CHAPTER TWO

2.1 Vectors as ordered pairs and triples.

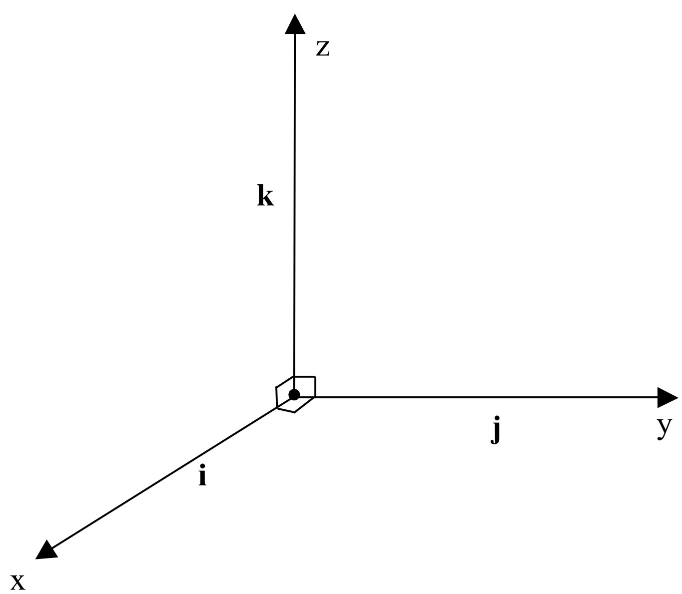
The most common set of basic vectors in 3-space is **i,j,k**

where

i represents a vector of magnitude 1 in the x direction

j represents a vector of magnitude 1 in the y direction

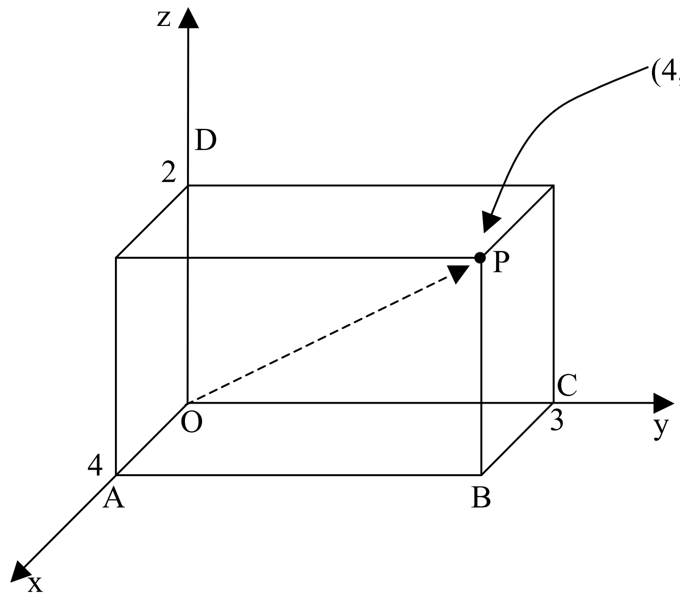
k represents a vector of magnitude 1 in the z direction



For example the vector **OA** where O is the origin and A is the point (0,2,0) is 2**j**.

Similarly 3**k** = **OB** where B is (0,0,3)

If we are considering R^2 then traditionally **i** = **OC** where C is (1,0) and **j** = **OD** where D is (0,1).



Let P be the point (4,3,2). Let O be the origin. In fact from now on O will **always** be the origin.

OP = **OA** + **AB** + **BP**

= **OA** + **OC** + **OD**

= 4**i** + 3**j** + 2**k**

We say the **position vector** of point P is 4**i** + 3**j** + 2**k**. There is clearly a 1-1 correspondence between the set of all points in R^3 and the set of all linear combinations of **i**, **j**, **k**.

Here (4,3,2) $\longleftrightarrow$ $4\vec{i} + 3\vec{j} + 2\vec{k}$

In general, (x,y,z) $\longleftrightarrow$ $x\vec{i} + y\vec{j} + z\vec{k}$

From now on (x,y,z) will refer to a point (say) P and **(x,y,z)** will refer to the position vector **OP** i.e. x**i** + y**j** + z**k**. A similar relationship exists between points in R^2 and the set of all linear combinations of **i**, **j.** The context of the question will determine whether, for example, **i** = (**1**,**0**) or (**1**,**0**,**0**).

Example

Let R be (2,3,4) and S be (4,7,11).

Then **OR** = (**2,3,4**) and **OS** = (**4,7,11**)

But **RS** = **OS** - **OR** (remember this is true regardless of the positions of O,R,S)

Therefore **RS** = (**4,7,11**) – (**2,3,4**)

= (4**i**, 7**j**, 11**k**) – (2**i**, 3**k**, 4**j**)

= 2**i**, 4**j**, 7**k**

= (**2,4,7**)

i.e. given any two points, we can obtain the vector joining them by subtracting their co-ordinates.

i.e. if P is $(p_1,p_2,p_3)$and Q is (q_1,q_2,q_3) then

PQ = $(q_1 - p_1,\ q_2 - p_2,\ q_3 - p_3)$

Similarly,

QP = $(p_1 - q_1,\ p_2 - q_2,\ p_3 - q_3)$

i.e. for example the vector (say) (**5,6,7**) can be thought of as the vector from the origin to the point (5,6,7) or as a vector joining two points whose x,y,z co-ordinates differ by 5,6,7 respectively.

When referring to vectors the co-ordinates are called **COMPONENTS.**

e.g. P has co-ordinates (2,3) and **OP** has components (**2,3**) i.e. 2**i** + 3**j**

It is clear that scalar multiplication satisfies the properties expected

viz. s(**a,b,c**) = s(a**i** + b**j** + c**k**)

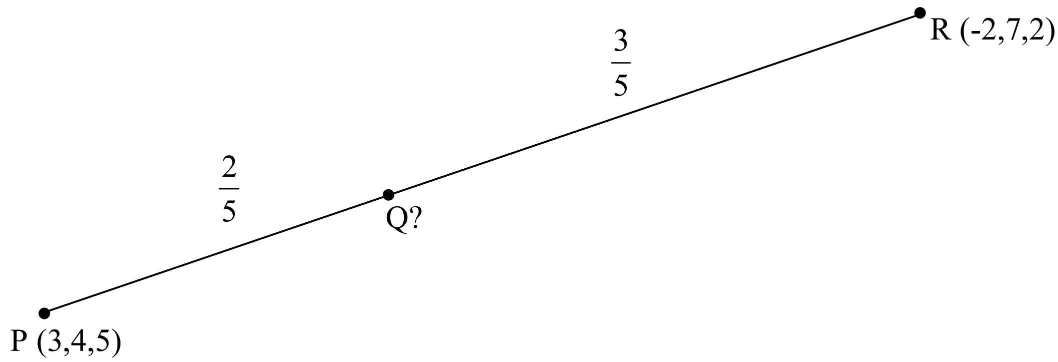
= sa**i** + sb**j** + sc**k**

= (**sa,sb,sc**) for any scalar s.

If (**a,b,c**) = (**d,e,f**) then a**i** + b**j** + c**k** = d**i** + e**j** + f**k** and hence it follows that a = d, b = e, c = f as expected.

<u>Example Question</u>

Let P be (3,4,5) and R be (-2,7,2). Find point Q dividing PR into the ratio $\frac{2}{5}$ to $\frac{3}{5}$.



Note that it is not necessary to visualize the diagram in 3-space.

<u>Solution</u>

OP = (**3,4,5**) and **OR** = (**-2,7,2**)

$$\mathbf{OQ} = \frac{3}{5}(\mathbf{3,4,5}) + \frac{2}{5}(\mathbf{-2,7,2})$$

$$= (\mathbf{1}, \mathbf{\frac{26}{9}}, \mathbf{\frac{19}{5}})$$

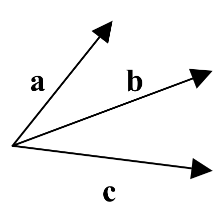
Therefore Point Q is $(1, \frac{26}{5}, \frac{19}{5})$.

<u>Notes:</u>

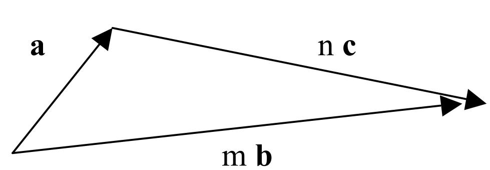
If 3 vectors **a**, **b** and **c** are coplanar then it is possible to write one of them in terms of the other two vectors.

