Manifolds with G_2 Holonomy

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Abstract

Riemannian manifolds with exceptional holonomy groups G_2 and Spin(7) have enjoyed a recent bout of popularity in the last few years, both in mathematics and in physics. In physics, their popularity stems from the fact that G_2-manifolds arise as the natural space on which to compactify 11-dimensional M-theory. In this essay we review Berger's classification of the holonomy groups and the status of the exceptional holonomy groups. We also review the basic algebraic and geometric properties of the Lie group G_2 and its relation to the octonion algebra. Finally, we highlight some of the properties of 7-manifolds with G_2 holonomy.

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1 Introduction

The classification of Riemannian holonomy groups (see Theorem 3 below) includes two exceptional cases: that of G_2 holonomy on a 7-dimensional manifold and Spin(7) holonomy on an 8-manifold. These holonomy groups are called the exceptional holonomy groups. Although the list has been around for a long time, little was known about the exceptional cases until quite recently. One of the basic difficulties is that there is no basic existence theorem for the exceptional holonomy groups like Yau's proof of the Calabi conjecture for manifolds with $\mathrm{SU}(n)$ holonomy. So examples have to be constructed on a case by case basis, and these are not easy to come by. One should note that even though existence has been proven there are still no known explicit examples of metrics with G_2 holonomy (or, for that matter, $\mathrm{SU}(n)$ holonomy) on a compact manifold.

Physicists studying string theory have been extremely interested in manifolds of special holonomy for the primary reason that they are natural spaces on which to compactify the extra spatial dimensions present in those theories. In order to get the correct low energy physics in four dimensions (namely $N = 1$ supersymmetry) it is necessary that the manifold on which you compactify admit a nonzero parallel (covariantly constant) spinor, which in turn implies that the manifold be Ricci-flat (see Lemma 4). For 10-dimensional string theory the necessary space is a Calabi-Yau 3-fold (a 6-manifold with SU(3) holonomy), while for 11-dimensional supergravity or M-theory the correct space turns out, as we shall see, to be a 7-manifold with G_2 holonomy.

Given the importance of G_2-manifolds in physics and their recent popularity in mathematics we decided to review in this essay the basic concepts needed to understand these ideas in greater detail. In section 2 we review the notion of holonomy groups and present Berger's classification theorem. In section 3 we discuss the definition and some of the basic properties of the, perhaps unfamiliar, exceptional Lie group G_2 and discuss its relation to the octonion algebra. Finally, in section 4 we discuss manifolds with G_2 holonomy.

The standard reference for this material is the monograph by Joyce [1] although the much briefer review [2] of the construction of compact G_2 manifolds is also useful. A good introduction to exceptional holonomy in string theory is Gubser [3]. Most of the material on the octonions and the algebraic structure of G_2 was taken from Baez [4] and Harvey [5].

2 Holonomy Groups

Let (M, g) be an n-dimensional Riemannian manifold and let ∇ be the Levi-Civita connection on M. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$. Parallel transport then defines an linear map $P_\gamma\colon T_xM \to T_yM$. For the Levi-Civita connection this map is actually an isometry as the metric is constant, $\nabla g = 0$. If γ is a loop then parallel transport defines a self-isometry of T_xM. The set of all loops based at x then gives rise to a group of isometries of T_xM called the **holonomy group** based at x, denoted $\mathrm{Hol}_x(g)$. It is a subgroup of the group of all isometries at x, which is isomorphic to $\mathrm{O}(n)$. On a connected manifold the holonomy groups based at different points are conjugate subgroups of $\mathrm{O}(n)$:

$$P_\gamma \operatorname{Hol}_x(g) P_\gamma^{-1} = \operatorname{Hol}_y(g) \tag{1}$$

where γ is any path from x to y in M. This fact allows us to drop reference to the base point and simply define the holonomy group $\operatorname{Hol}(g)$ of M as a subgroup of $\mathrm{O}(n)$ defined up to conjugation.

Often one is interested only in the local (or restricted) holonomy group $\operatorname{Hol}^0(g)$ which is obtained by restricting the parallel transport to topologically trivial loops (i.e. loops that can be contracted to a point). We state here, without proof, some basic properties of holonomy groups:

Proposition 1. *Let (M, g) be a connected Riemannian manifold of dimension n. Then*

- $\operatorname{Hol}(g)$ *is a subgroup of* $\mathrm{O}(n)$*; it is a subgroup of* $\mathrm{SO}(n)$ *iff M is orientable,*
- $\operatorname{Hol}^0(g)$ *is a closed, connected Lie subgroup of* $\mathrm{SO}(n)$*,*
- $\operatorname{Hol}^0(g)$ *is the identity component of* $\operatorname{Hol}(g)$*,*
- *There is a natural, surjective group homomorphism* $\phi\colon \pi_1(M) \to \operatorname{Hol}(g)/\operatorname{Hol}^0(g)$*,*
- $\operatorname{Hol}(\tilde{g}) \cong \operatorname{Hol}^0(g)$ *where $(\tilde{M}, \tilde{g})$ is the universal Riemannian cover of (M, g),*
- $\operatorname{Hol}^0(g)$ *is trivial iff g is flat.*

2.1 Classification Theorem

We know that the holonomy group of a Riemannian manifold is necessarily a subgroup on $\mathrm{O}(n)$, but one might wonder which subgroups of $\mathrm{O}(n)$ actually occur. A complete classification, at least for the simply connected case, was completed by Berger in 1955. Before we get to that theorem, however, we need a few preliminaries.

Proposition 2. *Let $(M_1 \times M_2, g_1 \times g_2)$ be a Riemannian product manifold. Then* $\operatorname{Hol}(g_1 \times g_2) = \operatorname{Hol}(g_1) \times \operatorname{Hol}(g_2)$.

A Riemannian metric is said to be **locally reducible** if it is locally isometric to a product metric, otherwise the metric is said to be **irreducible**. In classifying the possible holonomy groups it makes sense to restrict oneself to the irreducible case, for if g is locally reducible its holonomy will be a product of holonomy groups in lower dimensions.

Cartan concerned himself with the problem of classifying the holonomy groups of Riemannian manifolds M with $\nabla R = 0$, called **locally symmetric spaces**. Locally symmetric spaces turn out to be locally isometric to symmetric homogenous spaces G/H with restricted holonomy $\operatorname{Hol}^0 = H$ acting by the adjoint representation. By applying his classification of simple Lie groups, Cartan was able to classify the holonomy groups of all irreducible, simply-connected, symmetric spaces. For the (long) list of the possibilities we refer the reader to [6].

