},\vec{b},\vec{c}$ are coplanar. From now on we shall use $\vec{a}\bullet(\vec{b}\times\vec{c}) = 0$ as a test for coplanarity of vectors $\vec{a},\vec{b},\vec{c}$ in $R^3$. Note furthermore from the proof of theorem 3.3 that $|\vec{b}\bullet\vec{c}\times\vec{a}|$ and $|\vec{c}\bullet\vec{b}\times\vec{a}|$ equally well are expressions for the volume of the parallelepiped.

$\vec{a}\bullet(\vec{b}\times\vec{c})$ is called the triple scalar product of $\vec{a},\vec{b},\vec{c}$

<u>Example</u>

Find m so that A (1,1,1), B (2,2,2), C (3,4,5), D (1,3,m) are coplanar points.

$$\overrightarrow{AB} = \overrightarrow{(1,1,1)},\quad \overrightarrow{AC} = \overrightarrow{(2,3,4)},\quad \overrightarrow{AD} = \overrightarrow{(0,2,m-1)}$$

But A,B,C,D coplanar implies that $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are coplanar and so $\overrightarrow{AB}\bullet\overrightarrow{AC}\times\overrightarrow{AD} = 0$

i.e. $\overrightarrow{(1,1,1)}\bullet\overrightarrow{(2,3,4)}\times\overrightarrow{(0,2,m-1)} = 0$

i.e. $\overrightarrow{(1,1,1)}\bullet\overrightarrow{(3m-11,2-2m,4)} = 0$

i.e. $m - 5 = 0$

i.e. $m = 5$

Note that this is a much better and quicker way of handling coplanarity questions then previously developed.

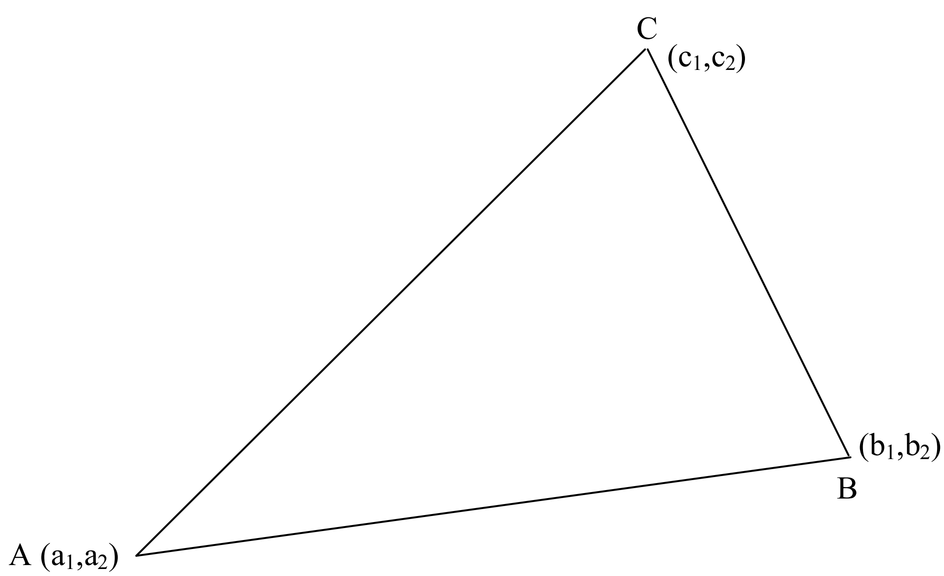
Note also that since $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$

and $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

then $\left|\vec{a} \bullet \vec{b} \times \vec{c}\right| = \left|\vec{b} \times \vec{c} \bullet \vec{a}\right| = \left|\vec{c} \bullet \vec{b} \times \vec{a}\right|$ etc. (see note – after Theorem 3.3)

<u>Example</u>

To find area of triangle with vertices A $(a_1,a_2)$, B $(b_1,b_2)$ and C $(c_1,c_2)$




We have established a formula for area of a triangle in $\mathbf{\underline{R}^3}$ and to find area of triangle ABC in question we merely "embed" $\Delta ABC$ in $R^3$ by letting A,B,C have co-ordinates $(a_1,a_2,0)$ $(b_1,b_2,0)$ $(c_1,c_2,0)$ respectively.