For example

given



it is clear that by repositioning the vectors we can make a triangle as shown below:



Clearly **a** can be written in terms of the other two vectors. We say **a** = m**b** - n**c** for some scalars m and n.

m**b** + n**c** is called a linear combination of **b** and **c**.

In general if **a** = m**b** + n**c** for some scalars m and n, then **a**, **b** and **c** are coplanar vectors.

Example Question

Investigate whether the four points A(-1,3,1), B(1,1,1), C(2,1,0) and D(1,0,2) are coplanar.

Solution

$\overrightarrow{AB} = \overrightarrow{(2,-2,0)}$

$\overrightarrow{AC} = \overrightarrow{(3,-2,-1)}$

$\overrightarrow{AD} = \overrightarrow{(2,-3,1)}$

Let $\overrightarrow{AB} = m\overrightarrow{AC} + n\overrightarrow{AD}$ and investigate whether this has a solution for m and n. If it does then $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ will be coplanar and hence A,B,C,D will be coplanar.

i.e. investigate $\overrightarrow{(2,-2,0)} = m\overrightarrow{(3,-2,-1)} + n\,\overrightarrow{(2,-3,1)}$

$$\begin{aligned} \text{i.e. } 2 &= 3m + 2n \\ -2 &= -2m + -3n \\ 0 &= -m + n \end{aligned}$$

Here we have three questions in two unknowns which **in general** would **not** have a solution satisfying all three equations. However for our example m = $\frac{2}{5}$ and n = $\frac{2}{5}$ does satisfy all three equations and hence $\overrightarrow{AB}$ is a linear combination of $\overrightarrow{AC}$ and $\overrightarrow{AD}$ i.e. $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar vectors meaning A,B,C,D **are** coplanar points.

Exercise 2.1

1. If P is (2,3,4) and Q is (5,-2,4) find $\overrightarrow{PQ}$.
2. Find m so that P(2,3,4), Q(5,-2,5) and R(8,-7,m) are collinear points. Which point is between the other two?
3. Find m so that (1,1,1), (1,2,0),(0,1,2),(m,4,m) are coplanar points.
4. B is (1,2,3) $\overrightarrow{AB}$ is $2\vec{i} + \vec{j} - \vec{k}$. State the co-ordinates of A.
5. True or False?
   i) (1,0,0), (0,1,0), (0,0,1) are coplanar.
   ii) $\overrightarrow{(1,0,0)}, \overrightarrow{(0,1,0)}, \overrightarrow{(0,0,1)}$ are coplanar
   iii) $\vec{i}, \vec{j}, \vec{k}$ are coplanar
   iv) (1,0) and (0,1) are collinear
   v) $\overrightarrow{(1,0)}$ and $\overrightarrow{(0,1)}$ are collinear
   vi) $\vec{i}$ and $\vec{j}$ are collinear
6. Does the line through the points A(2,-1,-3) and B(-6,3,8) pass through the origin? Does it pass through (2,7,20)?
7. R is (1,1), D is (3,3), P is (5,1). Find a fourth point G so that the four points form vertices of a parallelogram. How many possible positions does point G have?
8. M,N,P are the mid-points of edges of a triangle. M is (1,1,0) N is (-2,4,5) and P is (3,2,-1). Find the co-ordinates of the vertices of the triangle.
9. True or False?
   a) $|i + j| = 2$
   b) $|i + k| = 2$
   c) | **(3,4,12)** | = 13
   d) If A is (-1, 0, 4) and **AB** = (**2, 1, 5**) then B is (1, 1, 9).
   e) **i**+ **j** is perpendicular to the z axis
   f) **OA** = 2**OB** + **OC** implies that O, A, B and C are coplanar.
   g) |**OA**| = 2|**OB** |+ |**OC**| implies that O, A, B and C are coplanar.
10. Find P if AP : PB = 2 : 7 where A is (1, 2, 3) and B is (-3, 9, 2).

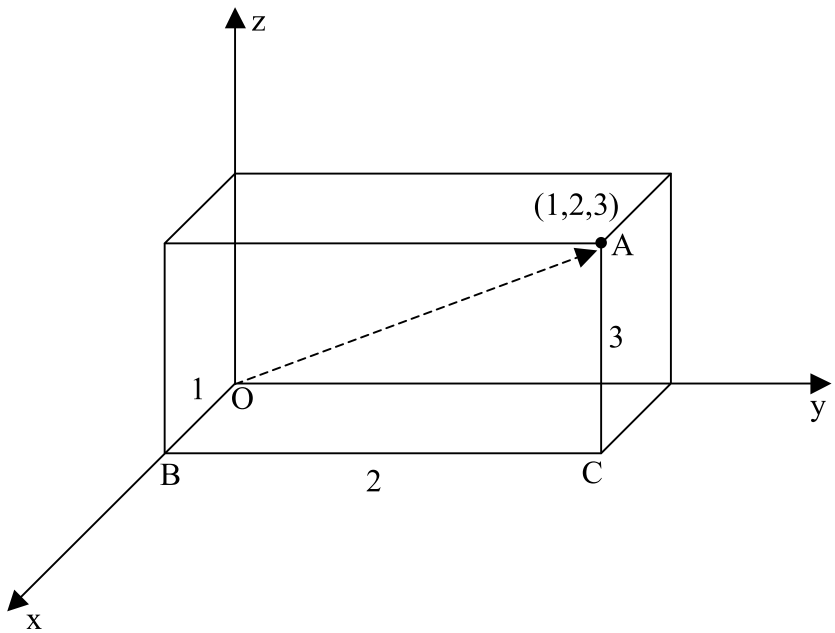
11. Find m, n if A, B and C are collinear where A is (1, 2, 3), B is (4, 7, -2) and C is (5, n, m).
12. Find m if A, B, C and D are coplanar where A is (m, m, m), B is (1, 2, 3), C is (3, 1, 2) and D is (2, 3, 1).

Exercise 2.1Answers

1. (**3,-5,0**)
2. m = 6.   P-Q-R
3. m = $-\frac{1}{2}$
4. (-1,1,4)
5. i) True   ii) False   iii) False   iv) True   v)False   vi) False
6. No No
7. G is (7,3) or (-1,3) or (3,-1)
8. (-4,3,6) (6,-1,-6) (0,5,4)
9. a) F   b) F   c) T   d) T   e) T   f) T   g) F
10. $\left(\frac{1}{9},\frac{32}{9},\frac{25}{9}\right)$
11. $n=\frac{26}{3}$, $m=\frac{-11}{3}$
12. $m=2$

2.2 Magnitude of a vector

If $\vec{a} = \overrightarrow{(1,2,3)}$ then $|\vec{a}|$= distance OA



In ΔOBC

OC = $\sqrt{5}$ (Pythagoras)

In ΔOCA

$$OA = \sqrt{OC^2 + CA^2} = \sqrt{5+3^2} = \sqrt{14}$$

i.e. $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 3^2}$

In general if $\vec{a} = \overrightarrow{(a_1, a_2, a_3)}$ then $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Note of course that $|\vec{a}|$ **is a positive real number** (or zero).

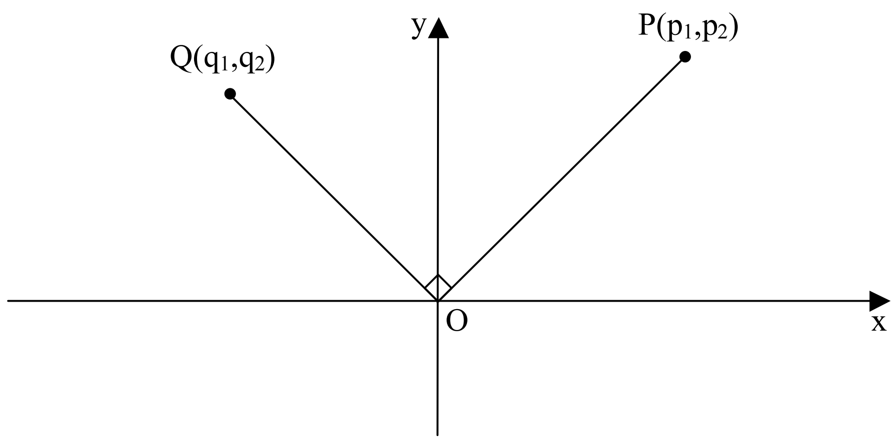
A **UNIT VECTOR** is a vector whose magnitude is 1 e.g. $\overrightarrow{(1,0,0)}, \overrightarrow{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)}$ are unit vectors.