If M is not locally symmetric ($\nabla R \neq 0$) then the possible holonomy turns out to be very restricted. The basic reason is that holonomy group determines the curvature tensor R and by symmetries of that tensor most groups would actually force $\nabla R = 0$. The remaining possibilities are as follows:

Theorem 3 (Berger). *Let (M, g) be a simply-connected, irreducible, nonsymmetric n-dimensional manifold. Then one of the following cases holds*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$ *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$,

(iii) $n = 2m$ *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$,

(iv) $n = 4m$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$,

(v) $n = 4m$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$,

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$,

Note that if one wishes to relax the requirement that M be simply-connected then the theorem still holds for the restricted holonomy group $\mathrm{Hol}^0(g)$. In cases (ii)–(v) we should really restrict $m \geq 2$ to avoid redundancy. Cases (vi) and (vii) are exceptional in this list and so the holonomy groups G_2 and Spin(7) are referred to as the **exceptional holonomy groups**. The group G_2 is a particular simply-connected, compact Lie group that will be defined in §3.2.

At the time Berger constructed his list it was not known whether or not all of the groups on the list actually appeared as holonomy groups of nonsymmetric manifolds.[1] This question has now been answered in the affirmative. The two exceptional cases held out until 1987 when Bryant [7] first proved their existence. In 1989, Bryant and Salamon [8] found examples of *complete* metrics with with G_2 and Spin(7) holonomy, and Joyce constructed the first examples on *compact* manifolds in 1996 [9, 10, 1].

2.2 Parallel Spinors and Holonomy

Much of the interest in manifolds with special holonomy, especially in physics, stems from the fact that they are precisely the manifolds which admit nonzero constant spinors.

Let (M, g) be an oriented, Riemannian n-manifold. Then there is a principal subbundle $P \to M$ of the frame bundle with fiber $\mathrm{SO}(n)$ called the oriented, orthonormal frame bundle. As the structure group $\mathrm{SO}(n)$ is not simply-connected, one is concerned with whether or not there exists a suitable "double-cover" of P with structure group $\mathrm{Spin}(n)$ (the universal cover of $\mathrm{SO}(n)$). Such a cover is called a spin structure on M. If M admits a spin structure then it is said to be a spin manifold. The spinor representation of $\mathrm{Spin}(n)$ then gives rise to an associated vector bundle $S \to M$ called the spin bundle. Sections of the spin bundle are called spinors. Finally, the Levi-Civita connection ∇ of g induces a natural connection on S, also denoted by ∇, called the spin connection. The holonomy group associated to the spin connection will, in general, be a subgroup of $\mathrm{Spin}(n)$ (rather than $\mathrm{SO}(n)$). It may be isomorphic to to $\mathrm{Hol}(g)$ or it may be a double cover.

Given a spin manifold M, a **parallel spinor** (or in the physics literature: a covariantly constant spinor) on M is a spinor ψ such that

$$\nabla \psi = 0 \tag{2}$$

[1] Actually, an eighth case, $n = 16$ and $\mathrm{Hol}(g) = \mathrm{Spin}(9)$, was eliminated when it was shown that every manifold with Spin(9) holonomy was symmetric.

Parallel spinors are simply those that are fixed by parallel transport in the spin bundle and so their existence naturally restricts the holonomy group of g and, in turn, the curvature. The key result is the following [11, 12]:

Lemma 4. *Every Riemannian spin manifold admitting a nonzero parallel spinor is Ricci-flat.*

Suppose M is an irreducible, Ricci-flat manifold. It cannot be locally symmetric since Ricci-flat, locally symmetric manifolds are flat and hence locally reducible (at least for $n > 1$). Berger's classification (Theorem 3) then tells us what the possible holonomy groups can be. Wang [12] applied this reasoning to show that for a simply-connected, irreducible manifold the only possible holonomy groups are[2]

$$\begin{array}{ll} \mathrm{SU}(m) & n = 2m \\ \mathrm{Sp}(m) & n = 4m \\ G_2 & n = 7 \\ \mathrm{Spin}(7) & n = 8 \end{array}$$

Furthermore, manifolds with one of the above holonomy groups are always spin, with a preferred spin structure, and each such manifold will have at least one nonzero parallel spinor. Therefore, an irreducible Riemannian manifold admits a nonzero parallel spinor if and only if it has one of the above Ricci-flat holonomy groups.

3 The Exceptional Lie Group G_2

Cartan's classification of the simple Lie algebras includes four infinite families of classical algebras, $\mathfrak{su}(n), \mathfrak{so}(2n+1), \mathfrak{sp}(n), \mathfrak{so}(2n)$, as well as five exceptional cases, $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. The classical algebras and their corresponding Lie groups

$$\mathrm{SO}(n),\ \mathrm{SU}(n),\ \text{and}\ \mathrm{Sp}(n)$$

are all related to the classical division algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. Specifically, they arise as groups preserving a natural inner product on vector spaces built from these algebras. They also arise as isometry groups of the corresponding projective spaces.

The exceptional Lie algebras and the corresponding groups are all related in one way or another to the fourth and largest normed division algebra, the **octonions**[3] $\mathbb{O}$. Unlike its better known cousins the octonions have the misfortune of being non-associative. For which reason John Baez calls them "the crazy old uncle nobody lets out of the attic" [4]. It is also for this reason that it is difficult to construct groups out of them—groups must always be associative. However, there are a few meaningful ways in which one can do so, giving rise to the exceptional Lie groups. In fact, Cartan showed in 1914 that the smallest exceptional Lie group, G_2, is nothing more than the automorphism group of $\mathbb{O}$.

[2]In the non-simply-connected case the restricted holonomy group will still belong to this list but the entire group may be larger [13].

[3]The octonion algebra is sometimes called the **Cayley algebra** even though it was first discovered by Graves in 1843.

Since our interest in G_2 is as the holonomy group of a 7-manifold, it will be sufficient to realize G_2 as a particular subgroup of SO(7), or equivalently as an abstract group with a particular 7-dimensional orthogonal representation. In light of this we will simply *define* G_2 to be the automorphism group of the octonions together with its natural representation on the 7-dimensional space of pure imaginary octonions, $\mathrm{Im}\,\mathbb{O}$. We will not make any attempt to connect this group with the exceptional Lie group defined in Cartan's classification, though they are, in fact, equivalent. We refer the interested reader to [14] for details.

