IFUM-830-FT

A simple parametrization for G_2

Sergio L. Cacciatori[1,2*]

[1] Dipartimento di Matematica dell'Università di Milano,
Via Saldini 50, I-20133 Milano, Italy.

[2] INFN, Sezione di Milano,
Via Celoria 16, I-20133 Milano, Italy.

Abstract

We give a simple parametrization of the G_2 group, which is consistent with the structure of G_2 as a $SU(3)$ fibration. We also explicitly compute the (bi)invariant measure, which turns out to have a simple expression.

*sergio.cacciatori@mi.infn.it

1 Introduction

Group theory plays an important role in physics. When a group appears as a symmetry, as for example a gauge symmetry, one needs to integrate over the group; to have an unbroken symmetry in a path-integral formulation of the theory, the integration measure must be invariant under the group action.
For semisimple groups there is a unique (up to normalization constants) invariant measure, obtained pulling back the Killing form on the algebra via the left action of the group over itself. In general however difficulties arise in finding an explicit parametrization of the group elements which give a simple expression for the invariant measure and also permit a full determination of the range of parameters. This is in fact what one needs to do explicit computations.

In [1], a solution of this problem was found for the group G_2. There the group parametrization emphasizes the structure of the group as a fibration with $SO(4)$ as fiber and the space $\mathcal{H}$ of quaternionic linear subalgebras of octonions as base. Here I give another parametrization for G_2, based on the well known fact that G_2 can be seen as an $SU(3)$ fibration over the six sphere S^6.[1] The resulting measure turn out to have a very simple expression, with all the parameters varying in a $14-$dimensional hypercube.

2 The representation algebra and the group ansatz

For the algebra we choose the fundamental ($\mathbf{7}$) representation as in [1]. We repeat in appendix C only the commutators matrix $B_{IJ} := [C_I, C_J]$. Now note that $\{C_i\}_{i=1}^8$ generate the $su(3)$ algebra. Thus we want to use this fact to construct a new parametrization for the G_2 elements, which underlines the $SU(3)$ subgroup.
To this end let us note that C_9 commutators with $su(3)$ generate all the remaining generators of the g_2 algebra. We can then hope to find a parametrization of the group miming the Euler parametrization for $SU(n)$. That is we write the generic element of G_2 in the form

$$g = SU(3)[\alpha_1, \dots, \alpha_8] e^{\beta C_9} SU(3)[\gamma_1\ , \dots\ , \gamma_8] \tag{2.1}$$

where $SU(3)[\alpha_1, \dots, \alpha_8]$ is the generic Euler parametrization of $SU(3)$ shown in appendix A.
However here we have three redundant parameters, the dimension of G_2 being 14. We can easily eliminate it as follows. In (2.1) the left $SU(3)$ term has the form

$$SU(3)[\alpha_1, \dots, \alpha_8] = h[\alpha_1, \dots, \alpha_5] e^{\alpha_6 C_3} e^{\alpha_7 C_2} e^{\alpha_8 C_3}\ . \tag{2.2}$$

Now $C_1\ , C_2\ , C_3$ commute with C_9 so that we can absorb $\alpha_1\ , \alpha_2\ , \alpha_3$ in the right $SU(3)$ in (2.1). We finally obtain the ansatz

$$g[\alpha_1\ , \dots\ , \alpha_6\ ; \gamma_1\ , \dots\ , \gamma_8] = \Sigma[\alpha_1\ , \dots\ , \alpha_6] SU(3)[\gamma_1\ , \dots\ , \gamma_8] \tag{2.3}$$

with

$$\Sigma[\alpha_1\ , \dots\ , \alpha_6] = e^{\alpha_1 C_3} e^{\alpha_2 C_2} e^{\alpha_3 C_3} e^{\frac{\sqrt{3}}{2}\alpha_4 C_8} e^{\alpha_5 C_5} e^{\frac{\sqrt{3}}{2}\alpha_6 C_9} \tag{2.4}$$

To show that this ansatz solves our problem we must show that we can choose the range of the parameters such to cover one time all G_2 (up to a subset of vanishing measure).