To **NORMALISE** a vector means to multiple a vector by a scalar to make its magnitude 1.

e.g. $\overrightarrow{(1,2,3)}$ has a magnitude $\sqrt{14}$. Therefore $\frac{1}{\sqrt{14}}\overrightarrow{(1,2,3)}$ has magnitude 1.

i.e. $\overrightarrow{(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}})}$ is a unit vector.

Perpendicularity of vectors



To find a relationship between components of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ determining that OP is perpendicular to OQ.

Since OP is perpendicular to OQ then (slope of OP) x (slope of OQ) = -1

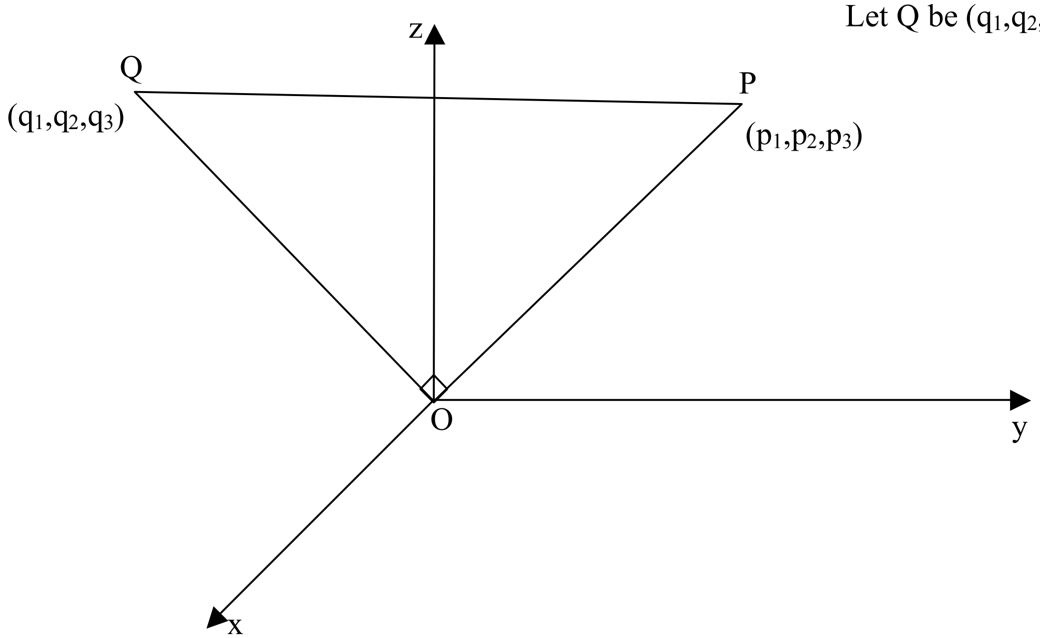
i.e. $\frac{p_2}{p_1}\times\frac{q_2}{q_1}=-1$

i.e. $p_2q_2 = -\, p_1q_1$

i.e. $p_1q_1 + p_2q_2 = 0$

To investigate perpendicularity of vectors in R^3

Let P be (p_1,p_2,p_3)
Let Q be (q_1,q_2,q_3)



Here in R^3, we have no notion of slope, so we use Pythagoras in ΔOPQ.

i.e $OP^2 + OQ^2 = PQ^2$

i.e. $p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2$

i.e. $0 = -2p_1q_1 - 2p_2q_2 - 2p_3q_3$

i.e. $p_1q_1 + p_2q_2 + p_3q_3 = 0$

Note the similarity of result with that of the perpendicularity in the R^2 case.

Example

A is (1,-1,1), B is (2,2,2), C is (4,-2,1)

i) Investigate whether $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$.

If $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$ then 1x2 + 3x(-4) + 1x(-1) = 0

This is false. Therefore $\overrightarrow{AB}$ is not perpendicular to $\overrightarrow{BC}$,

ii) Investigate whether $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$

$\overrightarrow{AB} = \overrightarrow{(1,3,1)}$ and $\overrightarrow{AC} = \overrightarrow{(3,-1,0)}$

Here 1x3 + 3x(-1) + 1x0 does equal 0 and therefore

$\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ i.e. $\angle A = 90°$

Dot Product (or Scalar Product)

If $\vec{a} = \overrightarrow{(a_1, a_2, a_3)}$ and $\vec{b} = \overrightarrow{(b_1, b_2, b_3)}$ then

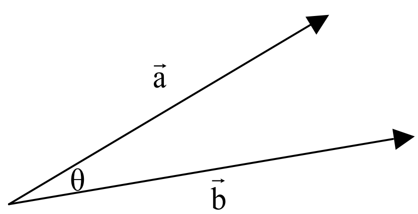
$$\boxed{\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

This is called a dot b. Note $\vec{a} \bullet \vec{b}$ is a **NUMBER** (not a vector). $\vec{a} \bullet \vec{b}$ is called the dot product or scalar product of $\vec{a}$ and $\vec{b}$.

$$\boxed{\vec{a} \text{ is perpendicular to } \vec{b} \text{ iff } \vec{a} \bullet \vec{b} = 0}$$

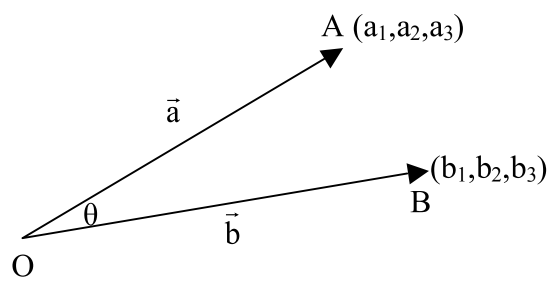
Theorem 2.1

$\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$.



Proof

Label diagram as below.



By the Cosine Rule in ΔOAB,

$$|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$$

i.e. $(b_1-a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + a_3^2 - 2|\vec{a}||\vec{b}|\cos\theta$

i.e. $-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = 2|\vec{a}||\vec{b}|\cos\theta$

i.e. $a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos\theta$

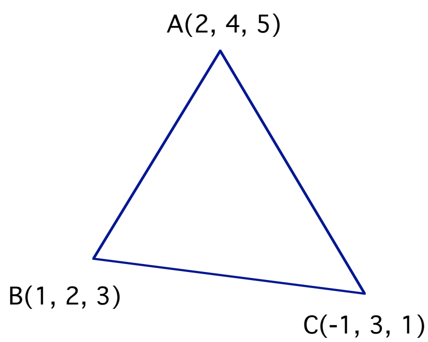
i.e $\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

In R^2 the proof is almost identical, since $\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2$ where $\vec{a} = \overrightarrow{(a_1, a_2)}$ and $\vec{b} = \overrightarrow{(b_1, b_2)}$.

Note that $\vec{a}$ perpendicular to $\vec{b}$ is an example of this theorem where $\theta = 90°$. Since $\cos 90° = 0$ it follows that $\vec{a} \bullet \vec{b} = 0$.

Example

To find $\angle ABC$ where A is (2, 4, 5), B is (1, 2, 3) and C is (-1, 3, 1).



Now **BA • BC = |BA| |BC|** $\cos\angle ABC$

Therefore **(1, 2, 2)** •**(-2, 1, 2)** = |**(1, 2, 2)**| |**(-2, 1, 2)**| $\cos\angle ABC$

$4 = (3)(3)\cos\angle ABC$

$\frac{4}{9} = \cos\angle ABC$

$63.6° = \angle ABC$.