Clearly this produces a triangle congruent to $\Delta ABC$.

Then area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$= \frac{1}{2}\left|\overrightarrow{(b_1 - a_1, b_2 - a_2, 0)} \times \overrightarrow{(c_1 - a_1, c_2 - a_2, 0)}\right|$$

$$= \frac{1}{2}\left|a_1 b_2 + b_1 c_2 + c_1 a_2 - a_1 c_2 - b_1 a_2 - c_1 b_2\right|$$

A mnemonic for this area is $\frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ a_1 & a_2 \end{vmatrix}$ adding $\searrow$ products and subtracting the $\nearrow$ products.

Note that this formula can be applied to any polygon in $R^2$ by simply listing the vertices in the proper sequence.

<u>Example</u>

To show:

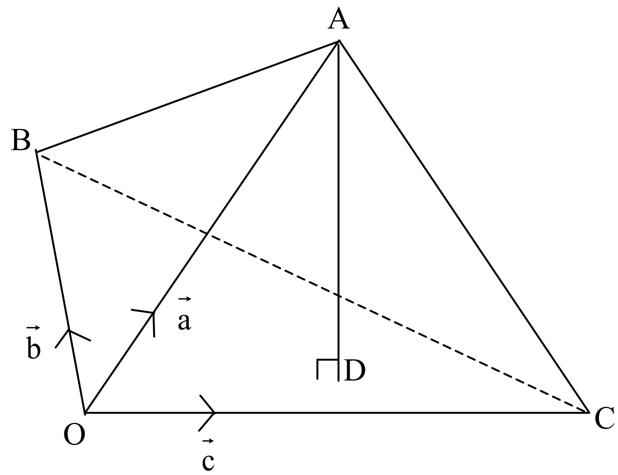
$$\vec{a} \times (\vec{b} \times \vec{c}) \equiv (\vec{a} \bullet \vec{c})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c}$$

Note that $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to $(\vec{b} \times \vec{c})$ and hence lies in the plane of $\vec{b}$ and $\vec{c}$. This means that $\vec{a} \times (\vec{b} \times \vec{c})$ can be represented as a linear combination of $\vec{b}$ and $\vec{c}$ which, note, $(\vec{a} \bullet \vec{c})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c}$ is.

The proof of this identity is rather lengthy but involves no more than the mechanical applications in computing $\vec{a} \times (\vec{b} \times \vec{c})$ and is hence left as an exercise.

Example

To show that the volume of a tetrahedron, three of whose concurrent edges are $\vec{a}, \vec{b}, \vec{c}$, is $\frac{1}{6}\left|\vec{a} \bullet (\vec{b} \times \vec{c})\right|$




Let the tetrahedron be as in the diagram. Its volume is $\frac{1}{3}$ the area of the base times the height (since a tetrahedron is a pyramid).

$$\text{i.e. the volume of OABC} = \frac{1}{3} \times (\text{area } \Delta\text{OBC}) \times (\text{height AD})$$

$$= \frac{1}{3}\left(\frac{1}{2}\left|\vec{b} \times \vec{c}\right|\right)\left(\left|\vec{a}\right| \cos\theta\right)$$

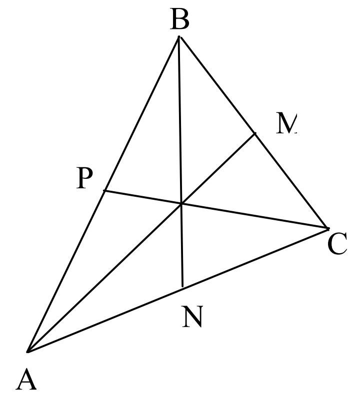
$$= \frac{1}{6}\left|\vec{a}\right|\left|\vec{b} \times \vec{c}\right| \cos\theta$$

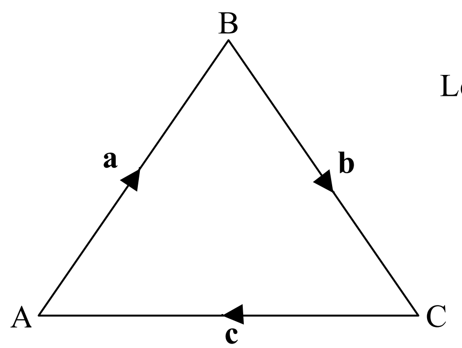
But note $\theta$ is the angle between $\vec{a}$ and $(\vec{b} \times \vec{c})$ since $\overline{AD}$ is perpendicular to $\Delta$OBC and hence is a multiple of $\vec{b} \times \vec{c}$.