[1] See for example [2].

2.1 Fibration and computation of the metric

We said that the quotient of G_2 with the $SU(3)$ subgroup is a sphere S^6. We will use this information to establish the range of the parameters. For brevity we will call $S := SU(3)[\gamma_1, \dots, \gamma_8]$. The metric on the group can be obtained from the left invariant currents

$$Jg = g^{-1}dg = \sum_{I=1}^{14} Jg^I C_I \tag{2.5}$$

via[2]

$$ds^2_{G_2} = -\frac{1}{4}Trace\{Jg \otimes Jg\} \ . \tag{2.6}$$

If we now write

$$J_\Sigma = \Sigma^{-1} \cdot d\Sigma = \sum_{I=1}^{14} J^I_\Sigma C_I \ , \tag{2.7}$$

$$J_S = dS \cdot S = \sum_{I=1}^{8} J^I_S C_I \ , \tag{2.8}$$

it is straightforward to show that

$$ds^2_{G_2} = \sum_{I=1}^{8} \left(J^I_S + J^I_\Sigma\right)^2 + \sum_{I=9}^{14} \left(J^I_\Sigma\right)^2 \ . \tag{2.9}$$

Now Σ parametrizes the quotient space between G_2 and the $SU(3)$ orbits generated by S. Fixing the base point Σ we recover the $SU(3)$ invariant metric

$$ds^2_{SU(3)} = \sum_{I=1}^{8} \left(J^I_S\right)^2 \ . \tag{2.10}$$

To find the induced metric on the base one must compute the current $\Sigma^{-1} \cdot d\Sigma$ and project out the components tangent to the $su(3)$ directions. Doing so one finds

$$ds^2_{BASE} = \sum_{I=9}^{14} \left(J^I_\Sigma\right)^2 \ . \tag{2.11}$$

If all goes right this must be the metric of a six sphere. We now show that this is exactly the case. Using the results shown in appendix B for the currents, one finds

$$\frac{4}{3}ds^2_{BASE} = d\alpha_6^2 + \sin^2\alpha_6 \left\{ d\alpha_5^2 + \cos^2\alpha_5 d\alpha_4^2 + \sin^2\alpha_5 \left[s_1^2 + s_2^2 + \left(s_3 + \frac{1}{2}d\alpha_4\right)^2 \right] \right\} \tag{2.12}$$

[2]The factor $-\frac{1}{4}$ is due to the normalization $Trace\{C_I C_J\} = -4\delta_{IJ}$. Choosing a metric $(C_I|C_J) = \delta_{IJ}$ on the algebra is equivalent to fix normalizations such that long and short roots of g_2 have length 2 and $\frac{2}{\sqrt{3}}$ respectively [1].

where

$$
\begin{aligned}
s_1 &= -\sin(2\alpha_2)\cos(2\alpha_3)d\alpha_1 + \sin(2\alpha_3)d\alpha_2 \\
s_2 &= \sin(2\alpha_2)\sin(2\alpha_3)d\alpha_1 + \cos(2\alpha_3)d\alpha_2 \\
s_3 &= \cos(2\alpha_2)d\alpha_1 + d\alpha_3
\end{aligned}
\tag{2.13}
$$