Exercise 2.2

1. When does $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|$?
2. Does $(\vec{a} \bullet \vec{b}) \bullet \vec{c} = \vec{a} \bullet (\vec{b} \bullet \vec{c})$?
3. Does $(\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b}) = \vec{a} \bullet \vec{a} - \vec{b} \bullet \vec{b}$?
4. Does $\dfrac{\vec{a} \bullet \vec{b}}{\vec{a} \bullet \vec{c}}$ make sense? If so, does it equal $\dfrac{\vec{b}}{\vec{c}}$ or $\dfrac{|\vec{b}|}{|\vec{c}|}$?
5. Find m so that ΔABC is right-angled at B where

   A is (1,2,m), B is (3,4,5), C is (6,8,-2)
6. A is (5,0,0), B is $(6,1,\sqrt{2})$ and C is (4, -1, $\sqrt{2}$). Find $\angle BAC$.
7. Show that $\vec{a} \bullet \vec{a} = |\vec{a}|^2$
8. True or False?

   i) $\vec{a} \bullet \vec{a} = \vec{b} \bullet \vec{b}$ implies $\vec{a} = \vec{b}$

   ii) $\vec{a} \bullet \vec{a} = \vec{b} \bullet \vec{b}$ implies $|\vec{a}| = |\vec{b}|$

   iii) $2\vec{i} - \vec{k} = \overrightarrow{(2,-1)}$

   iv) $\vec{a} \bullet \vec{a} = 0$ implies $\vec{a} = \mathbf{0}$

   v) $\vec{a} \bullet \vec{a} \geq 0$ for all vectors $\vec{a}$

   vi) $\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$

   vii) $\overrightarrow{(1,1,1)}$ is a unit vector

   viii) $\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|$ implies **a** = m**b** for some scalar m

   ix) $\vec{a} \bullet \vec{b} \leq |\vec{a}||\vec{b}|$

   x) $(3\vec{a}) \bullet \vec{b} = \vec{a} \bullet (3\vec{b})$

   xi) $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$

9. A is (1,1,1) B is (2,3-5) C is (4,0,-2) Is ΔABC right-angled?

10. Let $\vec{a} = \overrightarrow{(1,1)}$. Give an example for $\vec{b}$ and $\vec{c}$ where $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$ but $\vec{b} \neq \vec{c}$. i.e. the ‘Cancellation Law’ for dot product is not valid.

11. Find a vector perpendicular to both $(\overrightarrow{2,3,-5})$ and $(\overrightarrow{1,-2,1})$
12. Find the distance between (1,-1,1) and (-1,3,-2)
13. Given that A is (1,4,5), B is (3,5,4) and C is (2,3,4)
    i) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{AC}|$
    ii) Find $\overrightarrow{AB} \bullet \overrightarrow{AC}$ and hence cos $\angle$BAC
    iii) Find sin$\angle$BAC and hence find the area of ΔABC
    iv) Find the distance from C to the line through A and B (use iii).

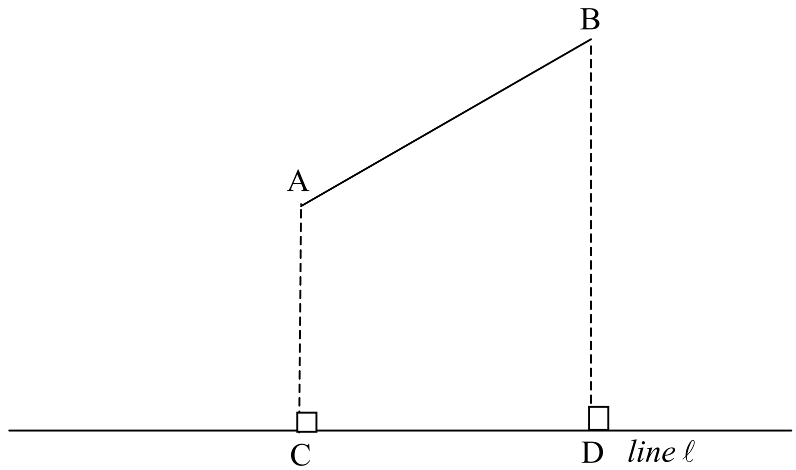
14. ABCD is a rhombus. A is (1,1,1), B is (2,3,3), C is (3,1,n) and D is (m,p,r). Find n,m,p,r.
15. Show that $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2\vec{a}\bullet\vec{b} + |\vec{b}|^2$

<u>Exercise 2.2 Answers</u>

1. When $\vec{u}$ and $\vec{v}$ have the same direction
2. No. 3. Yes. 4. Yes. No. No. 5. m = 3
6. 90°
8. i) False ii) True iii) False iv) True v) True vi) True vii) False viii) True ix) True x) True xi) True xii) True
9. Yes. $\angle C = 90°$
10. $\vec{b} = (\overrightarrow{3,4})$, $\vec{c} = (\overrightarrow{2,5})$ 11. $(\overrightarrow{1,1,1})$ 12. $\sqrt{29}$
13. i) $\sqrt{6}, \sqrt{3}$ ii) 2 $\cos\angle BAC = \dfrac{\sqrt{2}}{3}$ iii) $\sin\theta = \dfrac{\sqrt{7}}{3}$, Area $= \dfrac{\sqrt{14}}{2}$

    iv) $\dfrac{\sqrt{21}}{3}$
14. n = 5, m = 2, p = -1, r = 3 **<u>or</u>** n =1, m = 2, p = -1, r = -1.

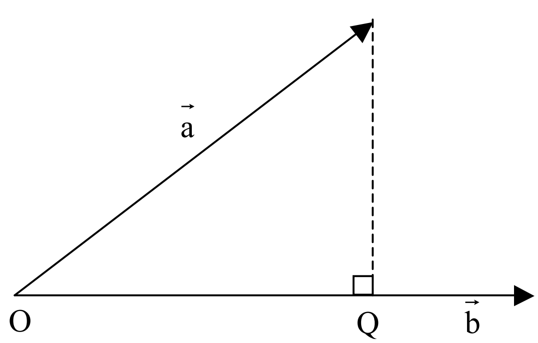
2.3 Projection



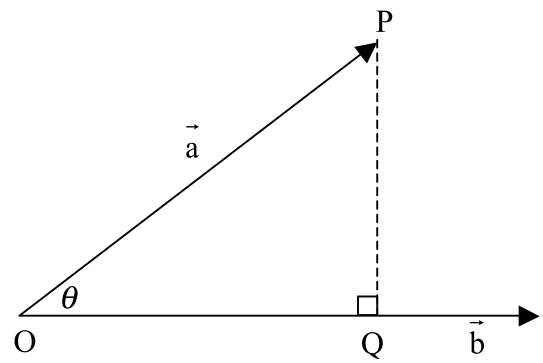
Let AB be a line segment. The length of the projection of AB onto the *line* ℓ is defined to be CD where AC and BD are individually perpendicular to ℓ with C and D lying on ℓ.

This applies even if AB and ℓ are not coplanar. Note that CD can be thought of as the shadow cast by AB on ℓ from a strip of lighting parallel to ℓ. This assumes AB and ℓ are coplanar of course.

For example the length of projection of $\vec{a}$ onto $\vec{b}$ in the diagram is the length OQ.



To find the length of projection of $\vec{a}$ onto $\vec{b}$.



Let θ be the acute angle between $\vec{a}$ and $\vec{b}$ in ΔOPQ

Then $\cos\theta = \dfrac{|\overrightarrow{OQ}|}{|\overrightarrow{OP}|} = \dfrac{|\overrightarrow{OQ}|}{|\vec{a}|}$

i.e. $\left|\overrightarrow{OQ}\right| = \left|\vec{a}\right|\cos\theta$. ☆

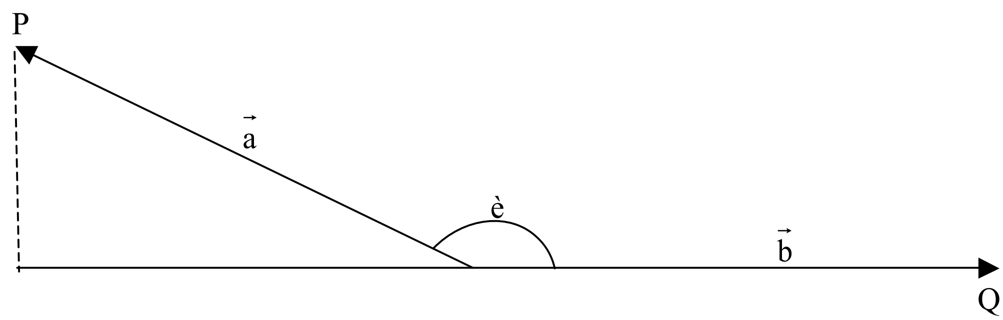
But $\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ (Theorem 2.1)

Therefore $\cos\theta = \dfrac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|}$

Substituting for $\cos\theta$ in ☆ above we have

$$\left|\overrightarrow{OQ}\right| = \left|\vec{a}\right| \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|}$$

In this text we will only be concerned with the length of the projection. In the case where $90° < \theta < 180°$, $\vec{a} \bullet \vec{b}$ will be negative but since we are only interested in lengths, i.e. magnitudes of vectors concerned, we will consider the absolute value of $\vec{a} \bullet \vec{b}$.



$$\text{i.e. Length of projection of } \vec{a} \text{ onto } \vec{b} = \frac{|\vec{a} \bullet \vec{b}|}{|\vec{b}|}$$

At this stage this result is not of great significance but in later chapters where we wish to find distances between lines and planes etc. it will be of great assistance.