Remark 5. We introduce the octonions in the next section primarily as a tool for describing the exceptional group G_2. However, before we lose our physics readership in what may appear to be a sea of mundane algebra we offer some extra words of motivation. The octonions and structures related to them seem be ubiquitous in string theory, M-theory, and supersymmetry. In particular, all of the exceptional Lie groups make an appearance somewhere. Most notable, perhaps, is the E_8 heterotic string which, together with its E_6 subgroup, is important in grand unification schemes. Other structures related to the octonions such as Spin(7) and Spin(8), as well as the octonionic projective line $\mathbb{OP}$ and plane $\mathbb{OP}^2$ also crop up in strange places. For these reasons and others, there has been some recent interest [15] in studying the role of the octonions in M-theory. The hope being that if M-theory is truly a unique theory it should depend in some critical way on unique mathematical structures.

3.1 The Octonions

Just as the quaternions can be constructed from pairs of complex numbers, $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$, the octonions can be defined as $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$ where ℓ is a new imaginary anticommuting with i, j and k. This process is known as the **Cayley-Dickson construction**. Multiplication in $\mathbb{O}$ is defined by

$$(a, b)(c, d) = (ac - d\bar{b}, \overline{a}d + cb) \tag{3}$$

for $a, b, c, d \in \mathbb{H}$.

The octonions $\mathbb{O}$ form an eight-dimensional algebra with basis $\{1, i, j, k, \ell, \ell i, \ell j, \ell k\}$. The seven-dimensional subspace spanned by the imaginary basis elements is denoted $\mathrm{Im}\,\mathbb{O}$. The seven imaginary basis elements form an anticommuting set with each element squaring to -1.

The multiplication rules for the octonions can be succinctly described with a diagram know as the **Fano plane** (Figure 1). This diagram has exactly seven points and seven lines (the circle through i, j, k is considered a line). The seven points correspond to the seven standard basis elements of $\mathrm{Im}\,\mathbb{O}$. Note that each pair of distinct points lies on a unique line and each line runs through exactly three points. Let $(\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r)$ be an ordered triple of points lying on a given line with the order specified by the direction of the arrow. Then multiplication is given by

$$\mathbf{e}_p\mathbf{e}_q = \mathbf{e}_r \qquad \mathbf{e}_q\mathbf{e}_p = -\mathbf{e}_r \tag{4}$$

together with cyclic permutations. These rules together with

- 1 is the multiplicative identity,

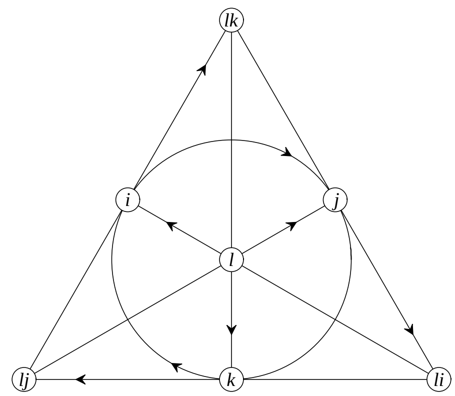

Figure 1: The Fano plane

- $\mathbf{e}_p^2 = -1$ for each point in the diagram

completely defines the algebraic structure of the octonions. Note that each of the seven lines generates a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

Using this diagram we can easily see that the octonions are non-associative:

$$\ell(ij) = k' \qquad (\ell i)j = -k' \tag{5}$$

where $k' = \ell k$. They do, however, satisfy a weaker form of associativity. We say that an algebra $\mathbb{K}$ is **alternative** if the subalgebra generated by any two elements is associative. There is a theorem due to Artin that says an algebra is alternative if any two of the following three conditions hold:

$$a(ab) = (a^2)b \tag{6a}$$

$$(ab)b = a(b^2) \tag{6b}$$

$$a(ba) = (ab)a \tag{6c}$$

The remaining condition then follows automatically.

Proposition 6. *The octonions form an alternative algebra.*

In fact, one can show that the subalgebra of $\mathbb{O}$ generated by any two elements is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. For a proof consult [5] or [16].

For any algebra $\mathbb{K}$ defined by the Cayley-Dickson construction we can define **conjugation** by

$$\overline{a} = \mathrm{Re}(a) - \mathrm{Im}(a)$$

for $a \in \mathbb{K}$. It has the properties that

$$\overline{\overline{a}} = a \qquad \overline{ab} = \overline{b}\,\overline{a}$$

The **norm** of an element $a \in \mathbb{K}$ is given by

$$\|a\|^2 = a\overline{a} = \overline{a}a$$

One can show that $\|\cdot\|^2$ is a real-valued, positive-definite, quadratic form on $\mathbb{K}$. The existence of a positive-definite norm shows that every nonzero element of $\mathbb{K}$ has a multiplicative inverse:

$$a^{-1} = \frac{\overline{a}}{\|a\|^2}$$

We can also define a symmetric, bilinear form—that is, an **inner product**—on $\mathbb{K}$ via the polarization identity. In other words, we set $c = a + b$ in the formula $\|c\|^2 = \langle c, c\rangle$ to obtain

$$\|a+b\|^2 = \|a\|^2 + 2\langle a, b\rangle + \|b\|^2$$

or

$$\begin{aligned} \langle a, b\rangle &= \tfrac{1}{2}\left(\|a+b\|^2 - \|a\|^2 - \|b\|^2\right) \\ &= \tfrac{1}{2}(a\overline{b} + b\overline{a}) \\ &= \mathrm{Re}(a\overline{b}) = \mathrm{Re}(\overline{a}b) \end{aligned} \tag{7}$$

The norm and inner product defined above are just the standard Euclidean norm and inner product on $\mathbb{K}$ considered as a real vector space. While all of this holds for any Cayley-Dickson algebra $\mathbb{K}$ there is a special property shared by only the first four algebras in the sequence:

$$\|ab\| = \|a\|\|b\| \tag{8}$$

A normed algebra for which (8) holds is called a **normed division algebra**. Note that a normed division algebra cannot have zero-divisors.[4]

Theorem 7 (Hurwitz). *The only normed division algebras are* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *and* $\mathbb{O}$.

The next algebra in the sequence, the sedenions, is neither alternative nor a division algebra—it has zero-divisors.

Forms

It is convenient to introduce two structures on $\mathbb{O}$ that measure the failure of $\mathbb{O}$ to be commutative or associative. The **commutator** is given by

$$[a, b] = ab - ba \tag{9}$$

and the **associator** by

$$[a, b, c] = (ab)c - a(bc) \tag{10}$$

for $a, b, c \in \mathbb{O}$. Both functions are separately linear in each of their arguments. Moreover, they both vanish when any argument is real. Together these facts imply that we can always take the arguments to be pure imaginary.