$\therefore$ volume of OABC = $\frac{1}{6}\left|\vec{a} \bullet (\vec{b} \times \vec{c})\right|$ using theorem 2.1.

Example

Consider triangle ABC with medians AM, BN, CP as shown



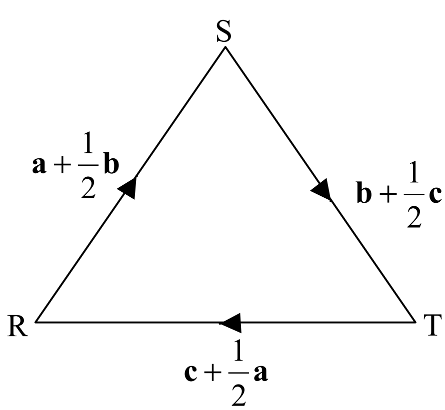

Let **AB** = **a**, **BC** = **b** and **CA** = **c**

Now medians **AM**, **BN**, **CP** are $\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{b} + \frac{1}{2}\mathbf{c}$ and $\mathbf{c} + \frac{1}{2}\mathbf{a}$ respectively. Note that

$$\mathbf{AM} + \mathbf{BN} + \mathbf{CP} = (\mathbf{a} + \frac{1}{2}\mathbf{b}) + (\mathbf{b} + \frac{1}{2}\mathbf{c}) + (\mathbf{c} + \frac{1}{2}\mathbf{a}) = 1\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = 1\frac{1}{2}\mathbf{0} = \mathbf{0}$$

It follows that the medians AM, BN, CP without rotations or magnification, can be individually translated to form a triangle. Call this triangle Δ RST.

To find area of Δ RST formed by medians



$$\text{Now Area } \Delta\text{RST} = \frac{1}{2}\left|(a+\frac{1}{2}b)\times-(c+\frac{1}{2}a)\right|$$

$$= \frac{1}{2}\left|(a+\frac{1}{2}b)\times-(-a-b+\frac{1}{2}a)\right|$$

$$= \frac{1}{2}\left|(a+\frac{1}{2}b)\times(\frac{1}{2}a+b)\right|$$

$$= \frac{1}{2}\left|a\times\frac{1}{2}a+a\times b+\frac{1}{2}b\times\frac{1}{2}a+\frac{1}{2}b\times b\right|$$

$$= \frac{1}{2}\left|\mathbf{0}+a\times b-(\frac{1}{2}a\times\frac{1}{2}b)+0\right|$$

$$= \frac{1}{2}\left|\frac{3}{4}(a\times b)\right|$$

$$= \frac{3}{4}\left|\frac{1}{2}(a\times b)\right|$$

$$= \frac{3}{4} \text{ Area } \Delta\text{ABC}$$

Therefore the medians form a triangle whose area is $\frac{3}{4}$ of the area of the original triangle.

<u>Exercise 3.2</u>

1. Find $\overrightarrow{(1,-3,2)}\times\overrightarrow{(4,1,2)}$
2. Find a unit vector perpendicular to both $\overrightarrow{(-1,5,3)}$ and $\overrightarrow{(2,4,1)}$
3. Explain why $\vec{a}\bullet\vec{b}\times\vec{c}$ is well defined, i.e. unambiguous.
4. True or False? Assume vectors are non-zero.