We can now recognize the metric of a six sphere S^6 with coordinates $(\alpha_6\ ,\vec{X})$, where α_6 is the azimuthal coordinate, $\alpha_6 \in [0,\pi]$, and $\vec{X}$ cover a five sphere. We can look at this sphere as immersed in $\mathbb{C}^3$ via

$$
\vec{X} = (z_1\ ,z_2\ ,z_3) = \left(\cos\alpha_5 e^{i\alpha_4}\ ,\sin\alpha_5\cos\alpha_2 e^{i\left(\alpha_1+\alpha_3+\frac{\alpha_4}{2}\right)}\ ,\sin\alpha_5\sin\alpha_2 e^{i\left(\alpha_1-\alpha_3-\frac{\alpha_4}{2}\right)}\right)\ ,
$$
$$
\alpha_1 \in [0\ ,\pi]\ , \qquad \alpha_2 \in \left[0\ ,\frac{\pi}{2}\right]\ , \qquad \alpha_3 \in [0\ ,2\pi]\ , \qquad \alpha_4 \in [0\ ,2\pi]\ , \qquad \alpha_5 \in \left[0\ ,\frac{\pi}{2}\right]\ .
$$

Computing the metric $ds^2_{S^5} = |dz_1|^2 + |dz_2|^2 + |dz_3|^2$ in these coordinates we find

$$
\frac{4}{3}ds^2_{BASE} = d\alpha_6^2 + \sin^2\alpha_6\left\{ds^2_{S^5}\right\}\ . \tag{2.14}
$$

This complete our identification.

3 Conclusions

We can thus parametrize the elements of the group G_2 as

$$
g = e^{\alpha_1 C_3}e^{\alpha_2 C_2}e^{\alpha_3 C_3}e^{\frac{\sqrt{3}}{2}\alpha_4 C_8}e^{\alpha_5 C_5}e^{\frac{\sqrt{3}}{2}\alpha_6 C_9}SU(3)[\gamma_1\ ,\dots\ ,\gamma_8] \tag{3.1}
$$

with

$$
\begin{aligned}
&\alpha_1 \in [0\ ,\pi]\ , \qquad \alpha_2 \in \left[0\ ,\frac{\pi}{2}\right]\ , \qquad \alpha_3 \in [0\ ,2\pi]\ , \qquad \alpha_4 \in [0\ ,2\pi]\ , \\
&\alpha_5 \in \left[0\ ,\frac{\pi}{2}\right]\ , \qquad \alpha_6 \in [0\ ,\pi]\ , \qquad \gamma_1 \in [0\ ,2\pi]\ , \qquad \gamma_2 \in \left[0\ ,\frac{\pi}{2}\right]\ , \\
&\gamma_3 \in [0\ ,\pi]\ , \qquad \gamma_4 \in \left[0\ ,\frac{\pi}{2}\right]\ , \qquad \gamma_5 \in [0\ ,2\pi]\ , \qquad \gamma_6 \in [0\ ,\pi]\ , \\
&\gamma_7 \in \left[0\ ,\frac{\pi}{2}\right]\ , \qquad \gamma_8 \in [0\ ,\pi]\ .
\end{aligned}
\tag{3.2}
$$

From (2.9) one could easily find the biinvariant metric. The corresponding invariant measure is

$$
d\mu_{G_2} = \frac{27}{32}\sin^5\alpha_6\cos\alpha_5\sin^3\alpha_5\sin(2\alpha_2)d\mu_{SU(3)}d\alpha_6 d\alpha_5 d\alpha_4 d\alpha_3 d\alpha_2 d\alpha_1\ . \tag{3.3}
$$

$d\mu_{SU(3)}$ being the invariant measure over $SU(3)$ as given in App.A.
This is a simple parametrization for G_2, which underlines the structure of G_2 as a $SU(3)$ fibration over S^6. It could be used to implement analytical or numerical computations in lattice gauge theory and in random matrix models.
Here we used a geometrical approach to determine the range of parameters, but the topological method of [1] could also be used, giving the same results.

Acknowledgments

I would like to thank Professor Ruggero Ferrari which introduced me to the problem, Bianca Letizia Cerchiai for computer checks, Professor Antonio Scotti for his participation and my friend Aldo Clericò for encouragements.
I also thank Dietmar Klemm, Alberto Della Vedova, Giovanni Ortenzi and Giuseppe Berrino for useful discussions.
This work was partially supported by INFN, COFIN prot. 2003023852_008 and the European Commission RTN program MRTN–CT–2004–005104 in which S. C. is associated to the University of Milano–Bicocca.

