14-Sets Algebra

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Abstract

A novel 14-D algebra based upon Kuratowski Monoid's Closure and Complement operators is uncovered. 14×14 Matrix Representation, Conditions for inverse elements, 0-Commutator and Coproduct computed using *Mathematica* 9.0 . Almost Random Almost Incompressible large-length coefficients are found to have inverse. Matrix representation has the form of a Stochastic Matrix, and its infinite power has a limit with three 1-valued eigenvalues with determinant 0. As a consequence of this algebra, a topological (Twist on $\mathbb{Z}_2^3$) construction of the Octonions developed. An approximate Logarithmic algorithm devised for mapping Kuratowski's topological operators to 182 dimensional Lie Algebra of 14×14 Stochastic Matrices, thus a concept of pre-geometry is brought forth. De Novo semantical meanings attributed to these 14-D vectors described, in categoreal form, borrowing from the seminal work of Alfred North Whitehead's Process and Reality.

Prelude

We 'feel' the space. Just like we feel the cold or feel the texture of a fabric. Alfred North Whitehead formulated these feelings as vectorial entities which also deal with the appropriation of resources for an organism to exist (continue to exist) and Subjective Forms as 'how' the vector is feeling, or the process underlying that particular feeling.

In this paper attempt is made to create an algebraic model for the concepts of feeling and in particular the feel for space e.g. I feel I am 'inside' the room or I feel I am 'inside the same' room and so on.

Every concept is coded both symbolically and numerically in *Mathematica* 9.0 so the interested readers could manipulate the algebraic expressions to better understand the concepts.

There is no way to compute almost any of the algebraic expressions below by hand, specially the Lie Algebra of dimension 182 for the Logarithm. This algebra as well as many others are gone un-studied due to the inability of the human researcher to perform such computations, therefore this work pioneers an innovative new approach for such investigations.

Theorems are discovered by using random coefficients and trying to compute many examples and look for patterns, these patterns of symbolic and numeric computations then turned into theorems [5]:

"The creative process of mathematics, both historically and individually, may be described as a counterpoint between theorems and examples. Although it would be hazardous to claim that the creation of significant examples is less demanding that the development of theory, we have discovered that focusing on examples is a particularly expeditious means of involving undergraduate mathematics students in actual research. Not only are examples more concrete than theorems, and thus more accessible, but they cut across individual theories and make it both appropriate and necessary for the student to explore the entire literature in journals as well as texts."

Prologue

After coding and writing this work, an odd idea came to be (See Remark 9.1.1): Topological Oscillators! These are wave forms which oscillate the Kuratowski operators and therefore the entire space, without the need for a position coordinates nor direction of propagation. These waves could possibly be good structures for constructing the likes of Casimir forces i.e. waves that are not made from any matter or energy but yet they are waves of some kind.

How To Read This Document

You need to sequentially go down the sections and execute the code and check the results for yourself.

English is slim, the computations provide the affinity for the concepts, therefore the results are obtained from the output of the computations:

http://reference.wolfram.com/mathematica/tutorial/UsingANotebookInterface.html

Table of Contents

1. Software and Symbolic Computations

Files
http://files.lossofgenerality.com/14sets.nb

Package Notation is needed therefore include this on top of the code:

```
<< Notation`
```

The following Cayley table [1] is simply turned into a series of function calls:

```
kuraTable[] := Module[{},
  {{σ0, σ1, σ2, σ3, σ4, σ5, σ6, σ7, σ8, σ9, σ10, σ11, σ12, σ13},
   {σ1, σ0, σ4, σ5, σ2, σ3, σ8, σ9, σ6, σ7, σ12, σ13, σ10, σ11},
   {σ2, σ3, σ2, σ3, σ6, σ7, σ6, σ7, σ10, σ11, σ10, σ11, σ6, σ7},
   {σ3, σ2, σ6, σ7, σ2, σ3, σ10, σ11, σ6, σ7, σ6, σ7, σ10, σ11},
   {σ4, σ5, σ4, σ5, σ8, σ9, σ8, σ9, σ12, σ13, σ12, σ13, σ8, σ9},
   {σ5, σ4, σ8, σ9, σ4, σ5, σ12, σ13, σ8, σ9, σ8, σ9, σ12, σ13},
   {σ6, σ7, σ6, σ7, σ10, σ11, σ10, σ11, σ6, σ7, σ6, σ7, σ10, σ11},
   {σ7, σ6, σ10, σ11, σ6, σ7, σ6, σ7, σ10, σ11, σ10, σ11, σ6, σ7},
   {σ8, σ9, σ8, σ9, σ12, σ13, σ12, σ13, σ8, σ9, σ8, σ9, σ12, σ13},
   {σ9, σ8, σ12, σ13, σ8, σ9, σ8, σ9, σ12, σ13, σ12, σ13, σ8, σ9},
   {σ10, σ11, σ10, σ11, σ6, σ7, σ6, σ7, σ10, σ11, σ10, σ11, σ6, σ7},
   {σ11, σ10, σ6, σ7, σ10, σ11, σ10, σ11, σ6, σ7, σ6, σ7, σ10, σ11},
   {σ12, σ13, σ12, σ13, σ8, σ9, σ8, σ9, σ12, σ13, σ12, σ13, σ8, σ9},
   {σ13, σ12, σ8, σ9, σ12, σ13, σ12, σ13, σ8, σ9, σ8, σ9, σ12, σ13}}

 ]

(* Returns σk=σiσj for ith row and jth column *)
kuraProd[i0_, j0_] := Module[{M, i = i0, j = j0},
  M = kuraTable[];
  M[[i + 1]] [[j + 1]]
 ]

(* just returns the index number k for the σk=σiσj *)
kuraProd2[i0_, j0_] := Module[{M, i = i0, j = j0, expr},
  M = kuraTable[];
  expr = M[[i + 1]] [[j + 1]];
  expr = InputForm[expr];
  Extract[expr, {1, 2}]
 ]

(* PERMUTATED TABLE: Returns σk=σiσj for ith row and jth column, see 1.1 *)
kuraProdAUTO[i0_, j0_] := Module[{M, i = i0, j = j0},
  M = kuraTable[];
  M = M /. {σ2 -> σ5, σ3 -> σ4, σ4 -> σ3, σ5 -> σ2, σ6 -> σ9,
     σ7 -> σ8, σ8 -> σ7, σ9 -> σ6, σ10 -> σ13, σ11 -> σ12, σ12 -> σ11, σ13 -> σ10};

  M[[i + 1]] [[j + 1]]
 ]

(* PERMUTATED TABLE: just returns the index number k for the σk=σiσj , see 1.1 *)
kuraProdAUTO2[i0_, j0_] := Module[{M, i = i0, j = j0, expr},
  M = kuraTable[];
  M = M /. {σ2 -> σ5, σ3 -> σ4, σ4 -> σ3, σ5 -> σ2, σ6 -> σ9,
     σ7 -> σ8, σ8 -> σ7, σ9 -> σ6, σ10 -> σ13, σ11 -> σ12, σ12 -> σ11, σ13 -> σ10};
  expr = M[[i + 1]] [[j + 1]];
  expr = InputForm[expr];
  Extract[expr, {1, 2}]
 ]
```