   i) $\vec{i}\times\vec{j}=\vec{k}$

   ii) $\vec{i}\times\vec{k}=\vec{j}$

   iii) $\vec{a}\bullet(\vec{b}\times\vec{c})=0 \rightarrow \vec{a},\vec{b},\vec{c}$ are coplanar vectors

   iv) $\vec{a}\times\vec{a}=\vec{0}$

   v) $\vec{a}\times\vec{b}=\vec{a}\times\vec{c} \rightarrow \vec{b}=\vec{c}$

   vi) $\vec{a}=m\vec{b}+n\vec{c} \rightarrow \vec{a}\bullet\vec{b}\times\vec{c}=0$

vii) $|\vec{a} \bullet \vec{b}| = |\vec{a} \times \vec{b}| \rightarrow$ the angle between $\vec{a}$ and $\vec{b}$ is 45° or 135°.

viii) $|\vec{a} \bullet \vec{b} \times \vec{c}| = |\vec{a} \bullet \vec{c} \times \vec{b}|$

ix) $\vec{a} \bullet \vec{b} \times \vec{c} = \vec{a} \bullet \vec{c} \times \vec{b}$

x) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

xi) $(\vec{a} \times \vec{b}) \times \vec{a} = \vec{0}$

xii) $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}|$

xiii) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

xiv) $\vec{a} \times \vec{b} = \vec{0} \rightarrow \vec{a} = m\vec{b}$ for some scalar m.

xv) $n\vec{a} \times \vec{b} = \vec{a} \times \overrightarrow{nb}$

xvi) $|\vec{a} \bullet \vec{b}| = |\vec{a} \times \vec{b}|$ iff $\vec{a} = \vec{b}$

xvii) $|\vec{a} \times \vec{b}| \le |\vec{a}||\vec{b}|$

xviii) $\tan\theta = \dfrac{|\vec{a} \times \vec{b}|}{\vec{a} \bullet \vec{b}}$ where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$.

xix) $\vec{a} \times \vec{b} = (-\vec{b}) \times (-\vec{a})$

xx) $(\vec{a} \times \vec{b}) \bullet \vec{a} = \vec{b} \bullet (\vec{a} \times \vec{b})$

xxi) $\vec{a} \times \vec{a} = \vec{b} \times \vec{b} \rightarrow \vec{a} = \vec{b}$

xxii) $\{\vec{a}, \vec{b}, (\vec{a} \times \vec{b}) \times \vec{c}\}$ is a coplanar set of vectors.

xxiii) If a,b,c are not coplanar then $|\vec{a} \bullet \vec{b} \times \vec{c}| > 0$

xxiv) $|\vec{a} \bullet \vec{b}| \ge |\vec{a} \times \vec{b}|$

xxv) $\vec{a} \times \vec{b} = \vec{b} \rightarrow \vec{b} = 0$

xxvi) $\vec{a} \times \vec{i} = \vec{b} \times \vec{i}$ and $\vec{a} \times \vec{k} = \vec{b} \times \vec{k} \rightarrow \vec{a} = \vec{b}$

xxvii) It is possible for $|\vec{a} \times \vec{b}| = |\vec{a}|$?

xxviii) $|\vec{a} \times \vec{b}| = |\vec{b}|$ and $\vec{a} \bullet \vec{b} = 0 \Rightarrow |\vec{a}| = 1$

5. Find the area of parallelogram which has (0,0,0) (-2,4,3) (2,-4,1) and (0,0,4) as its vertices.
6. Find m so that (1,1,1) (1,2,0), (0,1,2) and (m,m,0) are coplanar points
7. i) Find area of $\Delta ABC$ where A is (1,-1,2) B is (0,2,1) and C is (2,0,-1).

   ii) Try to find a general expression for area of $\Delta ABC$ where A is (a,0,0) B is (0,b,0) and C is (0,0,c).
8. State whether the following expressions are V – a vector, S – a scalar or M – meaningless.