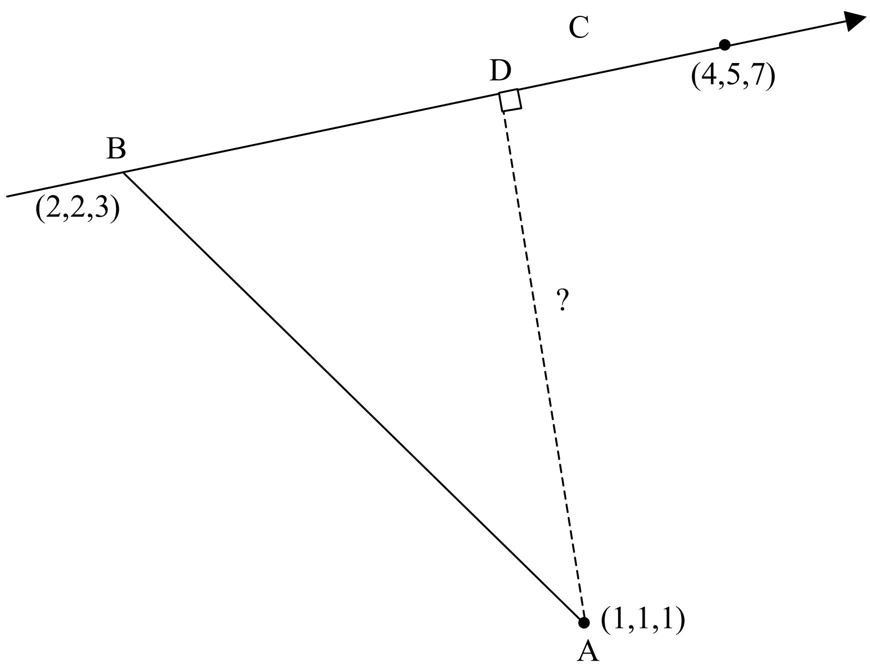
From a diagram point of view it is clear that the magnitude of $\vec{b}$, as expressed in the concept above, does not matter. We can show this algebraically by considering the length of projection of $\vec{a}$ onto $2\vec{b}$. This will be the same length as the length of projection of $\vec{a}$ onto $\vec{b}$.

The length of projection of $\vec{a}$ onto $2\vec{b} = \dfrac{|\vec{a} \bullet 2\vec{b}|}{|2\vec{b}|} = \dfrac{|2(\vec{a} \bullet \vec{b})|}{2|\vec{b}|}$

$= \dfrac{2|\vec{a} \bullet \vec{b}|}{2|\vec{b}|} = \dfrac{|\vec{a} \bullet \vec{b}|}{|\vec{b}|}$ = the length of projection of $\vec{a}$ onto $\vec{b}$ as required.

<u>Example Question</u>

Find the distance from (1,1,1) to the line passing through (2,2,3) and (4,5,7).



Let the points A,B,C,D be as in the diagram.

$$\overrightarrow{AB} = \overrightarrow{(1,1,2)} \text{ and } \overrightarrow{BC} = \overrightarrow{(2,3,4)}$$

The length of projection of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$, which is the length BD, $= \dfrac{\left|\overrightarrow{AB} \bullet \overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$

$$= \frac{|1\times 2 + 1\times 3 + 2\times 4|}{\sqrt{2^2 + 3^2 + 4^2}} = \frac{13}{\sqrt{29}}$$

Using Pythagoras in ΔABD we have

$$AD^2 = AB^2 - BD^2 = 6 - \frac{169}{29} = \frac{5}{29}$$

i.e. AD $= \dfrac{\sqrt{5}}{\sqrt{29}} = .415$ (aprrox).

i.e. The distance of (1,1,1) from the line passing through (2,2,3) and (4,5,7) is .415 (approx)

Exercise 2.3

1. Find the length of projection of $\overrightarrow{(2,3,4)}$ onto $\overrightarrow{(1,1,1)}$.
2. Find the angle between $\overrightarrow{(3,3,0)}$ and $\overrightarrow{(2,1,1)}$.
3. Find the angle which $\overrightarrow{(1,1,1)}$ makes with the x axis.
4. Find a point A on the line y = 2x in the xy plane so that AB is perpendicular to OA where B is (2,4,-2).
5. Find a vector perpendicular to both $3\vec{i} - \vec{j}$ and $\vec{j} - 2\vec{k}$
6. Find the distance from (0,0,0) to the line passing through (1,2,1) and (2,3,2)
7. Find m so that (1,1,1), (2,3,4), (2,5,3) and (3,5,m) are coplanar points.
8. Find the angles which $\overrightarrow{(1,1,\sqrt{2})}$ makes with the

   i) x axis   ii) y axis   iii) z axis
9. Find the angle between the diagonal of a cube and one of its edges.
10. A is (1,-3) and B is (-3,4). Find point C on the x axis so that $\angle ACB = 90°$.
11. Show that $\vec{a} \bullet \vec{b} = \dfrac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{b} - \vec{a}|^2}{2}$

Exercise 2.3 Answers

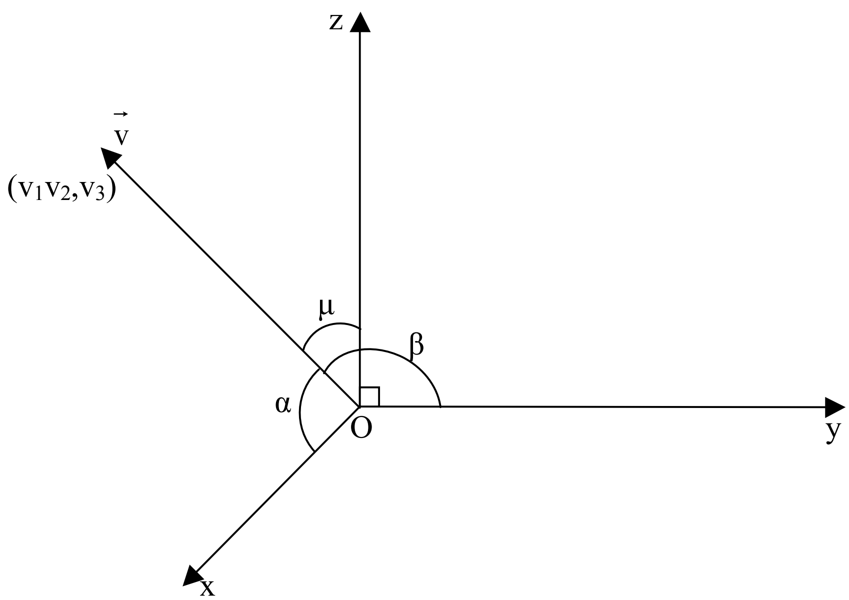
1. $3\sqrt{3}$
2. 30°
3. 54.7°
4. (2,4,0)
5. $\overrightarrow{(2,6,3)}$
6. $\dfrac{\sqrt{6}}{3}$
7. m = 7
8. i) 60° ii) 60° iii) 45°
9. 54.7°
10. (-5,0) or (3,0)

<u>Theorem 2.2</u>

If α, β, γ are the angles made by a vector $\vec{v}$ with the x,y,z axes respectively then

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma \equiv 1}$$

<u>Proof</u>



Let $\vec{v} = (\overrightarrow{v_1, v_2, v_3})$

<u>To find α</u>

$\vec{v} \bullet \vec{i} = |\vec{v}||\vec{i}|\cos\alpha$

Therefore $(\overrightarrow{v_1, v_2, v_3}) \bullet (\overrightarrow{1,0,0}) = |\vec{v}| \times 1 \times \cos\alpha$

i.e. $v_1 = |\vec{v}|\cos\alpha$

Therefore $\cos\alpha = \dfrac{v_1}{|\vec{v}|}$

Similarly $\cos\beta = \dfrac{v_2}{|\vec{v}|}$ and $\cos\gamma = \dfrac{v_3}{|\vec{v}|}$

Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{v_1^2}{|\vec{v}|^2} + \dfrac{v_2^2}{|\vec{v}|^2} + \dfrac{v_3^2}{|\vec{v}|^2} = \dfrac{v_1^2 + v_2^2 + v_3^2}{|\vec{v}|^2} = \dfrac{|\vec{v}|^2}{|\vec{v}|^2} = 1.$

■

Note that a **unit** vector can always be written as $\overrightarrow{(\cos\alpha,\ \cos\beta,\ \cos\gamma)}$ where α, β, γ are the angles made with the x,y,z axes respectively.