It is follows immediately from the definition that the commutator is antisymmetric, but what about the associator? Equations (6) tells us that the associator vanishes when any two of its arguments are equal. Polarization then implies that the associator is alternating. Conversely, if the associator for an algebra $\mathbb{K}$ is alternating equations (6) follow and $\mathbb{K}$ is alternative. That is,

[4] Recall that $a \in \mathbb{K}$ is a zero-divisor if there exists $b \neq 0$ in $\mathbb{K}$ such that $ab = 0$ or $ba = 0$.

Proposition 8. *The associator for an algebra $\mathbb{K}$ is totally antisymmetric if and only if $\mathbb{K}$ is alternative.*

Another useful fact which can be verified with some simple algebra is

Lemma 9. *Let $a, b, c \in \mathbb{O}$ then*

$$[a,b] \in \operatorname{Im}\mathbb{O} \quad \text{and} \quad \langle a, [a,b]\rangle = 0$$
$$[a,b,c] \in \operatorname{Im}\mathbb{O} \quad \text{and} \quad \langle a, [a,b,c]\rangle = 0$$

The last two statements, together with antisymmetry, say that the commutator and associator are always orthogonal to the space spanned by their arguments. This prompts the definition of two natural forms associated with $\mathbb{O}$:

$$\phi(x,y,z) = \left\langle \tfrac{1}{2}[x,y], z\right\rangle \tag{11}$$
$$\psi(w,x,y,z) = \left\langle \tfrac{1}{2}[w,x,y], z\right\rangle \tag{12}$$

called the **associative 3-form** and the **coassociative 4-form** respectively.[5] The fact that they are alternating follows from Lemma 9. It also follows that these forms vanish when any of their arguments are real. By linearity, we can then consider the forms restricted to $\operatorname{Im}\mathbb{O}$. That is, $\phi \in \Lambda^3(\operatorname{Im}\mathbb{O})$ and $\psi \in \Lambda^4(\operatorname{Im}\mathbb{O})$.

Let $\mathbf{e}_p$ for $p = 1, 2, \ldots 7$ denote the basis $\{i, j, k, \ell, \ell i, \ell j, \ell k\}$ of $\operatorname{Im}\mathbb{O}$. In terms of components we have

$$\{\mathbf{e}_p, \mathbf{e}_q\} = -2\delta_{pq} \tag{13a}$$
$$[\mathbf{e}_p, \mathbf{e}_q] = 2\phi_{pq}{}^{r}\,\mathbf{e}_r \tag{13b}$$
$$[\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r] = 2\psi_{pqr}{}^{s}\,\mathbf{e}_s \tag{13c}$$

where $\{\cdot,\cdot\}$ is the anti-commutator. Brute force calculation shows that

$$\phi = \mathbf{e}_{123} + \mathbf{e}_{415} + \mathbf{e}_{426} + \mathbf{e}_{437} + \mathbf{e}_{617} + \mathbf{e}_{725} + \mathbf{e}_{536} \tag{14}$$
$$\psi = \mathbf{e}_{4567} + \mathbf{e}_{6273} + \mathbf{e}_{7351} + \mathbf{e}_{5162} + \mathbf{e}_{3524} + \mathbf{e}_{1634} + \mathbf{e}_{2714} \tag{15}$$

where $\mathbf{e}_{pqr} = \mathbf{e}_p \wedge \mathbf{e}_q \wedge \mathbf{e}_r$, etc. These values actually follow fairly easily from the Fano plane diagram once you understand how to read it.

We see from the above expressions that the standard volume form on $\operatorname{Im}\mathbb{O}$ is given by

$$\Omega = \tfrac{1}{7}\phi \wedge \psi = \mathbf{e}_{1234567} \tag{16}$$

Using this orientation one can easily check that the 4-form ψ is simply the Hodge dual of the 3-form[6]

$$\psi = *\phi \tag{17}$$

[5] The reason for these names somewhat mysterious names is explained in [5]

[6] There seem to be as many choices of basis for $\operatorname{Im}\mathbb{O}$ as there are authors; some even chose a different orientation. In all cases however ψ is always chosen to be the dual of ϕ, so that ψ will differ by a sign when the orientation is reversed.

3.2 The Automorphism Group

The most natural and straightforward definition of the exceptional Lie group G_2 is as the automorphism group of the octonions:

$$G_2 \equiv \mathrm{Aut}(\mathbb{O}) = \{g \in \mathrm{GL}(\mathbb{O}) \mid g(ab) = g(a)g(b) \text{ for all } a, b \in \mathbb{O}\} \tag{18}$$

Note that since G_2 is a linear group we have $g(x) = xg(1) = x$ for all $x \in \mathbb{R} = \mathrm{Re}\,\mathbb{O}$ and $g \in G_2$. Hence G_2 acts trivially on $\mathbb{R} \subseteq \mathbb{O}$ and preserves the decomposition $\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{O}$. It follows that $\overline{g(x)} = g(\overline{x})$ and

$$\|g(x)\|^2 = g(x)g(\overline{x}) = g(x\overline{x}) = \|x\|^2 g(1) = \|x\|^2 \tag{19}$$

Hence G_2 preserves the bilinear form on $\mathbb{O}$ defined by the norm so that $G_2 \subseteq \mathrm{O}(\mathrm{Im}\,\mathbb{O}) \subseteq \mathrm{O}(\mathbb{O})$.

Proposition 10. *The group G_2 fixes both the associative 3-form ϕ and the coassociative 4-form ψ.*

Proof. Let $x, y, z \in \mathrm{Im}\,\mathbb{O}$. Then

$$\begin{aligned}\phi(g(x), g(y), g(z)) &= \left\langle \tfrac{1}{2}[g(x), g(y)], g(z) \right\rangle \\ &= \left\langle g(\tfrac{1}{2}[x, y]), g(z) \right\rangle \\ &= \left\langle \tfrac{1}{2}[x, y], z \right\rangle \\ &= \phi(x, y, z)\end{aligned}$$

and likewise for the 4-form ψ. □

Proposition 11. *The group G_2 is a compact Lie subgroup of the orientation preserving isometries of $\mathbb{O}$ leaving invariant the decomposition $\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{O}$:*

$$G_2 \subseteq \mathrm{SO}(\mathrm{Im}\,\mathbb{O}) \subseteq \mathrm{SO}(\mathbb{O})$$