In order to avoid function calls and attempting to use as much mathematical expressions as possible the following Notations are defined:

```
Off[Symbolize::rowboxh] (* this warning gets issued and thus suppressed *)
(* 1_14Sets is the multiplicative 1 of the 14-Sets algebra, and computed to be σ0 *)
Symbolize[ 14 Sets ]
Notation[ 1_(14 Sets) ⟹ σ0 ]
(* This assumes the a b product is commutative,
so we assume at most Complex numbers for coefficients *)
(* This can be changed with some effort. We
  used ** to make sure the product is non-commutative *)

Notation[ (a_ σi_) ** (b_ σj_) ⟹ a_ b_ kuraProd[i_, j_] ]

Notation[ σi_ ** σj_ ⟹ kuraProd[i_, j_] ]
(* use this or the other, but not at the same time,
since there is only one ** noncommutative operator in Mathematica *)
(*Notation[ (a_ σi_)**(b_ σj_) ⟹ a_ b_ kuraProdAUTO[i_,j_] ]*)

Notation[ (a_ σi_) (b_ σj_) ⟹ a_ b_ kuraProd[i_, j_] ]

(* short-hand for testing the table and simple experssions uses * *)
Notation[ σi_ σj_ ⟹ kuraProd[i_, j_] ]

(* This is necessary or Mathematica does
 not know how to further evaluate the expression *)
Notation[ σi_^2 ⟹ kuraProd[i_, i_] ]
Notation[ (a_ σi_)^2 ⟹ (a_^2) * kuraProd[i_, i_] ]

(* Matrix representation of 14-Sets Algebra *)
MatrixRep14Sets[cfs0_] :=
 Module[{mat, a, b, c, d, e, f, g, h, i, j, k, l, m, n, cfs = cfs0},
```

```
mat = (a b    0        0        0        0        0          0          0          0
       b a    0        0        0        0        0          0          0          0
       c d  a + c      0      b + d      0        0          0          0          0
       d c    0      a + c      0      b + d      0          0          0          0
       e f  b + e      0      a + f      0        0          0          0          0
       f e    0      b + e      0      a + f      0          0          0          0
       g h d + g + l   0    c + h + k    0    a + c + h + k    0     b + d + g + l    0
       h g    0    d + g + l    0    c + h + k      0    a + c + h + k    0     b + d + g +
       i j f + i + n   0    e + j + m    0    b + e + j + m    0     a + f + i + n    0
       j i    0    f + i + n    0    e + j + m      0    b + e + j + m    0     a + f + i +
       k l  h + k      0      g + l      0      d + g + l      0       c + h + k      0
       l k    0      h + k      0      g + l        0      d + g + l      0       c + h + k
       m n  j + m      0      i + n      0      f + i + n      0       e + j + m      0
       n m    0      j + m      0      i + n        0      f + i + n      0       e + j + m)

mat /. {a → cfs[[1]], b → cfs[[2]], c → cfs[[3]], d → cfs[[4]],
  e → cfs[[5]], f → cfs[[6]], g → cfs[[7]], h → cfs[[8]], i → cfs[[9]],
  j → cfs[[10]], k → cfs[[11]], l → cfs[[12]], m → cfs[[13]], n → cfs[[14]]}

]
$Failed
```

Bra ⟨| and Ket |⟩ notations and variable generators:

```
Notation[ ⅇ^x_ ⟹ MatrixExp[x_] ]
Notation[ < x_ | ⟹ id[x_] ]
(* Column use is for asthetics only,
this however is not working with matrix multiplicaiton so commented out *)
(*Notation[ | x_ > ⟹ Column[x_] ]*)
Notation[ < x_ | M_ | y_ > ⟹ Simplify[x_ . M_ . y_] ]
Notation[ < x_ | y_ > ⟹ Simplify[x_ . y_] ]
Notation[ < x_ | M_ ⟹ Simplify[x_ . M_ ] ]
Notation[ M_ | y_ > ⟹ Simplify[ M_ . y_] ]
Notation[ | x_ > ⟹ id[x_] ]
Notation[ a_
          n_  ⟹ ConstantArray[a_, n_] ]
Notation[ j_
          n_ a_ ⟹ makeVAR[a_, n_, j_] ]

makeVAR[x_, n_, j_] := Table[Subscript[x, i], {i, j, n - (1 - j)}]

id[x_] := x
```

Stochastic Lie Algebra generators, see [2]:

```
Notation[ L^(i_,j_)
            n_      ⟹ makeL[i_, j_, n_] ]

makeL[i0_, j0_, n0_] := Module[{i = i0, j = j0, n = n0, L},
  Table[KroneckerDelta[i, k] * KroneckerDelta[j, l] -
    KroneckerDelta[j, k] * KroneckerDelta[j, l], {k, 1, n}, {l, 1, n}]
 ]
```

Before doing anything we need to create the Cayley table for Kuratowski Monoid [1]:

Let X be a topological space. Denote A^{-}closure of a the set $A \subseteq X$ and A^{c} the Complement. A widely known fact due to K. Kuratowski [1] states that at most 14 distinct operations can be formed by such compositions of the two operators!