A Euler parametrization for $SU(3)$.

The Euler parametrization for $SU(3)$ can be easily obtained as[3]

$$SU(3)[\gamma_1,\dots,\gamma_8]=e^{\gamma_1 C_3}e^{\gamma_2 C_2}e^{\gamma_3 C_3}e^{\gamma_4 C_5}e^{\sqrt{3}\gamma_5 C_8}e^{\gamma_6 C_3}e^{\gamma_7 C_2}e^{\gamma_8 C_3}\ , \tag{A.1}$$

with range

$$\begin{aligned}&\gamma_1\in[0\ ,2\pi]\ ,\qquad \gamma_2\in\left[0\ ,\frac{\pi}{2}\right]\ ,\qquad \gamma_3\in[0\ ,\pi]\ ,\qquad \gamma_4\in\left[0\ ,\frac{\pi}{2}\right]\ ,\\ &\gamma_5\in[0\ ,2\pi]\ ,\qquad \gamma_6\in[0\ ,\pi]\ ,\qquad \gamma_7\in\left[0\ ,\frac{\pi}{2}\right]\ ,\qquad \gamma_8\in[0\ ,\pi]\ .\end{aligned} \tag{A.2}$$

In this way $\gamma_1,\gamma_2,\gamma_3$ cover an $SU(2)$ subgroup, γ_5 covers $U(1)$ and $\gamma_6,\gamma_7,\gamma_8$ cover $SO(3)$.
The resulting invariant measure is

$$d\mu_{SU(3)}=\sqrt{3}\sin(2\gamma_2)\sin^3\gamma_4\cos\gamma_4\sin(2\gamma_7)\prod_{i=1}^{8}d\gamma_i. \tag{A.3}$$

B J_h currents.

Here we give the currents J_h used to generate the metric on the base manifold. The relations we need follow from the commutators matrix given in App. C and are

$$\begin{aligned}
e^{-xC_3}C_2e^{xC_3}&=\cos(2x)C_2+\sin(2x)C_1\\
e^{-xC_3}C_1e^{xC_3}&=\cos(2x)C_1-\sin(2x)C_2\\
e^{-xC_2}C_3e^{xC_2}&=\cos(2x)C_3-\sin(2x)C_1\\
e^{-xC_5}C_1e^{xC_5}&=\cos xC_1+\sin xC_6\\
e^{-xC_5}C_2e^{xC_5}&=\cos xC_2+\sin xC_7\\
e^{-xC_5}C_3e^{xC_5}&=\frac{1}{4}(3+\cos(2x))C_3-\frac{1}{2}\sin(2x)C_4-\frac{\sqrt{3}}{2}\sin^2 xC_8\\
e^{-xC_5}C_8e^{xC_5}&=\frac{1}{4}(1+3\cos(2x))C_8-\frac{\sqrt{3}}{2}\sin(2x)C_4-\frac{\sqrt{3}}{2}\sin^2 xC_3\\
e^{-\sqrt{3}xC_9}C_4e^{\sqrt{3}xC_9}&=\cos^3 xC_4-\sin^3 xC_7+\sqrt{3}\cos x\sin^2 xC_{11}-\sqrt{3}\sin x\cos^2 xC_{14}\\
e^{-\sqrt{3}xC_9}C_5e^{\sqrt{3}xC_9}&=\cos^3 xC_5+\sin^3 xC_6+\sqrt{3}\cos x\sin^2 xC_{12}+\sqrt{3}\sin x\cos^2 xC_{13}\\
e^{-\sqrt{3}xC_9}C_6e^{\sqrt{3}xC_9}&=\cos^3 xC_6-\sin^3 xC_5+\sqrt{3}\cos x\sin^2 xC_{13}-\sqrt{3}\sin x\cos^2 xC_{12}\\
e^{-\sqrt{3}xC_9}C_7e^{\sqrt{3}xC_9}&=\cos^3 xC_7+\sin^3 xC_4+\sqrt{3}\cos x\sin^2 xC_{14}+\sqrt{3}\sin x\cos^2 xC_{11}\\
e^{-\sqrt{3}xC_9}C_8e^{\sqrt{3}xC_9}&=\cos(2x)C_8+\sin(2x)C_{10}\ ,
\end{aligned} \tag{B.1}$$