Proof. We have already established that G_2 acts trivially on $\mathbb{R} \in \mathbb{O}$ and that it preserves the metric on $\mathbb{O}$. The fact that it is a closed subgroup of the compact group $\mathrm{O}(\mathbb{O})$ implies that it is a compact Lie subgroup. Furthermore, it follows from Proposition 10 that the standard volume form (16) on $\mathrm{Im}\,\mathbb{O}$ is invariant under the action of G_2, so that the action preserves the orientation of $\mathrm{Im}\,\mathbb{O}$. □

The converse of Proposition 10 states that the total isotropy group of ϕ and ψ is nothing more than G_2. Actually, the isotropy group of ϕ is exactly G_2 but the isotropy group of ψ is slightly larger, namely $G_2 \times \mathbb{Z}_2$. This is because the parity transformation $-1 \in \mathrm{O}(7)$ preserves ψ (but not ϕ). Proving these claims takes a little more work. Basically the idea is that any $g \in \mathrm{GL}(\mathrm{Im}\,\mathbb{O})$ satisfying $g^*\phi = \phi$ can be shown to preserve the volume form Ω on $\mathrm{Im}\,\mathbb{O}$ and the inner product via the identity[7]

$$\frac{1}{6}(x \lrcorner \phi) \wedge (y \lrcorner \phi) \wedge \phi = \langle x, y \rangle\, \Omega \tag{20}$$

[7] Here $x \lrcorner \phi$ denotes the 2-form obtained by contracting x with ϕ, $(x \lrcorner \phi)(y, z) = \phi(x, y, z)$.

Thus $g \in \mathrm{SO}(\mathrm{Im}\,\mathbb{O})$. Any such g will then preserve the commutator as well (which only differs from ϕ by raising an index). Finally, one can show that any g which preserves the commutator is necessarily an automorphism. For a detailed proof the interested reader should consult [5] or [7]. It is for this reason that many authors simply *define* G_2 as the subgroup of $\mathrm{GL}_7\,\mathbb{R}$ that preserves the 3-form ϕ given by (14).

A **basic triple** or **Cayley triangle** in $\mathrm{Im}\,\mathbb{O}$ is an orthonormal 3-frame (v_1, v_2, v_3) in $\mathrm{Im}\,\mathbb{O}$ such that each element is orthogonal the the subalgebra generated by the other two. This is equivalent to saying that $\phi(v_1, v_2, v_3) = 0$. For example (i, j, ℓ) forms a basic triple. Define the set of all basic triples as

$$\mathcal{C} = \{(v_1, v_2, v_3) \in (\mathrm{Im}\,\mathbb{O})^3 \mid \langle v_i, v_j\rangle = \delta_{ij} \text{ and } \phi(v_1, v_2, v_3) = 0\} \tag{21}$$

The set $\mathcal{C}$ is a 14-dimensional submanifold of the 21-dimensional vector space $(\mathrm{Im}\,\mathbb{O})^3$. To see this note that v_1 can be chosen to any unit imaginary in $S^6 \subset \mathrm{Im}\,\mathbb{O}$ and v_2 can be chosen to be any unit in the 5-sphere orthogonal to v_1. Finally, v_3 can be taken as any element in the 3-sphere orthogonal to v_1, v_2, and v_1v_2. So

$$\dim \mathcal{C} = 6 + 5 + 3 = 14 \tag{22}$$

Because G_2 preserves the metric and 3-form on $\mathbb{O}$ every automorphism necessarily takes basic triples into basic triples. That is, G_2 acts on manifold $\mathcal{C}$. The action is necessarily free as each basic triple generates the entire algebra. Zorn showed that it is transitive as well:

Theorem 12 (Zorn). *The group G_2 acts freely and transitively on set of basic triples $\mathcal{C}$. Hence, G_2 is diffeomorphic to $\mathcal{C}$ and $\dim G_2 = 14$.*

By examining the isotropy groups of this action we obtain some useful corollaries:

Proposition 13. *The group G_2 acts transitively on $S^6 \subset \mathrm{Im}\,\mathbb{O}$ with isotropy group $\mathrm{SU}(3)$. That is, S^6 is a homogeneous G_2-space with*

$$G_2/SU(3) \cong S^6$$

Proof. If follows immediately from 12 that G_2 acts transitively on $S^6 \subset \mathrm{Im}\,\mathbb{O}$, we have only to establish the isotropy group. Fix $v_1 \in S^6$ and let H be the isotropy group of v_1. We note that $\dim H = \dim G_2 - \dim S^6 = 14 - 6 = 8$.

Let $V = {v_1}^{\perp} \cap \mathrm{Im}\,\mathbb{O}$ be the 6-dimensional subspace of $\mathrm{Im}\,\mathbb{O}$ orthogonal to v_1. H clearly takes V into itself. Define a function on $J\colon V \to V$ by

$$J(y) = \tfrac{1}{2}[v_1, y] = v_1 y$$

Note that $J(y)$ is indeed an element of V as the commutator is always pure imaginary and orthogonal to its arguments by Lemma 9. Also note that $J^2 = -1$,

$$J^2(y) = v_1(v_1 y) = (v_1)^2 y = -y$$

so that J defines a complex structure on V. The action of H on V clearly commutes with J. Also the inner product of $\mathbb{O}$ when restricted to V becomes a Hermitian form with respect to J,

$$\langle Jx, Jy\rangle = \langle v_1 x, v_1 y\rangle = \|v_1\|^2 \langle x, y\rangle = \langle x, y\rangle$$

The group H preserves this Hermitian form so that $H \subseteq \mathrm{U}(3) \subseteq \mathrm{SO}(6)$. Direct computation shows that H also preserves the orientation of V considered as a complex vector space so that $H \subseteq \mathrm{SU}(3)$. But because H is necessarily an 8-dimensional closed group we have $H \cong \mathrm{SU}(3)$. □

Likewise, we can consider the isotropy group of an orthonormal 2-frame (v_1, v_2) in $\mathrm{Im}\,\mathbb{O}$. Such a group is also the stabilizer of the entire quaternion subalgebra generated by v_1 and v_2. The set of all orthonormal 2-frames in $\mathrm{Im}\,\mathbb{O}$ is called a **Stiefel manifold** and is denoted

$$\mathcal{V}_{7,2} = \{(v_1, v_2) \in (\mathrm{Im}\,\mathbb{O})^2 \mid \langle v_i, v_j \rangle = \delta_{ij}\} \tag{23}$$

It is an 11-dimensional manifold.