Kuratowski Operations:

$\sigma_0(A) = A$ (The Identity), $\sigma_1(A) = A^{c}$ (The Complement),
$\sigma_2(A) = A^{-}$ (The Closure), $\sigma_3(A) = A^{c-}$,
$\sigma_4(A) = A^{-c}$, $\sigma_5(A) = A^{c-c}$ (The Interior),

$\sigma_6(A) = A^{-c-}$, $\sigma_7(A) = A^{c-c-}$,
$\sigma_8(A) = A^{-c-c}$, $\sigma_9(A) = A^{c-c-c}$,
$\sigma_{10}(A) = A^{-c-c-}$, $\sigma_{11}(A) = A^{c-c-c-}$,
$\sigma_{12}(A) = A^{-c-c-c}$, $\sigma_{13}(A) = A^{c-c-c-c}$,

Cancellation rules:

$A^{-c-} = A^{-c-c-c-}$ $A^{c-c-} = A^{c-c-c-c-}$

Call the Cayley table of the above 14 generators $\mathbb{M}$:

```
𝕄 = kuraTable[];
𝕄 // MatrixForm
```

$$\begin{pmatrix}
\sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_1 & \sigma_0 & \sigma_4 & \sigma_5 & \sigma_2 & \sigma_3 & \sigma_8 & \sigma_9 & \sigma_6 & \sigma_7 & \sigma_{12} & \sigma_{13} & \sigma_{10} & \sigma_{11} \\
\sigma_2 & \sigma_3 & \sigma_2 & \sigma_3 & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_3 & \sigma_2 & \sigma_6 & \sigma_7 & \sigma_2 & \sigma_3 & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_4 & \sigma_5 & \sigma_4 & \sigma_5 & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_5 & \sigma_4 & \sigma_8 & \sigma_9 & \sigma_4 & \sigma_5 & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} \\
\sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_7 & \sigma_6 & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} \\
\sigma_9 & \sigma_8 & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_{11} & \sigma_{10} & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_{13} & \sigma_{12} & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13}
\end{pmatrix}$$

Now let’s test variable generation and Bra and Ket:

```
(* Make an array of 14 1's *)
1
14
```

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

```
(* Make an array of 14 indexed σ's starting with 0 *)
 0
14 σ
```

$\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}\}$

Ket should output vertically but *Mathematica* does not:

```
(* keep a space between | and the numbers,
better yet first do |x> then substitue for x *)
(* FIME ME 1.1:|>
   has been commented out in the code due to some conflict in Mathematica *)
```

$|{}^{1}_{14}$ x >

x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
x_9
x_{10}
x_{11}
x_{12}
x_{13}
x_{14}

Instead *Mathematica* outputs both Bra and Ket horizontally:

$\left| \, {}^{1}_{14}\, \mathbf{x} \right>$

$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}$

$\left< \mathbf{1}_{14} \right|$

$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$

Inner product:

$\left< \mathbf{1}_{14} \middle| \, {}^{0}_{14}\, \mathbf{x} \right>$

$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13}$

Let’s play around with Kuratowski Monoid:

$\sigma_6\ \sigma_5$

σ_{11}

$\sigma_8\ \sigma_9$

σ_9

σ_6^2

σ_{10}

Check to see if the table is associative via brute force check:

```
res = True
assoc = Table[If[kuraProd2[kuraProd2[i, j], k] - kuraProd2[i, kuraProd2[j, k]] ≠ 0,
    res = False], {i, 0, 13}, {j, 0, 13}, {k, 0, 13}];
res
```

```
True
```

```
True
```

Let’s work with some small dimensional Stochastic matrices generated (by exponentiation) from their Lie Algebra, note that the columns add to 0:

$L_3^{1,3}$ // MatrixForm

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Compute the exponential of this matrices which is a Stochastic matrix, note that the columns add to 1:

$e^{L_3^{1,3}}$ // MatrixForm

$$\begin{pmatrix} 1 & 0 & \frac{-1+e}{e} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{e} \end{pmatrix}$$

Make a 3D vector

${}_3^0x$

$\{x_0, x_1, x_2\}$

Compute its linear form with all 1s for coefficients:

$\langle 1_3 \mid {}_3^0x \rangle$

$x_0 + x_1 + x_2$

Transform by the Stochastic matrix $e^{L_3^{1,3}}$ and note the linear form is preserved:

$< \underset{3}{1} \mid e^{L_3^{1,3}} \mid \underset{3}{\overset{0}{}} x >$

$x_0 + x_1 + x_2$

$< 1 \mid M = 1$:

$< \underset{3}{1} \mid e^{L_3^{1,3}}$

$\{1, 1, 1\}$

$e^{L_3^{1,3}} \mid \underset{3}{\overset{0}{}} x >$

$$\left\{x_0 + \frac{(-1 + e)\ x_2}{e},\ x_1,\ \frac{x_2}{e}\right\}$$

Note that the $\{x_0, x_1, x_2\}$ was transformed to $\left\{x_0 + \frac{(-1+e)\ x_2}{e},\ x_1,\ \frac{x_2}{e}\right\}$ by $e^{L_3^{1,3}}$, however both vectors lie on the plane perpendicular to $\{1, 1, 1\}$ or $x_0 + x_1 + x_2$ = Const . In other words the linear form $x_0 + x_1 + x_2$ is preserved even if a transformation applied.

Try the other Stochastic matrices of dimension 3, again note each the linear form is preserved :

$< \underset{3}{1} \mid e^{L_3^{1,2}} \mid \underset{3}{\overset{0}{}} x >$

$x_0 + x_1 + x_2$

$< \underset{3}{1} \mid e^{L_3^{2,1}} \mid \underset{3}{\overset{0}{}} x >$

$x_0 + x_1 + x_2$

$< \underset{3}{1} \mid e^{L_3^{2,3}} \mid \underset{3}{\overset{0}{}} x >$

$x_0 + x_1 + x_2$

$< \underset{3}{1} \mid e^{L_3^{3,1}} \mid \underset{3}{\overset{0}{}} x >$

$x_0 + x_1 + x_2$

$$< \underset{3}{1} \left| e^{L_3^{3,2}} \right| {}^{0}_{3}x >$$

$x_0 + x_1 + x_2$

Replace x by a x^2, the quadratic form is also preserved :

$$< \underset{3}{1} \left| e^{L_3^{3,2}} \right| {}^{0}_{3}x^2 >$$

$x^2{}_0 + x^2{}_1 + x^2{}_2$

Try other matrices sand same preservation:

$$< \underset{3}{1} \left| e^{L_3^{1,2}} \right| {}^{0}_{3}x^2 >$$

$x^2{}_0 + x^2{}_1 + x^2{}_2$

and so on.