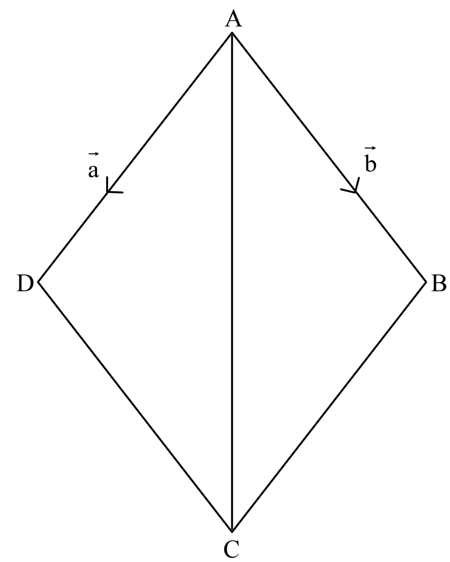
2.4 Geometric Examples of Use of Dot Product

Example 1.

Prove "Diagonals of a rhombus are perpendicular"

Proof

Let the rhombus be as in the diagram:



Let $\overrightarrow{AD} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

Since ABCD is a parallelogram $\overrightarrow{BC} = \vec{a}$ and $\overrightarrow{DC} = \vec{b}$. Also, since ABCD is a rhombus, $|\vec{a}| = |\vec{b}|$.

But $\overrightarrow{AC} = \vec{a} + \vec{b}$ and $\overrightarrow{DB} = \vec{b} - \vec{a}$.

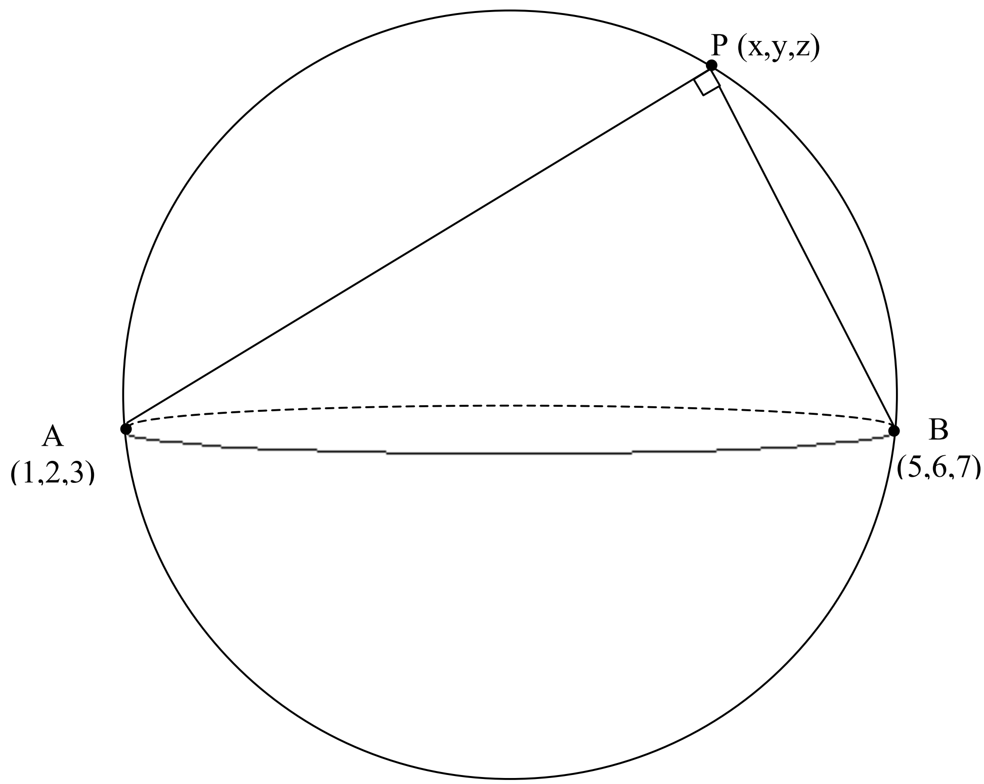
Consider
$$\begin{aligned}\overrightarrow{AC} \bullet \overrightarrow{DB} &= (\vec{b} + \vec{a}) \bullet (\vec{b} - \vec{a}) \\ &= \vec{b} \bullet \vec{b} - \vec{a} \bullet \vec{a} \\ &= |\vec{b}|^2 - |\vec{a}|^2 \\ &= 0 \quad \text{since } |\vec{b}| = |\vec{a}|\end{aligned}$$

Therefore AC is perpendicular to DB

i.e. 'diagonals of a rhombus are perpendicular'.

Example 2

To find an equation of the sphere having AB as a diameter where A is (1,2,3) and B is (5,6,7).



Let P (x,y,z) be an arbitrary point on the sphere.

Then $\angle APB = 90^\circ$ for any point P on the sphere.

$\therefore \overrightarrow{AP} \bullet \overrightarrow{BP} = 0$

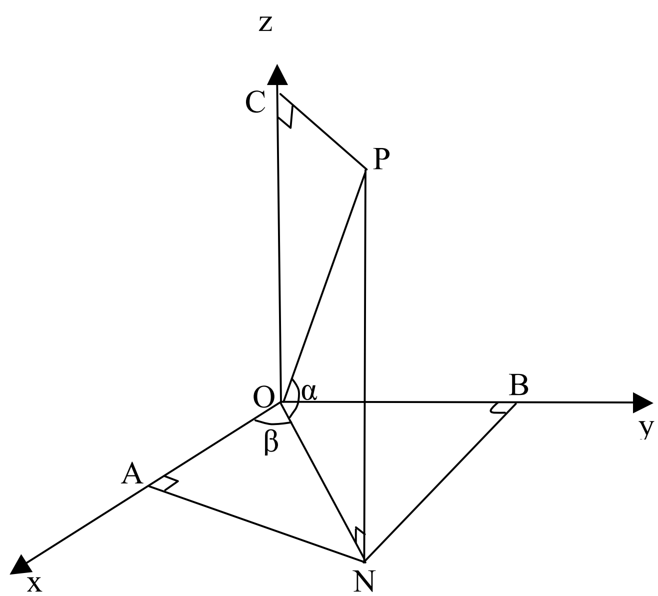
$\therefore \overrightarrow{(x-1, y-2, z-3)} \bullet \overrightarrow{(x-5, y-6, z-7)} = 0$

$\therefore (x-1)(x-5) + (y-2)(y-6) + (z-3)(z-7) = 0$

To find co-ordinates of a point on the surface of the Earth.

The radius of the earth is approximately 6366 kms. We will call it R.
Given a point on the surface of the Earth with latitude α(North) and longitude β (East), we can parametrise any point on the Earth as ($R\cos\alpha\cos\beta$, $R\cos\alpha\sin\beta$, $R\sin\alpha$)
(Note that $x^2 + y^2 + z^2 = R^2$)

To show the derivation of the parametrisation



Note that OA = x co-ordinate of P

$= ON\cos\beta$

$= (OP\cos\alpha)\cos\beta$

$= R\cos\alpha\cos\beta$

OB = y co-ordinate

$= ON\sin\beta$

$= (OP\cos\alpha)\sin\beta$

$= R\cos\alpha\sin\beta$

$OC = OP\cos(90 - \alpha) = R\sin\alpha$

$\therefore$ The parametrisation of P is ($R\cos\alpha\cos\beta$, $R\cos\alpha\sin\beta$, $R\sin\alpha$).