Proposition 14. *The group G_2 acts transitively on $\mathcal{V}_{7,2}$ with isotropy group* SU(2). *That is, $\mathcal{V}_{7,2}$ is a homogeneous G_2-space with*

$$G_2/\,\mathrm{SU}(2) \cong \mathcal{V}_{7,2}$$

The proof proceeds in much the same way as the previous one, with the exception that the complex structure is given by the associator instead of the commutator. In summary, we have a sequence of closed Lie subgroups in G_2:

$$1 \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{SU}(3) \longrightarrow G_2 \tag{24}$$

with actions on $\mathrm{Im}\,\mathbb{O}$ as given above.

It follows from Proposition 13 that G_2 is both connected and simply-connected. To see this, note that the long exact sequence for fibrations says that for a homogeneous space G/H, $\pi_k(G) = \pi_k(H)$ if $\pi_k(G/H) = \pi_{k+1}(G/H) = 0$. Since $\pi_i(S^6) = 0$ for $i = 0, \ldots, 5$ we can conclude that that

$$\begin{aligned} \pi_i(G_2) &= 0 \qquad i = 0, 1, 2, \text{ and } 4 \\ \pi_3(G_2) &= \mathbb{Z} \end{aligned} \tag{25}$$

We summarize the results regarding G_2 in the following:

Theorem 15. *G_2 is a 14-dimensional compact, connected, simply-connected, Lie subgroup of* SO(7).

Remark 16. The group G_2 is tied up in a rather interesting way to a property of Spin(8) known as **triality**. We lack the time and space to go into the details, but the basic idea is as follows. The group Spin(8) has three 8-dimensional representations: the standard vector representation as well as the right and left-handed spinor representations. The triality automorphism, θ, is an outer automorphism of Spin(8) of order three that permutes these representations. It turns out that the subgroup left invariant by this automorphism is isomorphic to G_2,

$$G_2 \cong \{g \in \mathrm{Spin}(8) \mid \theta(g) = g\} \tag{26}$$

If we identify each of the representations with the octonions, $\mathbb{O}$, then G_2 will be the subgroup that fixes $1 \in \mathbb{O}$ in each of the three representations. For more details see [16, 5, 4].

4 G_2 Holonomy

4.1 General Features

We now want to consider 7-dimensional manifolds with holonomy group contained in G_2. To this end it is useful to recall the notion of a G-structure on a smooth n-manifold M. Recall that the frame bundle of M is a principal bundle $F \to M$ with fiber $\mathrm{GL}_n \mathbb{R}$. We define a **G-structure** on M as a principal subbundle of F with fiber $G \subseteq \mathrm{GL}_n \mathbb{R}$. Many of the interesting geometric structures that can be placed on manifolds can be formulated as G-structures. For example, a Riemannian structure on M can be defined as a $\mathrm{O}(n)$-structure. The metric on M is completely determined by the $\mathrm{O}(n)$-structure. A complex manifold M can be defined as a $2n$-manifold together with a $\mathrm{GL}_n \mathbb{C}$-structure satisfying some suitable torsion-freeness condition.

If we demand that a G-structure on M be compatible with a given connection on M we find that the holonomy of the connection restricts what structures are allowed:

Proposition 17. *Let M be a smooth n-dimensional manifold and let ∇ be a connection on M. For each Lie subgroup $G \subseteq \mathrm{GL}_n \mathbb{R}$, there exists a G-structure P on M compatible with ∇ if and only if $\mathrm{Hol}(\nabla) \subseteq G$. If P exists it is unique.*

Thus to every Riemannian 7-manifold (M, g) with $\mathrm{Hol}(g) \subseteq G_2$ there exists a unique G_2-structure on M compatible with the Levi-Civita connection. Hence a necessary condition for a manifold to have G_2 holonomy is that it admit a G_2-structure. We will see shortly that a sufficient condition is that this structure be *torsion-free* in some suitable sense.

Recall that the group G_2 preserves the flat Euclidean metric

$$g_0 = dx_1^2 + \cdots + dx_7^2 \tag{27}$$

and associative 3-form[8]

$$\phi_0 = d\mathbf{x}_{123} + d\mathbf{x}_{415} + d\mathbf{x}_{426} + d\mathbf{x}_{437} + d\mathbf{x}_{617} + d\mathbf{x}_{725} + d\mathbf{x}_{536} \tag{28}$$

arising from the octonionic structure on $\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$. Hence, every oriented G_2-structure on a 7-manifold M gives rise to a Riemannian metric g and a 3-form ϕ on M, such that each tangent space, $T_m M$, admits an orientation-preserving isomorphism with $\mathrm{Im}\,\mathbb{O}$ identifying g with g_0 and ϕ with ϕ_0. Given g and ϕ we can also construct the associated 4-form $\psi = *\phi$ on M. By a slight abuse of notation we will refer to the pair (g, ϕ) as a G_2-structure on M. Though this is not exactly rigorous, one can show that a properly chosen 3-form ϕ (called a positive 3-form) uniquely determines a G_2-structure on M.

Proposition 18. *Let M be a 7-manifold and let (g, ϕ) be a G_2-structure on M, and let ∇ be the Levi-Civita connection of g. Then the following statements are equivalent:*

(i) $\mathrm{Hol}(g) \subseteq G_2$ *and ϕ is the induced 3-form,*

(ii) $\nabla\phi = 0$,

(iii) $d\phi = d{*}\phi = 0$

[8] Note that our choice of basis for $\mathrm{Im}\,\mathbb{O}$ differs from Joyce's [1] by a sign change in x_1 and x_2.

Define the **torsion**[9] of a G_2-structure (g, ϕ) to be the tensor $\nabla\phi$. A G_2-structure is said to be **torsion-free** if $\nabla\phi = 0$. By Proposition 18, a manifold with holonomy contained in G_2 is precisely one which admits a torsion-free G_2-structure. We then define a **G_2-manifold**, (M, g, ϕ), as a 7-manifold together with a torsion-free G_2-structure (g, ϕ). Recalling our discussion in §2.2 we note the following:

Theorem 19. *Let (M, g, ϕ) be a G_2-manifold. Then M is Ricci-flat. Moreover, M is a spin manifold with a preferred spin structure. The space of parallel spinors on M has dimension one. If S is the spin bundle of M, then there is a natural decomposition*

$$S \cong \Lambda^0(M) \oplus \Lambda^1(M)$$

Note that, as we have defined it, a G_2-manifold does not necessarily have G_2 holonomy. The holonomy group may actually be a proper subgroup of G_2. But from Berger's classification theorem and our discussion of the subgroups of G_2 in §3.2 we can deduce

Proposition 20. *Let (M, g, ϕ) be a G_2-manifold. The restricted holonomy group* $\mathrm{Hol}^0(g)$ *is one of*

$$1 \subset \mathrm{SU}(2) \subset \mathrm{SU}(3) \subset G_2$$

The holonomy representation on TM for each of these groups is just the standard one described in §3.2. The next result shows that for a compact G_2-manifold the holonomy can be determined solely from the topology of M.