1.1 Automorphism

Consider the permutation [1]:

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_0 & \sigma_1 & \sigma_5 & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_9 & \sigma_8 & \sigma_7 & \sigma_6 & \sigma_{13} & \sigma_{12} & \sigma_{11} & \sigma_{10} \end{pmatrix}$$

It determines the automorphism $\mathbb{A} : \mathbb{M} \longrightarrow \mathbb{M}$.

Theorem 1.1: *The Identity and $\mathbb{A}$ are the only automorphisms of $\mathbb{M}$. Furthermore $\mathbb{A}$ is isomorphic to $\mathbb{Z}_2$.*

Proof: Establish $\mathbb{A}$ as a permutation and compute its cycles:

```
cyc = FindPermutation[head[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14],
  head[1, 2, 6, 5, 4, 3, 10, 9, 8, 7, 14, 13, 12, 11]]
Cycles[{{3, 6}, {4, 5}, {7, 10}, {8, 9}, {11, 14}, {12, 13}}]
```

Calculate the power 2 of the permutation:

```
PermutationPower[ cyc, 2]
Cycles[{}]
```

Therefore the $\mathbb{A}^2 = \text{Identity}$. For the rest of the technicalities of the proof see [1].

2. Basis $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}\}$

Imagine σ_i as vector basis for a vector space over a (commutative) Field e.g. $\mathbb{R}$ or $\mathbb{C}$:

$$v = \sum_{i=0}^{13} a_i \sigma_i \qquad \text{(EQ 2.1)}$$

Let's define such two vectors v1 and v2 in *Mathematica*, and define a vector product based upon an abstract $**$ non-commutative product with distributive property:

```
v1 = a σ_0 + b σ_1 + c σ_2 + d σ_3 + e σ_4 + f σ_5 + g σ_6 + h σ_7 + i σ_8 + j σ_9 + k σ_10 + l σ_11 + m σ_12 + n σ_13;
v2 = a2 σ_0 + b2 σ_1 + c2 σ_2 + d2 σ_3 + e2 σ_4 + f2 σ_5 +
   g2 σ_6 + h2 σ_7 + i2 σ_8 + j2 σ_9 + k2 σ_10 + l2 σ_11 + m2 σ_12 + n2 σ_13;

(* In Mathematica ** is a non-commutative anstract binary operator (product) *)
(* Distribute[ ] is a function that executes algebraic distributive law *)
Distribute[v1 ** v2]
```

```
(a σ_0) ** (a2 σ_0) + (a σ_0) ** (b2 σ_1) + (a σ_0) ** (c2 σ_2) + (a σ_0) ** (d2 σ_3) + (a σ_0) ** (e2 σ_4) +
 (a σ_0) ** (f2 σ_5) + (a σ_0) ** (g2 σ_6) + (a σ_0) ** (h2 σ_7) + (a σ_0) ** (i2 σ_8) + (a σ_0) ** (j2 σ_9) +
 (a σ_0) ** (k2 σ_10) + (a σ_0) ** (l2 σ_11) + (a σ_0) ** (m2 σ_12) + (a σ_0) ** (n2 σ_13) +
 (b σ_1) ** (a2 σ_0) + (b σ_1) ** (b2 σ_1) + (b σ_1) ** (c2 σ_2) + (b σ_1) ** (d2 σ_3) + (b σ_1) ** (e2 σ_4) +
 (b σ_1) ** (f2 σ_5) + (b σ_1) ** (g2 σ_6) + (b σ_1) ** (h2 σ_7) + (b σ_1) ** (i2 σ_8) + (b σ_1) ** (j2 σ_9) +
 (b σ_1) ** (k2 σ_10) + (b σ_1) ** (l2 σ_11) + (b σ_1) ** (m2 σ_12) + (b σ_1) ** (n2 σ_13) +
 (c σ_2) ** (a2 σ_0) + (c σ_2) ** (b2 σ_1) + (c σ_2) ** (c2 σ_2) + (c σ_2) ** (d2 σ_3) + (c σ_2) ** (e2 σ_4) +
 (c σ_2) ** (f2 σ_5) + (c σ_2) ** (g2 σ_6) + (c σ_2) ** (h2 σ_7) + (c σ_2) ** (i2 σ_8) + (c σ_2) ** (j2 σ_9) +
 (c σ_2) ** (k2 σ_10) + (c σ_2) ** (l2 σ_11) + (c σ_2) ** (m2 σ_12) + (c σ_2) ** (n2 σ_13) +
 (d σ_3) ** (a2 σ_0) + (d σ_3) ** (b2 σ_1) + (d σ_3) ** (c2 σ_2) + (d σ_3) ** (d2 σ_3) + (d σ_3) ** (e2 σ_4) +
 (d σ_3) ** (f2 σ_5) + (d σ_3) ** (g2 σ_6) + (d σ_3) ** (h2 σ_7) + (d σ_3) ** (i2 σ_8) + (d σ_3) ** (j2 σ_9) +
 (d σ_3) ** (k2 σ_10) + (d σ_3) ** (l2 σ_11) + (d σ_3) ** (m2 σ_12) + (d σ_3) ** (n2 σ_13) +
 (e σ_4) ** (a2 σ_0) + (e σ_4) ** (b2 σ_1) + (e σ_4) ** (c2 σ_2) + (e σ_4) ** (d2 σ_3) + (e σ_4) ** (e2 σ_4) +
 (e σ_4) ** (f2 σ_5) + (e σ_4) ** (g2 σ_6) + (e σ_4) ** (h2 σ_7) + (e σ_4) ** (i2 σ_8) + (e σ_4) ** (j2 σ_9) +
 (e σ_4) ** (k2 σ_10) + (e σ_4) ** (l2 σ_11) + (e σ_4) ** (m2 σ_12) + (e σ_4) ** (n2 σ_13) +
 (f σ_5) ** (a2 σ_0) + (f σ_5) ** (b2 σ_1) + (f σ_5) ** (c2 σ_2) + (f σ_5) ** (d2 σ_3) + (f σ_5) ** (e2 σ_4) +
 (f σ_5) ** (f2 σ_5) + (f σ_5) ** (g2 σ_6) + (f σ_5) ** (h2 σ_7) + (f σ_5) ** (i2 σ_8) + (f σ_5) ** (j2 σ_9) +
 (f σ_5) ** (k2 σ_10) + (f σ_5) ** (l2 σ_11) + (f σ_5) ** (m2 σ_12) + (f σ_5) ** (n2 σ_13) +
 (g σ_6) ** (a2 σ_0) + (g σ_6) ** (b2 σ_1) + (g σ_6) ** (c2 σ_2) + (g σ_6) ** (d2 σ_3) + (g σ_6) ** (e2 σ_4) +
 (g σ_6) ** (f2 σ_5) + (g σ_6) ** (g2 σ_6) + (g σ_6) ** (h2 σ_7) + (g σ_6) ** (i2 σ_8) + (g σ_6) ** (j2 σ_9) +
 (g σ_6) ** (k2 σ_10) + (g σ_6) ** (l2 σ_11) + (g σ_6) ** (m2 σ_12) + (g σ_6) ** (n2 σ_13) +
 (h σ_7) ** (a2 σ_0) + (h σ_7) ** (b2 σ_1) + (h σ_7) ** (c2 σ_2) + (h σ_7) ** (d2 σ_3) + (h σ_7) ** (e2 σ_4) +
 (h σ_7) ** (f2 σ_5) + (h σ_7) ** (g2 σ_6) + (h σ_7) ** (h2 σ_7) + (h σ_7) ** (i2 σ_8) + (h σ_7) ** (j2 σ_9) +
 (h σ_7) ** (k2 σ_10) + (h σ_7) ** (l2 σ_11) + (h σ_7) ** (m2 σ_12) + (h σ_7) ** (n2 σ_13) +
 (i σ_8) ** (a2 σ_0) + (i σ_8) ** (b2 σ_1) + (i σ_8) ** (c2 σ_2) + (i σ_8) ** (d2 σ_3) + (i σ_8) ** (e2 σ_4) +
 (i σ_8) ** (f2 σ_5) + (i σ_8) ** (g2 σ_6) + (i σ_8) ** (h2 σ_7) + (i σ_8) ** (i2 σ_8) + (i σ_8) ** (j2 σ_9) +
 (i σ_8) ** (k2 σ_10) + (i σ_8) ** (l2 σ_11) + (i σ_8) ** (m2 σ_12) + (i σ_8) ** (n2 σ_13) +
 (j σ_9) ** (a2 σ_0) + (j σ_9) ** (b2 σ_1) + (j σ_9) ** (c2 σ_2) + (j σ_9) ** (d2 σ_3) + (j σ_9) ** (e2 σ_4) +
 (j σ_9) ** (f2 σ_5) + (j σ_9) ** (g2 σ_6) + (j σ_9) ** (h2 σ_7) + (j σ_9) ** (i2 σ_8) + (j σ_9) ** (j2 σ_9) +
 (j σ_9) ** (k2 σ_10) + (j σ_9) ** (l2 σ_11) + (j σ_9) ** (m2 σ_12) + (j σ_9) ** (n2 σ_13) +
 (k σ_10) ** (a2 σ_0) + (k σ_10) ** (b2 σ_1) + (k σ_10) ** (c2 σ_2) + (k σ_10) ** (d2 σ_3) +
 (k σ_10) ** (e2 σ_4) + (k σ_10) ** (f2 σ_5) + (k σ_10) ** (g2 σ_6) + (k σ_10) ** (h2 σ_7) +
 (k σ_10) ** (i2 σ_8) + (k σ_10) ** (j2 σ_9) + (k σ_10) ** (k2 σ_10) + (k σ_10) ** (l2 σ_11) +
 (k σ_10) ** (m2 σ_12) + (k σ_10) ** (n2 σ_13) + (l σ_11) ** (a2 σ_0) + (l σ_11) ** (b2 σ_1) +
 (l σ_11) ** (c2 σ_2) + (l σ_11) ** (d2 σ_3) + (l σ_11) ** (e2 σ_4) + (l σ_11) ** (f2 σ_5) +
 (l σ_11) ** (g2 σ_6) + (l σ_11) ** (h2 σ_7) + (l σ_11) ** (i2 σ_8) + (l σ_11) ** (j2 σ_9) +
 (l σ_11) ** (k2 σ_10) + (l σ_11) ** (l2 σ_11) + (l σ_11) ** (m2 σ_12) + (l σ_11) ** (n2 σ_13) +
 (m σ_12) ** (a2 σ_0) + (m σ_12) ** (b2 σ_1) + (m σ_12) ** (c2 σ_2) + (m σ_12) ** (d2 σ_3) +
 (m σ_12) ** (e2 σ_4) + (m σ_12) ** (f2 σ_5) + (m σ_12) ** (g2 σ_6) + (m σ_12) ** (h2 σ_7) +
 (m σ_12) ** (i2 σ_8) + (m σ_12) ** (j2 σ_9) + (m σ_12) ** (k2 σ_10) + (m σ_12) ** (l2 σ_11) +
 (m σ_12) ** (m2 σ_12) + (m σ_12) ** (n2 σ_13) + (n σ_13) ** (a2 σ_0) + (n σ_13) ** (b2 σ_1) +
 (n σ_13) ** (c2 σ_2) + (n σ_13) ** (d2 σ_3) + (n σ_13) ** (e2 σ_4) + (n σ_13) ** (f2 σ_5) +
 (n σ_13) ** (g2 σ_6) + (n σ_13) ** (h2 σ_7) + (n σ_13) ** (i2 σ_8) + (n σ_13) ** (j2 σ_9) +
 (n σ_13) ** (k2 σ_10) + (n σ_13) ** (l2 σ_11) + (n σ_13) ** (m2 σ_12) + (n σ_13) ** (n2 σ_13)
```