To find the great circle distance from A (20°N 40°E) to B(50°N 75°E)

Remember that a great circle is a circle on the Earth whose centre is the centre of the Earth. A line of latitude (except the Equator) is not a great circle but all lines of longitude are great circles.

A is R(cos20ºcos40º, cos20ºsin40º, sin20°)

B is R(cos50°cos75°, cos50°sin75°, sin50°)

Where R is the radius of the Earth.

Let O be the centre of the circle , i.e., the centre of the Earth.

Then to find $\angle$AOB we use

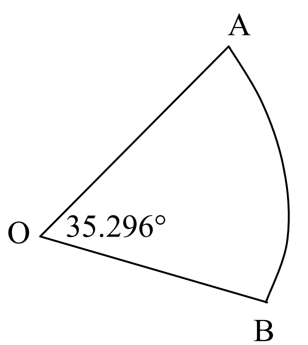
OA•OB = |**OA**||**OB**|cos$\angle$AOB

Then,

$R^2(\cos 20°\cos 40°\cos 50°\cos 75° + \cos 20°\sin 40°\cos 50°\sin 75° + \sin 20°\sin 50°) = R^2\cos\angle AOB$

$\therefore 0.81618 = \cos\angle AOB$

$35.296° = \angle AOB$



$\therefore$ The great circle distance $arc\ AB = \dfrac{35.296}{360} \times 2\pi \times 6366$ kms

$= 3922$ kms

Exercise 2.4

1. $\vec{a}$ and $\vec{b}$ are sides of an equilateral triangle. Find $\left|\vec{a} \bullet \vec{b}\right|$
2. If $\left|\vec{a}\right| = 5$, $\left|\vec{b}\right| = 4$ and $\vec{a} \bullet \vec{b} = 10$ what is the angle between $\vec{a}$ and $\vec{b}$?
3. True or False?

   i) $\overrightarrow{OA} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}$ implies A is the mid point of BC.

   ii) $\overrightarrow{AC} = \overrightarrow{EF} + \overrightarrow{DC}$ implies $\overrightarrow{EF} = \overrightarrow{AD}$

   iii) $(\vec{a} \bullet \vec{b})\vec{c} = \vec{a}(\vec{b} \bullet \vec{c})$

   vi) $\vec{a} \bullet \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|$ implies $\vec{a}$ and $\vec{b}$ are parallel.

   v) $\vec{a} \bullet \vec{b} = \vec{c} \bullet \vec{b}$ implies $\vec{a} = \vec{c}$ or $\vec{b} = 0$

   vi) $(\vec{a} + \vec{b})$ perpendicular to $(\vec{a} - \vec{b})$ implies $\left|\vec{a}\right| = \left|\vec{b}\right|$.
4. Use dot product to prove that the angle in a semi-circle is 90°.
5. Find a vector perpendicular to $\vec{i}$, $\vec{i} + \vec{j}$ and $\vec{i} + 2\vec{j}$. (Yes it is possible)
6. Explain why a vector in R^3 cannot make angles of 30° with both the x axis and the y axis.
7. A vector in R^3 makes angles of 60° with both the x axis and the y axis. Find the angle it makes with the x axis.
8. Show that (0,0,0), (1,2,0), (3,2,-1) and (m,m,m) can never be coplanar for any value of m other than zero.
9. A is (1,1,1), B is (2,3,-5) an C is (4,0,-2). Show that ΔABC is right-angled and hence find its area.
10. Find the length of projection of $\overrightarrow{(1,2,3)}$ onto $\overrightarrow{(4,5,6)}$
11. Show that if A,B,C,D have position vectors $\vec{a}, \vec{b}, \vec{c}, \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b} + \frac{1}{3}\vec{c}$ then A,B,C,D are coplanar.
12. Prove that $\left|\vec{u}\right|\vec{v} + \left|\vec{v}\right|\vec{u}$ and $\left|\vec{u}\right|\vec{v} - \left|\vec{v}\right|\vec{u}$ are perpendicular.
13. A is (a_1,a_2,a_3), B is (b_1,b_2,b_3) and C is (c_1,c_2,c_3). If (a_1,a_2), (b_1,b_2) and (c_1,c_2) are collinear and (a_2,a_3), (b_2,b_3), (c_2,c_3) are collinear does it follow that A,B,C are collinear? Does it make any difference if a_2,b_2,c_2 are all the same?

14. If a triangle has two medians of equal length show that the triangle is isosceles. (Hint: $\vec{a} \bullet \vec{a} = |\vec{a}|^2$)

15. Find the great circle distance from Toronto (44º N, 79ºW) to Singapore (1º N, 104ºE).

Exercise 2.4 Answers

1. $\frac{1}{2}|\vec{a}|^2$
2. 60°
3. i) True ii) True iii) False iv) True v) False vi) True
5. $\vec{k}$
6. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ would not be true.
7. 45° or 135°
9. 10.22
10. 3.65
13. No, for example (2,3,1) (1,3,2) (0,3,0). Yes the points are now collinear.
15. 14 991 kms

Chapter Review Questions

1. True or False?

   i) $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$ implies $\vec{b} = \vec{c}$

   ii) $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$

   iii) $m\vec{a} \bullet \vec{b} = \vec{a} \bullet m\vec{b}$

   iv) $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$ AND $\vec{b} \neq \vec{c}$ implies $\vec{a} = \vec{0}$.

   v) $\vec{a} \bullet \vec{i} = \vec{b} \bullet \vec{i}$ implies $\vec{a} \bullet \vec{j} = \vec{b} \bullet \vec{j}$

   vi) $\vec{a} \bullet \vec{b} = 0$ and $\vec{a} \bullet \vec{c} = 0$ implies $\vec{b} = m\vec{c}$ for some scalar m.

   vii) $\vec{a} \bullet \vec{b} = 0$ and $\vec{a} \bullet \vec{i} = 0$ implies $\vec{b} = m\vec{i}$ for some scalar m.

   viii) $\vec{a} \bullet \vec{a} = |\vec{a}|^2$

ix) $\vec{a} \bullet \vec{b} = 0$ implies $\vec{a}$ is perpendicular to $\vec{b}$.

x) $|\vec{a}|(\vec{b} + (\vec{c} \bullet \vec{d})\vec{e})$ is a vector.

xi) $\vec{a} \bullet (\overrightarrow{ab}) = (\vec{a} \bullet \vec{a})\vec{b}$

xii) $(\vec{a} + \vec{b}) \bullet \vec{c} = (\vec{a} \bullet \vec{c}) + (\vec{b} \bullet \vec{c})$

xiii) $\vec{a} \bullet \vec{a} = 0 \rightarrow \vec{a} = 0$

xiv) $(\vec{a} \bullet \vec{b})(\vec{c} \bullet \vec{d}) = (\vec{a} \bullet \vec{d})(\vec{b} \bullet \vec{c})$

xv) $\sqrt{\vec{a} \bullet \vec{a}} = \vec{a}$

xvi) $(\vec{a} - |\vec{c}|) \bullet (\vec{a} + |\vec{c}|)$ is a scalar.

2. Find the length of projection of $\overrightarrow{(3,2,1)}$ on $\overrightarrow{(2,\text{-}1,3)}$.
3. Find the area of ΔABC where A is (1,1,1), B is (2,2,2) and C is (3,4,5).
4. Find a vector perpendicular to both $2\vec{i} - \vec{j} - \vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$.
5. $\overrightarrow{(3,1,2)}, \overrightarrow{(2,0,1)}$ and $\overrightarrow{(2,m,m)}$ are coplanar. Find m.
6. Find the distance from (1,2,3) to the line passing through (-1,3,4) and (1,5,-3).
7. i) Find the angle between $\overrightarrow{(1,3)}$ and $\overrightarrow{(\text{-}1,2)}$.

   ii) Find the angle between $\overrightarrow{(3,3,0)}$ and $\overrightarrow{(2,1,1)}$.
8. Find a point A on the line y = x in the xy plane so that AB is perpendicular to AO where B is (2,4-3).
9. $\overrightarrow{AB} = 2\vec{i} - \vec{j} + \vec{k}$. A is (-7,3,-5). Let $n \in \Re$ be scalar such that $\overrightarrow{AC} = n\overrightarrow{AB}$ where C lies on the xy plane. Find n.
10. If $\overrightarrow{(\frac{1}{2},m,0)}$ is a unit vector, find m.
11. (1,1,-1),(0,1,-2),(-1,0,0) and (1,3,m) are coplanar points. Find m.
12. Prove $|\vec{a} + \vec{b}|^2 = \vec{a} \bullet \vec{a} + \vec{b} \bullet \vec{b} + 2\vec{a} \bullet \vec{b}$
13. Explain how it is possible for A,B,C,D to be coplanar, B,C,D,E to be coplanar and yet A,B,C,E are not coplanar.