Proposition 21. *Let (M, g, ϕ) be a compact G_2-manifold. Then M admits a finite cover isometric to $N \times \mathbb{T}^k$ where $\mathbb{T}^k$ is a flat torus, N is a compact, simply-connected Riemannian manifold, and*

- $k = 0$ *when* $\mathrm{Hol}^0(g) = G_2$
- $k = 1$ *when* $\mathrm{Hol}^0(g) = \mathrm{SU}(3)$
- $k = 3$ *when* $\mathrm{Hol}^0(g) = \mathrm{SU}(2)$
- $k = 7$ *when* $\mathrm{Hol}^0(g) = 1$

Thus $\pi_1(M) \cong H \ltimes \mathbb{Z}^k$ where H is a finite group. In particular, $\mathrm{Hol}(g) = G_2$ *if and only if $\pi_1(M)$ is finite.*

Based on the above it is easy to see how one might construct compact G_2-manifolds with holonomy less then G_2 given that one understands how to construct compact $2n$-manifolds with holonomy $\mathrm{SU}(n)$. Such manifolds are called Calabi-Yau n-folds, and have been extensively studied. Calabi-Yau manifolds are always compact, Ricci-flat, Kähler manifolds with trivial canonical bundle.

Proposition 22. *Let Y be a Calabi-Yau 3-fold with metric h, Kähler form ω, and holomorphic volume form θ. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle with periodic coordinate x. Define a metric g and three-form ϕ on $\mathbb{T} \times Y$ by*

$$g = dx^2 + h$$

and

$$\phi = dx \wedge \omega + \mathrm{Re}\,\theta$$

then $(\mathbb{T} \times Y, g, \phi)$ is a G_2-manifold with holonomy $\mathrm{SU}(3)$.

[9] This torsion has nothing to do with the torsion of the Levi-Civita connection, ∇, which vanishes as always.

An analogous result for 2-folds is the following,

Proposition 23. *Let Y be a Calabi-Yau 2-fold with metric h, Kähler form ω, and holomorphic volume form θ. Let $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be the flat 3-torus with periodic coordinates (x_1, x_2, x_3). Define a metric g and three-form ϕ on $\mathbb{T}^3 \times Y$ by*

$$g = dx_1^2 + dx_2^2 + dx_3^2 + h$$

and

$$\phi = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega + dx_2 \wedge \operatorname{Re}\theta - dx_3 \wedge \operatorname{Im}\theta$$

then $(\mathbb{T}^3 \times Y, g, \phi)$ is a G_2-manifold with holonomy $\mathrm{SU}(2)$.

Of course, the simplest example of a compact G_2-manifold is just $\mathbb{T}^7$ with trivial holonomy. The metric and three-form are just those inherited form $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$. Constructing a compact G_2-manifold with the full G_2 holonomy is much more difficult. By Proposition 21 this amounts to finding a compact 7-manifold with a torsion-free G_2 structure whose fundamental group is finite. This is the topic of the next section.

4.2 Joyce's Construction

Joyce's construction [1, 9, 10] of compact 7-manifolds with G_2 holonomy proceeds as follows:

1. Start with a flat G_2-structure (g_0, ϕ_0) on a 7-torus, $\mathbb{T}^7$. Quotient $\mathbb{T}^7$ by a finite group Γ of isometries preserving ϕ_0. The quotient space $\mathbb{T}^7/\Gamma$ will be a singular, compact 7-manifold, also known as a **orbifold**. The singularities of the orbifold will correspond to the fixed point set of Γ.
2. If Γ is chosen carefully, one can resolve the singularities of $\mathbb{T}^7/\Gamma$ in natural way to obtain a nonsingular, compact 7-manifold M.
3. Define on M a 1-parameter family of G_2-structures (g_t, ϕ_t) for $t \in (0, \epsilon)$. These G_2-structures will not be torsion-free, but they will have small torsion for small t.
4. Show that for sufficiently small t, one can deform (g_t, ϕ_t) to a nearby torsion-free G_2-structure (g, ϕ). If $\pi_1(M)$ is finite, then M is a compact 7-manifold with G_2 holonomy, as desired.

This principal difficulty in this process lies in step 2, resolving the orbifold singularities. We do not know how to do this in general. However, if the singularity is of a special type one can use methods of complex geometry to find a suitable resolution. In particular, let S be the singular set of $\mathbb{T}^7/\Gamma$. Suppose that every connected component of S is locally isomorphic to either

- $\mathbb{T}^3 \times \mathbb{C}^2/G$, for a finite subgroup $G \subseteq \mathrm{SU}(2)$, or
- $\mathbb{T}^1 \times \mathbb{C}^3/G$, for a finite subgroup $G \subseteq \mathrm{SU}(3)$ acting freely on $\mathbb{C}^3 - \{0\}$.

Using algebraic geometry one can find resolutions of $\mathbb{C}^2/G$ and $\mathbb{C}^3/G$ called **crepant resolutions**. A crepant resolution of $\mathbb{C}^n/G$ is a noncompact, complex manifold Y with vanishing first Chern class, $c_1(Y) = 0$, together with a resolving map $\pi\colon Y \to \mathbb{C}^n/G$. One can show that there exist Calabi-Yau metrics (i.e. Ricci-flat, Kähler metrics), one in each Kähler class, on Y such that the metrics approximate at infinity the flat Euclidean metric on $\mathbb{C}^n/G$. Such metrics are said to **asymptotically locally Euclidean** (ALE). Furthermore, one can show that the ALE Calabi-Yau metrics on Y necessarily have holonomy $\mathrm{SU}(n)$. Hence, we can think of Y as a noncompact version of a Calabi-Yau n-fold.

If Y is a crepant resolution of $\mathbb{C}^2/G$ or $\mathbb{C}^3/G$ then we can use either $\mathbb{T}^3 \times Y$ or $\mathbb{T}^1 \times Y$ as a local model for repairing the singularities described above. Furthermore, Propositions 22 and 23 remain true when Y is noncompact, so that both $\mathbb{T}^3 \times Y$ and $\mathbb{T}^1 \times Y$ admit torsion-free G_2-structures. Thus these spaces allow us to repair the singular G_2-structure on $\mathbb{T}^7/\Gamma$ as well.