We need to run the output of the Distribute [ ] one more time to have the kuraTable figure out the products:

v1v2 = (a σ_0) ** (a2 σ_0) + (a σ_0) ** (b2 σ_1) + (a σ_0) ** (c2 σ_2) + (a σ_0) ** (d2 σ_3) +
(a σ_0) ** (e2 σ_4) + (a σ_0) ** (f2 σ_5) + (a σ_0) ** (g2 σ_6) + (a σ_0) ** (h2 σ_7) + (a σ_0) ** (i2 σ_8) +
(a σ_0) ** (j2 σ_9) + (a σ_0) ** (k2 σ_{10}) + (a σ_0) ** (l2 σ_{11}) + (a σ_0) ** (m2 σ_{12}) +
(a σ_0) ** (n2 σ_{13}) + (b σ_1) ** (a2 σ_0) + (b σ_1) ** (b2 σ_1) + (b σ_1) ** (c2 σ_2) +
(b σ_1) ** (d2 σ_3) + (b σ_1) ** (e2 σ_4) + (b σ_1) ** (f2 σ_5) + (b σ_1) ** (g2 σ_6) + (b σ_1) ** (h2 σ_7) +
(b σ_1) ** (i2 σ_8) + (b σ_1) ** (j2 σ_9) + (b σ_1) ** (k2 σ_{10}) + (b σ_1) ** (l2 σ_{11}) +
(b σ_1) ** (m2 σ_{12}) + (b σ_1) ** (n2 σ_{13}) + (c σ_2) ** (a2 σ_0) + (c σ_2) ** (b2 σ_1) +
(c σ_2) ** (c2 σ_2) + (c σ_2) ** (d2 σ_3) + (c σ_2) ** (e2 σ_4) + (c σ_2) ** (f2 σ_5) + (c σ_2) ** (g2 σ_6) +
(c σ_2) ** (h2 σ_7) + (c σ_2) ** (i2 σ_8) + (c σ_2) ** (j2 σ_9) + (c σ_2) ** (k2 σ_{10}) +
(c σ_2) ** (l2 σ_{11}) + (c σ_2) ** (m2 σ_{12}) + (c σ_2) ** (n2 σ_{13}) + (d σ_3) ** (a2 σ_0) +
(d σ_3) ** (b2 σ_1) + (d σ_3) ** (c2 σ_2) + (d σ_3) ** (d2 σ_3) + (d σ_3) ** (e2 σ_4) + (d σ_3) ** (f2 σ_5) +
(d σ_3) ** (g2 σ_6) + (d σ_3) ** (h2 σ_7) + (d σ_3) ** (i2 σ_8) + (d σ_3) ** (j2 σ_9) +
(d σ_3) ** (k2 σ_{10}) + (d σ_3) ** (l2 σ_{11}) + (d σ_3) ** (m2 σ_{12}) + (d σ_3) ** (n2 σ_{13}) +
(e σ_4) ** (a2 σ_0) + (e σ_4) ** (b2 σ_1) + (e σ_4) ** (c2 σ_2) + (e σ_4) ** (d2 σ_3) + (e σ_4) ** (e2 σ_4) +
(e σ_4) ** (f2 σ_5) + (e σ_4) ** (g2 σ_6) + (e σ_4) ** (h2 σ_7) + (e σ_4) ** (i2 σ_8) + (e σ_4) ** (j2 σ_9) +
(e σ_4) ** (k2 σ_{10}) + (e σ_4) ** (l2 σ_{11}) + (e σ_4) ** (m2 σ_{12}) + (e σ_4) ** (n2 σ_{13}) +
(f σ_5) ** (a2 σ_0) + (f σ_5) ** (b2 σ_1) + (f σ_5) ** (c2 σ_2) + (f σ_5) ** (d2 σ_3) + (f σ_5) ** (e2 σ_4) +
(f σ_5) ** (f2 σ_5) + (f σ_5) ** (g2 σ_6) + (f σ_5) ** (h2 σ_7) + (f σ_5) ** (i2 σ_8) + (f σ_5) ** (j2 σ_9) +
(f σ_5) ** (k2 σ_{10}) + (f σ_5) ** (l2 σ_{11}) + (f σ_5) ** (m2 σ_{12}) + (f σ_5) ** (n2 σ_{13}) +
(g σ_6) ** (a2 σ_0) + (g σ_6) ** (b2 σ_1) + (g σ_6) ** (c2 σ_2) + (g σ_6) ** (d2 σ_3) + (g σ_6) ** (e2 σ_4) +
(g σ_6) ** (f2 σ_5) + (g σ_6) ** (g2 σ_6) + (g σ_6) ** (h2 σ_7) + (g σ_6) ** (i2 σ_8) + (g σ_6) ** (j2 σ_9) +
(g σ_6) ** (k2 σ_{10}) + (g σ_6) ** (l2 σ_{11}) + (g σ_6) ** (m2 σ_{12}) + (g σ_6) ** (n2 σ_{13}) +
(h σ_7) ** (a2 σ_0) + (h σ_7) ** (b2 σ_1) + (h σ_7) ** (c2 σ_2) + (h σ_7) ** (d2 σ_3) + (h σ_7) ** (e2 σ_4) +
(h σ_7) ** (f2 σ_5) + (h σ_7) ** (g2 σ_6) + (h σ_7) ** (h2 σ_7) + (h σ_7) ** (i2 σ_8) + (h σ_7) ** (j2 σ_9) +
(h σ_7) ** (k2 σ_{10}) + (h σ_7) ** (l2 σ_{11}) + (h σ_7) ** (m2 σ_{12}) + (h σ_7) ** (n2 σ_{13}) +
(i σ_8) ** (a2 σ_0) + (i σ_8) ** (b2 σ_1) + (i σ_8) ** (c2 σ_2) + (i σ_8) ** (d2 σ_3) + (i σ_8) ** (e2 σ_4) +
(i σ_8) ** (f2 σ_5) + (i σ_8) ** (g2 σ_6) + (i σ_8) ** (h2 σ_7) + (i σ_8) ** (i2 σ_8) + (i σ_8) ** (j2 σ_9) +
(i σ_8) ** (k2 σ_{10}) + (i σ_8) ** (l2 σ_{11}) + (i σ_8) ** (m2 σ_{12}) + (i σ_8) ** (n2 σ_{13}) +
(j σ_9) ** (a2 σ_0) + (j σ_9) ** (b2 σ_1) + (j σ_9) ** (c2 σ_2) + (j σ_9) ** (d2 σ_3) + (j σ_9) ** (e2 σ_4) +
(j σ_9) ** (f2 σ_5) + (j σ_9) ** (g2 σ_6) + (j σ_9) ** (h2 σ_7) + (j σ_9) ** (i2 σ_8) + (j σ_9) ** (j2 σ_9) +
(j σ_9) ** (k2 σ_{10}) + (j σ_9) ** (l2 σ_{11}) + (j σ_9) ** (m2 σ_{12}) + (j σ_9) ** (n2 σ_{13}) +
(k σ_{10}) ** (a2 σ_0) + (k σ_{10}) ** (b2 σ_1) + (k σ_{10}) ** (c2 σ_2) + (k σ_{10}) ** (d2 σ_3) +
(k σ_{10}) ** (e2 σ_4) + (k σ_{10}) ** (f2 σ_5) + (k σ_{10}) ** (g2 σ_6) + (k σ_{10}) ** (h2 σ_7) +
(k σ_{10}) ** (i2 σ_8) + (k σ_{10}) ** (j2 σ_9) + (k σ_{10}) ** (k2 σ_{10}) + (k σ_{10}) ** (l2 σ_{11}) +
(k σ_{10}) ** (m2 σ_{12}) + (k σ_{10}) ** (n2 σ_{13}) + (l σ_{11}) ** (a2 σ_0) + (l σ_{11}) ** (b2 σ_1) +
(l σ_{11}) ** (c2 σ_2) + (l σ_{11}) ** (d2 σ_3) + (l σ_{11}) ** (e2 σ_4) + (l σ_{11}) ** (f2 σ_5) +
(l σ_{11}) ** (g2 σ_6) + (l σ_{11}) ** (h2 σ_7) + (l σ_{11}) ** (i2 σ_8) + (l σ_{11}) ** (j2 σ_9) +
(l σ_{11}) ** (k2 σ_{10}) + (l σ_{11}) ** (l2 σ_{11}) + (l σ_{11}) ** (m2 σ_{12}) + (l σ_{11}) ** (n2 σ_{13}) +
(m σ_{12}) ** (a2 σ_0) + (m σ_{12}) ** (b2 σ_1) + (m σ_{12}) ** (c2 σ_2) + (m σ_{12}) ** (d2 σ_3) +
(m σ_{12}) ** (e2 σ_4) + (m σ_{12}) ** (f2 σ_5) + (m σ_{12}) ** (g2 σ_6) + (m σ_{12}) ** (h2 σ_7) +
(m σ_{12}) ** (i2 σ_8) + (m σ_{12}) ** (j2 σ_9) + (m σ_{12}) ** (k2 σ_{10}) + (m σ_{12}) ** (l2 σ_{11}) +
(m σ_{12}) ** (m2 σ_{12}) + (m σ_{12}) ** (n2 σ_{13}) + (n σ_{13}) ** (a2 σ_0) + (n σ_{13}) ** (b2 σ_1) +
(n σ_{13}) ** (c2 σ_2) + (n σ_{13}) ** (d2 σ_3) + (n σ_{13}) ** (e2 σ_4) + (n σ_{13}) ** (f2 σ_5) +
(n σ_{13}) ** (g2 σ_6) + (n σ_{13}) ** (h2 σ_7) + (n σ_{13}) ** (i2 σ_8) + (n σ_{13}) ** (j2 σ_9) +
(n σ_{13}) ** (k2 σ_{10}) + (n σ_{13}) ** (l2 σ_{11}) + (n σ_{13}) ** (m2 σ_{12}) + (n σ_{13}) ** (n2 σ_{13})