Chapter 2 Review Exercises Answers

1. i) False ii) True iii) True iv) False v) False vi) False
   vii) False viii) True ix) True x) True xi) False xii) True
   xiii) True xiv) False xv) False xvi) False
2. 2.94
3. 1.225
4. $\overrightarrow{(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})}$
5. m = 2
6. $\frac{\sqrt{261}}{\sqrt{57}} = 2.14$
7. i) 45° ii) 30°
8. (3,3,0)
9. n = 5
10. m= $\pm\frac{\sqrt{3}}{2}$
11. m = -3

Practice Test 1 on Chapter 2

1. True or False

   a) $\vec{a} \bullet \vec{b} = 0$ implies $\vec{a}$ is perpendicular to $\vec{b}$

   b) If $\overrightarrow{OA} \bullet \overrightarrow{OB} = 0$ then slope of OA is the negative reciprocal of slope of OB.

   c) (3, 4, 5), (4, 5, 6) and (5, 6, 7) are collinear.

   d) If ABCD is a rhombus then **AC** • **BD** = 0

   e) If ABCD is a rhombus then **AB • AB = AD • AD**

   f) Given any set of four vectors in $\Re^3$, then one can be written in terms of the others

   g) The position vector of the point (1,2,3) is $\vec{i} + 2\vec{j} + 3\vec{k}$

   h) $\overrightarrow{(1,2,3)} \bullet \overrightarrow{(3,2,1)} = \overrightarrow{(3,4,3)}$

   i) (1,1,1) (2,1,3) (2.1,4.8,5.7) are coplanar points

j) $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ implies that $\vec{a},\vec{b},\vec{c}$ are coplanar

k) The length of projection of $\vec{i}+\vec{j}$ onto $\vec{j}+\vec{k}$ is zero

2. Find the value of m so that (2,1,-2), (2,0,-1), (1,-1,1) and (m,2,3) are coplanar points.
3. Explain in your own words why it is not true that $\dfrac{\vec{a}\bullet\vec{b}}{\vec{a}\bullet\vec{c}}=\dfrac{|\vec{b}|}{|\vec{c}|}$
4. Find a vector making angles of 60° with the x axis and 60° with the z axis in R^3.
5. Let V be the set of all vectors such that $\vec{v}\bullet\overrightarrow{(1,1,2)}=0$ Find the set V.
6. Prove that if $(\vec{a}-\vec{b})$ is perpendicular to $(\vec{a}+\vec{b})$ then $|\vec{a}|=|\vec{b}|$
7. $\vec{a}$ is a vector such that $\vec{a}\bullet\vec{i}=\vec{a}\bullet\vec{j}=\vec{a}\bullet\vec{k}$. Find the angle which $\vec{a}$ makes with the z axis.

<u>Practice Test 1 Answers</u>

1. a) T b) T c) T d) T e) T f) T g) T h) F i)T j)F k)F
2. m = -4
3. The equation is true when $\vec{b}$ and $\vec{c}$ are parallel
4. Any multiple of $\overrightarrow{(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2})}$
5. The set is a plane of vectors all perpendicular to $\overrightarrow{(1,1,2)}$
7. $\cos^{-1}(\frac{1}{\sqrt{3}})$

Practice Test 2 on Chapter 2

1. True or False?
    a) (3,2,1) is perpendicular to (1,1,-5)
    b) $\overrightarrow{(3,2,1)}$ is perpendicular to $\overrightarrow{(1,1,-5)}$
    c) $\vec{i} \bullet \vec{j} = \vec{j} \bullet \vec{k}$
    d) $\vec{a} \bullet \vec{a} = 0$ implies $\vec{a} = 0$
    e) $\overrightarrow{(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}})}$ is a unit vector
    f) $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$
    g) $\vec{a} \bullet \vec{i} = \vec{b} \bullet \vec{i}$ and $\vec{a} \bullet \vec{j} = \vec{b} \bullet \vec{j}$ implies $\vec{a} = \vec{b}$
    h) If $\vec{a} \bullet \vec{i} = \vec{a} \bullet \vec{j} = \vec{a} \bullet \vec{k}$ and $\vec{a} \in R^3$ then $\vec{a}$ is the position vector for point A where A lies on the line x = y = z.
    i) $\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$
    j) $\vec{i} \bullet \vec{i} = \vec{k} \bullet \vec{k}$
    k) $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$ implies $\vec{b} = \vec{c}$ or $\vec{a} = 0$.
    l) $2\vec{a} \bullet \vec{b} = 2\vec{b} \bullet \vec{a}$

2. Find the value of m so that (1,1,m), (2,3,-1), (3,2,-1) and (3,-3,0) are coplanar points
3. R is (1,1,2), G is (2,3,4), M is (0,-2,7). Find point C so that RCGM is a parallelogram.
4. (1,1) (4,c) (3,3) and (a,b) are vertices of a rhombus. Find the values of a, b and c.
5. Find a point Q dividing PR in the ratio $\frac{2}{5}:\frac{3}{5}$ (i.e. PQ:QR = $\frac{2}{5}:\frac{3}{5}$) if P is (1,1,1) and R is (6,-9,-4)
6. Prove that if $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{a}$ is perpendicular to $\vec{c}$ then $\vec{a}$ is perpendicular to $2\vec{b} + 3\vec{c}$.

7. Let $\vec{a}$ be $\overrightarrow{(1,2,-3)}$ and let $\vec{b}$ be $\overrightarrow{(3,-2,-1)}$. Find the vector **(7, -2, m)** which is in the same plane as $\vec{a}$ and $\vec{b}$.

<u>Practice Test 2 on Chapter 2 Answers</u>

1. a) F b) T c) T d) T e) T f) T g) F ($\vec{a}$ and $\vec{b}$ could be in R^3)
   h) T i) T j) T k) F l) T
2. $m = -\frac{2}{5}$
3. (3,6,-1)
4. a = 0, b = 4 and c = 0
5. (3,-3,1)
7. m = 5

<u>Practice Test 3 on Chapter 2</u>

1. True or False?
   a) $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$ implies $\vec{b} = \vec{c}$ or $\vec{a} = 0$
   b) $\overrightarrow{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ makes an angle of 60° with the x axis
   c) $\overrightarrow{(\frac{1}{3},\frac{1}{3},\frac{1}{3})}$ is a unit vector
   d) A vector perpendicular to a plane is perpendicular to every vector in that plane.
   e) If a vector is perpendicular to two non-parallel vectors then the three vectors must lie in $\Re^3$.
   f) $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$ implies $\vec{a}$ is perpendicular to $\vec{b}$
2. Do you think that the set S = $\left\{\overrightarrow{(x,y,z)} : xz = 0\right\}$ is such that if $\vec{a} \in S$ and if $\vec{b} \in S$ then $\vec{a} + \vec{b} \in S$?

3. RFDP is a rhombus. R is (1,1,-m) F is (2,m,4) D is (3,r,t) and P is (s,3,4). Find the values of m,r,s,t.
4. The length of projection of $\vec{i} + m\vec{j}$ onto $\overrightarrow{(2,2,1)}$ is 4. Find the value(s) of m.
5. ABCD is a regular tetrahedron. E,F,G,H are mid-points of AB, CD, BC and AD respectively. Show that $\overrightarrow{EF} \bullet \overrightarrow{GH} = 0$
6. Show that $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2\vec{a} \bullet \vec{b} + |\vec{b}|^2$

Practice Test 3 on Chapter 2 Answers

1. a)F b)F c)F d)T f)F
2. No. S is not closed under addition. For example, (0,1,2) and (2,1,0) each belongs to S but their addition **(2,2,2)** does not belong to S.
3. m = -1, r = 1, s = 2 and t = 7 **OR** m = 3, r = 5, s = 2 and t = 11
4. m = 5 or – 7