Step 3 in the process is then to construct a suitable G_2-structure on the resolved manifold M. This is done by gluing together the torsion-free G_2-structures on $\mathbb{T}^7/\Gamma$ and $\mathbb{T}^3 \times Y$ (or $\mathbb{T}^1 \times Y$) using a partition of unity. The G_2-structure thus defined is not quite torsion-free as the gluing process introduces errors in the regions where the partition of unity changes. However, by writing down a 1-parameter family of ALE metrics on Y we can make the torsion as small as we like by asymptotically approaching the flat metric. The final step is then to prove, using analysis, that for sufficiently small torsion the G_2-structure can be deformed to a torsion-free one. The gory details of this proof, as well as rigorous definitions of the above steps, can be found in Joyce's book [1].

Using Joyce's construction one can construct a compact G_2-manifolds provided one can find an orbifold group Γ with the requisite singularities. We illustrate this with an example below. We should mention that there many be several topologically distinct choices in making the resolution of $\mathbb{T}^7/\Gamma$ so that one can potentially find numerous G_2-manifolds all arising from a single orbifold.

Example

Let $(x_1, \ldots, x_7)$ be periodic coordinates on $\mathbb{T}^7 = \mathbb{R}^7/\mathbb{Z}^7$, and let (g_0, ϕ_0) be the flat G_2-structure on $\mathbb{T}^7$ inherited from $\mathbb{R}^7$,

$$\phi_0 = d\mathbf{x}_{123} + d\mathbf{x}_{415} + d\mathbf{x}_{426} + d\mathbf{x}_{437} + d\mathbf{x}_{617} + d\mathbf{x}_{725} + d\mathbf{x}_{536} \tag{29}$$

Let $\Gamma \cong (\mathbb{Z}_2)^3$ be the finite group generated by the involutions

$$\alpha(x_1, \ldots, x_7) = (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7) \tag{30a}$$

$$\beta(x_1, \ldots, x_7) = \big(x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7\big) \tag{30b}$$

$$\gamma(x_1, \ldots, x_7) = \big(-x_1, x_2, -x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7\big) \tag{30c}$$

It is easily to check that $\alpha^2 = \beta^2 = \gamma^2 = 1$ and that α, β, and γ all commute. Furthermore, each of these generators preserves (g_0, ϕ_0) by a careful choice of which signs to change.

By analyzing the fixed point sets of α, β, and γ, as well as the behavior of Γ near the fixed point sets one can show the following,

Lemma 24. *The singular set S of $\mathbb{T}^7/\Gamma$ is a disjoint union of 12 copies of $\mathbb{T}^3$ with the singularity at each $\mathbb{T}^3$ locally modeled on $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$.*

This type of singularity is the simplest kind that we know how to resolve. The complex orbifold $\mathbb{C}^2/\mathbb{Z}^2$ can be considered as a cone on $\mathbb{RP}^3 = S^3/\mathbb{Z}^2$ with a singularity at the origin. The crepant resolution Y of $\mathbb{C}^2/\mathbb{Z}^2$ is called a **Eguchi-Hanson space**, denoted EH_2. Topologically, one can think of EH_2 as a cone on $\mathbb{RP}^3$ with the singularity at the tip replaced by a smooth S^2. Thus, near the origin EH_2 looks like $S^2 \times \mathbb{R}^2$, while asymptotically it looks like $\mathbb{RP}^3 \times \mathbb{R}$. One can show that EH_2 is biholomorphic to $T^*\mathbb{CP}^1$, the holomorphic cotangent bundle of the Riemann sphere.

We can introduce a 1-parameter family of ALE Kähler metrics on $Y = \mathrm{EH}_2$ as follows. Define a family of Kähler potentials on $Y - \pi^{-1}(0)$ by

$$\mathcal{K}_t = \sqrt{r^4 + t^4} + 2t^2 \log r - t^2 \log\left(\sqrt{r^4 + t^4} + t^2\right) \qquad (31)$$

where $r = \left(|z_1|^2 + |z_2|^2\right)^{1/2}$ is the radial coordinate on Y and $t \in (0, \epsilon)$ corresponds to the radius of the central S^2. Then we can locally define the Kähler form on $Y - \pi^{-1}(0)$ by $\omega_t = i\partial\bar{\partial}\mathcal{K}_t$. This extends uniquely and smoothly to a closed, positive $(1,1)$-form on all of Y which is then the Kähler form associated to a Kähler metric on Y. For large r (or small t) this metric approaches the standard Euclidean metric on $\mathbb{C}^2/\mathbb{Z}^2$.

To construct the G_2-manifold M we then cut out the 12 copies of $\mathbb{T}^3 \times \mathbb{C}^2/\mathbb{Z}^2$ and glue in 12 copies of $\mathbb{T}^3 \times \mathrm{EH}_2$. In order to claim that the resultant manifold M has G_2 holonomy we must show that $\pi_1(M)$ is finite. One can show that if the resolutions Y_i are simply-connected, then $\pi_1(M) = \pi_1(\mathbb{T}^7/\Gamma)$. This will always be the case for the crepant resolutions of $\mathbb{C}^2/G$ and $\mathbb{C}^3/G$ that we are considering here. Indeed, the reader should be able to verify that $\pi_1(\mathrm{EH}_2) = 1$. In addition, one can show that any of the non-trivial loops in $\mathbb{T}^7$ can be shrunk to a point using the identifications in (30), so that $\pi_1(M) = \pi_1(\mathbb{T}^7/\Gamma) = 1$. By Joyce's construction we can then conclude that M admits metrics with G_2 holonomy as desired. ■

Remark 25. We should comment briefly on the relevance of Joyce's construction to string theory/M-theory. From the mathematics point of view, the inherent appearance of orbifold singularities in the construction of G_2-manifolds leaves a little to be desired. One might like to see a more direct description in which there are no singularities to smooth out. But from the physics point of view this is actually an advantage. The reason goes back to Witten's 1983 argument [17] that compactification of 11-dimensional supergravity on any *smooth* 7-manifold cannot lead to chiral matter or nonabelian gauge theories in 4 dimensions, both of which are fundamental ingredients in the world as we know it. On the other hand, *singular* 7-manifolds can lead to both chiral matter and nonabelian gauge fields. It has been known for some time that strings can propagate quite smoothly on spaces possessing orbifold singularities. Thus the primary focus of studying G_2 holonomy in physics has been centered around studying the singularities in such manifolds where interesting physics can develop. See for example [18, 19].

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