Kuratowski Monoid then reduces the $\sigma_i \sigma_j$ to $\sigma_{i,j}$ in the Cayley table mentioned above:

a a2 σ_{0} + b b2 σ_{0} + a2 b σ_{1} + a b2 σ_{1} + a2 c σ_{2} + a c2 σ_{2} + c c2 σ_{2} + b2 d σ_{2} + b e2 σ_{2} + d e2 σ_{2} +
b2 c σ_{3} + a2 d σ_{3} + a d2 σ_{3} + c d2 σ_{3} + b f2 σ_{3} + d f2 σ_{3} + b c2 σ_{4} + a2 e σ_{4} + c2 e σ_{4} + a e2 σ_{4} +
b2 f σ_{4} + e2 f σ_{4} + b d2 σ_{5} + b2 e σ_{5} + d2 e σ_{5} + a2 f σ_{5} + a f2 σ_{5} + f f2 σ_{5} + c2 d σ_{6} + c e2 σ_{6} +
a2 g σ_{6} + c2 g σ_{6} + a g2 σ_{6} + c g2 σ_{6} + b2 h σ_{6} + e2 h σ_{6} + g2 h σ_{6} + b i2 σ_{6} + d i2 σ_{6} + g i2 σ_{6} +
e2 k σ_{6} + g2 k σ_{6} + d k2 σ_{6} + g k2 σ_{6} + c2 l σ_{6} + i2 l σ_{6} + k2 l σ_{6} + c m2 σ_{6} + h m2 σ_{6} + k m2 σ_{6} +
d d2 σ_{7} + c f2 σ_{7} + b2 g σ_{7} + d2 g σ_{7} + a2 h σ_{7} + f2 h σ_{7} + a h2 σ_{7} + c h2 σ_{7} + h h2 σ_{7} + b j2 σ_{7} +
d j2 σ_{7} + g j2 σ_{7} + f2 k σ_{7} + h2 k σ_{7} + d2 l σ_{7} + j2 l σ_{7} + d l2 σ_{7} + g l2 σ_{7} + l l2 σ_{7} + c n2 σ_{7} +
h n2 σ_{7} + k n2 σ_{7} + e e2 σ_{8} + c2 f σ_{8} + b g2 σ_{8} + e g2 σ_{8} + a2 i σ_{8} + c2 i σ_{8} + a i2 σ_{8} +
f i2 σ_{8} + i i2 σ_{8} + b2 j σ_{8} + e2 j σ_{8} + g2 j σ_{8} + f k2 σ_{8} + i k2 σ_{8} + e2 m σ_{8} + g2 m σ_{8} +
e m2 σ_{8} + j m2 σ_{8} + m m2 σ_{8} + c2 n σ_{8} + i2 n σ_{8} + k2 n σ_{8} + d2 f σ_{9} + e f2 σ_{9} + b h2 σ_{9} +
e h2 σ_{9} + b2 i σ_{9} + d2 i σ_{9} + a2 j σ_{9} + f2 j σ_{9} + h2 j σ_{9} + a j2 σ_{9} + f j2 σ_{9} + i j2 σ_{9} +
f l2 σ_{9} + i l2 σ_{9} + f2 m σ_{9} + h2 m σ_{9} + d2 n σ_{9} + j2 n σ_{9} + l2 n σ_{9} + e n2 σ_{9} + j n2 σ_{9} +
m n2 σ_{9} + e2 g σ_{10} + d g2 σ_{10} + g g2 σ_{10} + c2 h σ_{10} + c i2 σ_{10} + h i2 σ_{10} + a2 k σ_{10} + c2 k σ_{10} +
i2 k σ_{10} + a k2 σ_{10} + c k2 σ_{10} + h k2 σ_{10} + k k2 σ_{10} + b2 l σ_{10} + e2 l σ_{10} + g2 l σ_{10} + b m2 σ_{10} +
d m2 σ_{10} + g m2 σ_{10} + l m2 σ_{10} + f2 g σ_{11} + d2 h σ_{11} + d h2 σ_{11} + g h2 σ_{11} + c j2 σ_{11} + h j2 σ_{11} +
b2 k σ_{11} + d2 k σ_{11} + j2 k σ_{11} + a2 l σ_{11} + f2 l σ_{11} + h2 l σ_{11} + a l2 σ_{11} + c l2 σ_{11} + h l2 σ_{11} +
k l2 σ_{11} + b n2 σ_{11} + d n2 σ_{11} + g n2 σ_{11} + l n2 σ_{11} + f g2 σ_{12} + e2 i σ_{12} + g2 i σ_{12} + e i2 σ_{12} +
c2 j σ_{12} + i2 j σ_{12} + b k2 σ_{12} + e k2 σ_{12} + j k2 σ_{12} + a2 m σ_{12} + c2 m σ_{12} + i2 m σ_{12} + k2 m σ_{12} +
a m2 σ_{12} + f m2 σ_{12} + i m2 σ_{12} + b2 n σ_{12} + e2 n σ_{12} + g2 n σ_{12} + m2 n σ_{12} + f h2 σ_{13} + f2 i σ_{13} +
h2 i σ_{13} + d2 j σ_{13} + e j2 σ_{13} + j j2 σ_{13} + b l2 σ_{13} + e l2 σ_{13} + j l2 σ_{13} + b2 m σ_{13} + d2 m σ_{13} +
j2 m σ_{13} + l2 m σ_{13} + a2 n σ_{13} + f2 n σ_{13} + h2 n σ_{13} + a n2 σ_{13} + f n2 σ_{13} + i n2 σ_{13} + n n2 σ_{13}