   i) $(\vec{a} \times \vec{b}) \bullet (\vec{b} \times \vec{a})$

   ii) $\left|(\sqrt{\vec{a} \bullet \vec{a}})\vec{c} \bullet \vec{d}\right|$

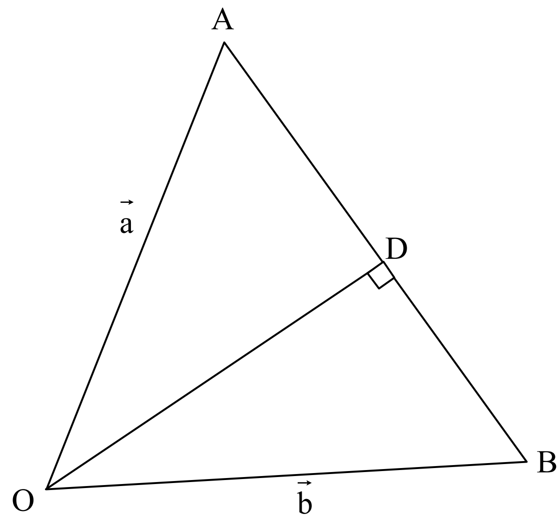
   iii) $(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$

   iv) $(\vec{a} - |\vec{c}|) \bullet (\vec{a} + |\vec{c}|)$

   v) $\vec{a} \bullet (\vec{a} + \vec{b} \bullet \vec{c})$
9. Find angle between $\overrightarrow{(1,2,0)}$ and $\overrightarrow{(1,-3,0)}$
10. The vertices of a tetrahedron are (0,0,0) (2,-1,0) (0,1,1) and (1,3,-1). Find its volume.
11. $\vec{a}$ is a vector making equal angles with each of the three axes in $R^3$. Find that angle.
12. Show that if any one of the vectors $\vec{a}, \vec{b}, \vec{c}$ is a linear combination of the other two vectors then $\vec{a} \bullet \vec{b} \times \vec{c} = 0$

Harder Questions –

13.




Given $\Delta OAB$ as in the diagram, with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, find the altitude $\overrightarrow{OD}$ in terms of $\vec{a}$ and $\vec{b}$.

14. Find circumcentre of $\Delta ABC$ if A is (0,1,4), B is (2,3,4) and C is (0,3,2).

Hint: i) circumcentre is coplanar with A,B,C.

ii) circumcentre is the intersection of the perpendicular bisectors of sides of the triangle.

15. Show $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$

16. Show $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \bullet \vec{b}\right|^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2$

17. Let A be (1,0,0) B be (0,2,0) and C be (0,0,3)

Find a point P coplanar with A,B,C so that $\overrightarrow{OP}$ is perpendicular to the plane containing A,B,C,P.

18. Find the distance from D to the plane containing A,B,C where A is (1,2,3) B is (2,2,2), C is (2,3,1) and D is (1,1,1)

19. Find the co-ordinates of the centre of gravity of the tetrahedron ABCD if A is (1,1,1), B is (2,3,4), C is (-1,3,-1) and D is (3,4,7).

20. ABCD is a square. A is (1,2). C, the centre of gravity of the square is (3,5). Find the co-ordinates of B,C and D.

21. Is $(\vec{a}+\vec{b})\times\vec{c}$ perpendicular to $\vec{a}$ ?

22. If $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ does it follow that $\vec{a}\bullet\vec{b}\times\vec{c}=0$ ?

23. In the twelve possible arrangements of $\vec{a}\bullet\vec{b}\times\vec{c},\vec{b}\times\vec{c}\bullet\vec{a},\vec{c}\bullet\vec{b}\times\vec{a}$ etc. find those which are equal and those which are negatives of each other.

<u>Exercise 3.2 Answers</u>

1. $\overrightarrow{(-8,6,13)}$
2. $\overrightarrow{(\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}})}$
3. Because $(\vec{a}\bullet\vec{b})\times\vec{c}$ makes no sense the only way to read $\vec{a}\bullet\vec{b}\times\vec{c}$ is $\vec{a}\bullet(\vec{b}\times\vec{c})$.
4. i) True ii) False iii) True iv) True v) False vi) True
   vii) True viii) True ix) False x) False xi) False xii) True
   xiii) True xiv) True xv) True xvi) False xvii) True
   xviii) True xix) False xix) False xx) True xxi) False
   xxii) True xxiii) True xxiv) False xxv) True xxvi) True
   xxvii) True xxviii) True
5. $8\sqrt{5}$
6. $m = 1\frac{1}{2}$
7. i) $2\sqrt{6}$ ii) $\frac{1}{2}\sqrt{a^2b^2+b^2c^2+a^2c^2}$.
8. i) S ii) S iii)S iv)M v) M
9. 135°
10. 1.5
11. arccos $\frac{1}{\sqrt{3}}$
13. $\dfrac{(\vec{a}\times\vec{b})\times(\vec{a}-\vec{b})}{\left|(\vec{b}-\vec{a})\right|^2}$