and the fact that C_9 commutes with C_1,C_2 and C_3.
If we put

$$\begin{aligned}s_1&=-\sin(2\alpha_2)\cos(2\alpha_3)d\alpha_1+\sin(2\alpha_3)d\alpha_2\\ s_2&=\sin(2\alpha_2)\sin(2\alpha_3)d\alpha_1+\cos(2\alpha_3)d\alpha_2\end{aligned}$$

[3]again the ranges could be found using the topological method, however we simply adapted the results shown in App. B of [3] to our case.

$$s_3 = \cos(2\alpha_2)d\alpha_1 + d\alpha_3 \tag{B.2}$$

the resulting currents are then

$$
\begin{aligned}
J_h^1 &= \cos\alpha_5 s_1\\
J_h^2 &= \cos\alpha_5 s_2\\
J_h^3 &= \frac{1}{4}(3+\cos(2\alpha_5))s_3 - \frac{3}{4}\sin^2\alpha_5 d\alpha_4\\
J_h^4 &= -\frac{1}{2}\sin(2\alpha_5)\cos^3\frac{\alpha_6}{2}\left[s_3+\frac{3}{2}d\alpha_4\right] + \sin a_5\sin^3\frac{\alpha_6}{2}s_2\\
J_h^5 &= -\sin\alpha_5\sin^3\frac{\alpha_6}{2}s_1 + \cos^3\frac{\alpha_6}{2}d\alpha_5\\
J_h^6 &= \sin\alpha_5\cos^3\frac{\alpha_6}{2}s_1 + \sin^3\frac{\alpha_6}{2}d\alpha_5\\
J_h^7 &= \frac{1}{2}\sin(2\alpha_5)\sin^3\frac{\alpha_6}{2}\left[s_3+\frac{3}{2}d\alpha_4\right] + \sin\alpha_5\cos^3\frac{\alpha_6}{2}s_2\\
J_h^8 &= \frac{\sqrt{3}}{2}\cos\alpha_6\left[\frac{1}{4}(1+3\cos(2\alpha_5))d\alpha_4 - \sin^2\alpha_5 s_3\right]\\
J_h^9 &= \frac{\sqrt{3}}{2}d\alpha_6\\
J_h^{10} &= \frac{\sqrt{3}}{2}\sin\alpha_6\left[\frac{1}{4}(1+3\cos(2\alpha_5))d\alpha_4 - \sin^2\alpha_5 s_3\right]\\
J_h^{11} &= -\frac{\sqrt{3}}{2}\sin(2\alpha_5)\cos\frac{\alpha_6}{2}\sin^2\frac{\alpha_6}{2}\left[s_3+\frac{3}{2}d\alpha_4\right] + \sqrt{3}\sin\alpha_5\sin\frac{\alpha_6}{2}\cos^2\frac{\alpha_6}{2}s_2\\
J_h^{12} &= -\sqrt{3}\sin\alpha_5\sin\frac{\alpha_6}{2}\cos^2\frac{\alpha_6}{2}s_1 + \sqrt{3}\cos\frac{\alpha_6}{2}\sin^2\frac{\alpha_6}{2}d\alpha_5\\
J_h^{13} &= \sqrt{3}\sin\alpha_5\cos\frac{\alpha_6}{2}\sin^2\frac{\alpha_6}{2}s_1 + \sqrt{3}\sin\frac{\alpha_6}{2}\cos^2\frac{\alpha_6}{2}d\alpha_5\\
J_h^{14} &= \frac{\sqrt{3}}{2}\sin(2\alpha_5)\sin\frac{\alpha_6}{2}\cos^2\frac{\alpha_6}{2}\left[s_3+\frac{3}{2}d\alpha_4\right] + \sqrt{3}\sin\alpha_5\cos\frac{\alpha_6}{2}\sin^2\frac{\alpha_6}{2}s_2\ .
\end{aligned}
\tag{B.3}
$$