Note that this vector is perpendicular to $(\vec{a} \times \vec{b})$ and hence a linear combination of $\vec{a}$ and $\vec{b}$, as required, and perpendicular to $(\vec{a} - \vec{b})$ the base BA.

14. $(\frac{2}{3}, \frac{7}{3}, \frac{10}{3})$

17. $(\frac{36}{49}, \frac{18}{49}, \frac{12}{49})$

18. $\sqrt{3}$

19. $(\frac{5}{4}, \frac{11}{4}, \frac{11}{4})$

20. (0,7) (5,8) (6,3)

21. No

22. Yes

Practice Test 1 on Chapter 3

1. True or False?

   a) $\vec{a}\times\vec{b} = \vec{0}$ implies $\vec{a} = m\vec{b}$ for some scalar m.

   b) $\vec{a}\bullet(\vec{b}\times\vec{c}) = 0$ implies $\vec{a} = m\vec{b} + n\vec{c}$.

   c) The length of projection of $\vec{j}$ onto $\vec{i}+\vec{k}$ is 0.

   d) $\vec{a}\times\vec{b} = \vec{b}$ implies $|\vec{a}| = 1$

   e) $\vec{a}\times\vec{b} = \vec{c}$ implies $\vec{a}\bullet\vec{c} = 0$

   f) $\vec{a}\times\vec{b} = \vec{c}$ implies $\vec{b}\bullet\vec{c} = 0$

   g) $\vec{i}\bullet\vec{i} = \vec{k}\bullet\vec{k}$

   h) $\vec{i}\times(\vec{j}\times\vec{k}) = (\vec{i}\times\vec{j})\times\vec{k}$

   i) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$

   j) $\vec{a}\times\vec{a} = \vec{b}\times\vec{b}$ implies $|\vec{a}| = |\vec{b}|$

   k) $\vec{a}\times\vec{b} = \vec{b}\times\vec{a}$ implies $\vec{a} = m\vec{b}$ for some scalar m.
2. Find the area of triangle of $\Delta ABC$ where A is (1,2,3) B is (2,3,4) and C is (4,3,2).
3. Find the shortest distance from (1,1,1) to the plane containing the points (1,2,3) (2,2,2) and (2,3,1)
4. Describe the distance from point A to the line passing through B and C in terms of **AC**,**BC**,**AC**, etc. using dot product, cross product.
5. a) If $\vec{b}$ and $\vec{c}$ are known, does $\vec{a}\times\vec{b} = \vec{c}$ always have a solution? Explain.

   b) Solve $\vec{a}\times\overrightarrow{(1,2,3)} = \overrightarrow{(4,5,-2)}$
6. a) Decide whether it is true that $m\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b})$

   b) Decide whether it is true that if $\vec{a}\times\vec{b} = \vec{c}\times\vec{b}$ then $\vec{a} = m\vec{c}$ for some scalar m.

Practice Test 1 on Chapter 3 Answers

1. a) T b) T c)T d)F e)T f)T g)T h)T i)F j)F k)T
2. $\sqrt{6}$

3. $\sqrt{3}$
4. $\dfrac{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}{\left|\overrightarrow{BC}\right|}$
5. a) No because $\vec{b}$ and $\vec{c}$ might not be perpendicular to each other. b) No solution.
6. a) Yes b) False

Practice Test 2 on Chapter 3

1. Find the area of triangle ABC where A is (1,2,3) B is (2,3,5) and C is (4,2,6)
2. Let {**a**,**b**,**c**} be a coplanar set of vectors. Describe what vector $(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})$ might be. For example is this vector a linear combination of $\vec{b}$ and $\vec{c}$, a multiple of $\vec{a}$, the zero vector, a vector perpendicular to each of $\vec{a},\vec{b},\vec{c}$?
3. The line passing through (1,2,3) and (2,3,4) is parallel to the line passing through (0,0,0) and (1,1,1). Find the shortest distance between these lines.
4. Find the shortest distance from the origin (0,0,0) to the plane containing the points (3,0,0) (0,3,0) and (0,0,3)
5. Find co-ordinates of point D so that OD is perpendicular to the plane containing the points (3,0,0) (0,3,3) and (0,0,3) where O is the origin (0,0,0)
6. Simplify $(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})$ if $\vec{a}=\vec{b}+\vec{c}$.
7. Is it possible for $\left|\vec{a}\bullet\vec{b}\right|$ to equal $\left|\vec{a}\times\vec{b}\right|$? If so, give an example. If not, explain why not.

Practice Test 2 on Chapter 3 Answers

1. $\dfrac{3\sqrt{3}}{2}$ 2. the zero vector 3. $\sqrt{2}$ 4. $\sqrt{3}$ 5. $(\frac{3}{2},0,\frac{3}{2})$
6. the zero vector
7. Yes, if the angle between $\vec{a}$ and $\vec{b}$ is 45° or 135°.

Practice Test 3 on Chapter 3

1. True or False?

   a) $\vec{a} \times \vec{a} = |\vec{a}|^2$

   b) $\vec{a} \bullet \vec{a} = |\vec{a}|^2$

   c) $\vec{a} \times \vec{a} \bullet \vec{a} = \vec{a} \times \vec{b} \bullet \vec{a}$

   d) $\vec{a} \times \vec{a} \bullet \vec{a} = \vec{a} \times \vec{b} \bullet \vec{a}$ implies $\vec{a} = \vec{b}$

   e) If $\vec{a} \times \vec{b} = \vec{0}$ then $\vec{a}, \vec{b}$ are collinear

   f) If $(\vec{a} \times \vec{b})$ is perpendicular to $\vec{c}$ then $\vec{c} = m\vec{a} + n\vec{b}$ for some values of m and n.

   g) $\vec{v} \times \overrightarrow{(1,1,2)} = \vec{0}$ implies $\vec{v} = m\overrightarrow{(1,1,2)}$ for some value of m.

   h) $\vec{a} \times \vec{b} = \vec{c}$ implies $\vec{a} \bullet \vec{c} = 0$

   i) $\vec{a} \times \vec{i} = \vec{b} \times \vec{i}$ implies $\vec{a} \bullet \vec{i} = \vec{b} \bullet \vec{i}$

   j) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

   k) $\vec{a} \times (\vec{b} \times \vec{c}) = m\vec{b} + n\vec{c}$ for some values of m and n

2. APFT is a parallelogram. A is (1,1,3) P is (2,3,5) F is (4,-1,2)

   a) Find the co-ordinates of point T.

   b) Find the area of APFT

3. Find the value(s) of n so that (1,1,2) (1,2,1) (2,1,1) and (0,0,n) are coplanar.

4. Investigate whether you think it is true that

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

5. Find a vector $\vec{a}$ so that $(\vec{a} + \vec{i}) \times \vec{j} = \vec{i}$

6. Let P be a point so that $\overrightarrow{OP} \times \overrightarrow{(1,2,3)} = \overrightarrow{(7,-5,1)}$

   a) Find a possible position for P

   b) Describe in your own words the set of all points P that satisfy the equation below.

7. Find a vector $\vec{v}$ so that

$$\overrightarrow{(1,1,1)} \times \vec{v} = \overrightarrow{(1,-2,1)}$$

and $\overrightarrow{(1,1,1)} \bullet \vec{v} = 6$

8. Prove that $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \bullet \vec{b}\right|^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2$

<u>Practice Test 3 on Chapter 3 Answers</u>

1. a) F b) T c) T d) F e) T f) T g) T h) T i) F j) F k) T
2. T is (3,-3,0) Area of APFT is $3\sqrt{13}$
3. n = 4
4. True
5. $(\overrightarrow{-1, b, -1})$ where b can be any number
6. P can be any point of the form (t, 2t –1, 3t –5)
7. v = (1,2,3)