These currents together the $SU(3)$ currents (as given in [3]) can be used in (2.9) to compute the full (bi-)invariant metric of G_2.

C The commutators matrix.

$$
B = \begin{pmatrix}
0 & 2C_3 & -2C_2 & C_7 & C_6 & -C_5 & -C_4 & 0 & 0 & 0 & C_{14} & C_{13} & -C_{12} & -C_{11} \\
* & 0 & 2C_1 & -C_6 & C_7 & C_4 & -C_5 & 0 & 0 & 0 & -C_{13} & C_{14} & C_{11} & -C_{12} \\
* & * & 0 & C_5 & -C_4 & C_7 & -C_6 & 0 & 0 & 0 & C_{12} & -C_{11} & C_{14} & -C_{13} \\
* & * & * & 0 & C_3 + \sqrt{3}C_8 & -C_2 & C_1 & -\sqrt{3}C_5 & -C_{14} & -C_{13} & 0 & 0 & C_{10} & C_9 \\
* & * & * & * & 0 & C_1 & C_2 & \sqrt{3}C_4 & C_{13} & -C_{14} & 0 & 0 & -C_9 & C_{10} \\
* & * & * & * & * & 0 & C_3 - \sqrt{3}C_8 & \sqrt{3}C_7 & -C_{12} & C_{11} & -C_{10} & C_9 & 0 & 0 \\
* & * & * & * & * & * & 0 & -\sqrt{3}C_6 & C_{11} & C_{12} & -C_9 & -C_{10} & 0 & 0 \\
* & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_{10} & -\frac{2}{\sqrt{3}}C_9 & -\frac{1}{\sqrt{3}}C_{12} & \frac{1}{\sqrt{3}}C_{11} & \frac{1}{\sqrt{3}}C_{14} & -\frac{1}{\sqrt{3}}C_{13} \\
* & * & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_8 & C_7 - \frac{2}{\sqrt{3}}C_{14} & \frac{2}{\sqrt{3}}C_{13} - C_6 & C_5 - \frac{2}{\sqrt{3}}C_{12} & \frac{2}{\sqrt{3}}C_{11} - C_4 \\
* & * & * & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_{13} + C_6 & \frac{2}{\sqrt{3}}C_{14} + C_7 & -\frac{2}{\sqrt{3}}C_{11} - C_4 & -\frac{2}{\sqrt{3}}C_{12} + C_5 \\
* & * & * & * & * & * & * & * & * & * & 0 & -\frac{1}{\sqrt{3}}C_8 + C_3 & \frac{2}{\sqrt{3}}C_{10} - C_2 & -\frac{2}{\sqrt{3}}C_9 + C_1 \\
* & * & * & * & * & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_9 + C_1 & \frac{2}{\sqrt{3}}C_{10} + C_2 \\
* & * & * & * & * & * & * & * & * & * & * & * & 0 & \frac{1}{\sqrt{3}}C_8 + C_3 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & 0
\end{pmatrix}
$$

References

[1] S. L. Cacciatori, B. L. Cerchiai, A. Della Vedova, G. Ortenzi and A. Scotti, "Euler angles for G2," arXiv:hep-th/0503106.

[2] J. F. Adams, "Lectures on exceptional Lie groups", The University of Chicago Press.

[3] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, "Cohomogeneity one manifolds of Spin(7) and G(2) holonomy," Phys. Rev. D **65** (2002) 106004 [arXiv:hep-th/0108245].