Kuratowski Monoid then reduces the $\sigma_i \sigma_j$ to $\sigma_{i,j}$ in the Cayley table 𝕄 mentioned above in section 1.

Collect and sort the coefficients for σ_i :

**Collect[a a2 σ_{0} + b b2 σ_{0} + a2 b σ_{1} + a b2 σ_{1} + a2 c σ_{2} + a c2 σ_{2} + c c2 σ_{2} + b2 d σ_{2} + b e2 σ_{2} +
d e2 σ_{2} + b2 c σ_{3} + a2 d σ_{3} + a d2 σ_{3} + c d2 σ_{3} + b f2 σ_{3} + d f2 σ_{3} + b c2 σ_{4} + a2 e σ_{4} + c2 e σ_{4} +
a e2 σ_{4} + b2 f σ_{4} + e2 f σ_{4} + b d2 σ_{5} + b2 e σ_{5} + d2 e σ_{5} + a2 f σ_{5} + a f2 σ_{5} + f f2 σ_{5} + c2 d σ_{6} +
c e2 σ_{6} + a2 g σ_{6} + c2 g σ_{6} + a g2 σ_{6} + c g2 σ_{6} + b2 h σ_{6} + e2 h σ_{6} + g2 h σ_{6} + b i2 σ_{6} + d i2 σ_{6} +
g i2 σ_{6} + e2 k σ_{6} + g2 k σ_{6} + d k2 σ_{6} + g k2 σ_{6} + c2 l σ_{6} + i2 l σ_{6} + k2 l σ_{6} + c m2 σ_{6} + h m2 σ_{6} +
k m2 σ_{6} + d d2 σ_{7} + c f2 σ_{7} + b2 g σ_{7} + d2 g σ_{7} + a2 h σ_{7} + f2 h σ_{7} + a h2 σ_{7} + c h2 σ_{7} + h h2 σ_{7} +
b j2 σ_{7} + d j2 σ_{7} + g j2 σ_{7} + f2 k σ_{7} + h2 k σ_{7} + d2 l σ_{7} + j2 l σ_{7} + d l2 σ_{7} + g l2 σ_{7} + l l2 σ_{7} +
c n2 σ_{7} + h n2 σ_{7} + k n2 σ_{7} + e e2 σ_{8} + c2 f σ_{8} + b g2 σ_{8} + e g2 σ_{8} + a2 i σ_{8} + c2 i σ_{8} + a i2 σ_{8} +
f i2 σ_{8} + i i2 σ_{8} + b2 j σ_{8} + e2 j σ_{8} + g2 j σ_{8} + f k2 σ_{8} + i k2 σ_{8} + e2 m σ_{8} + g2 m σ_{8} +
e m2 σ_{8} + j m2 σ_{8} + m m2 σ_{8} + c2 n σ_{8} + i2 n σ_{8} + k2 n σ_{8} + d2 f σ_{9} + e f2 σ_{9} + b h2 σ_{9} +
e h2 σ_{9} + b2 i σ_{9} + d2 i σ_{9} + a2 j σ_{9} + f2 j σ_{9} + h2 j σ_{9} + a j2 σ_{9} + f j2 σ_{9} + i j2 σ_{9} +
f l2 σ_{9} + i l2 σ_{9} + f2 m σ_{9} + h2 m σ_{9} + d2 n σ_{9} + j2 n σ_{9} + l2 n σ_{9} + e n2 σ_{9} + j n2 σ_{9} +
m n2 σ_{9} + e2 g σ_{10} + d g2 σ_{10} + g g2 σ_{10} + c2 h σ_{10} + c i2 σ_{10} + h i2 σ_{10} + a2 k σ_{10} + c2 k σ_{10} +
i2 k σ_{10} + a k2 σ_{10} + c k2 σ_{10} + h k2 σ_{10} + k k2 σ_{10} + b2 l σ_{10} + e2 l σ_{10} + g2 l σ_{10} + b m2 σ_{10} +
d m2 σ_{10} + g m2 σ_{10} + l m2 σ_{10} + f2 g σ_{11} + d2 h σ_{11} + d h2 σ_{11} + g h2 σ_{11} + c j2 σ_{11} + h j2 σ_{11} +
b2 k σ_{11} + d2 k σ_{11} + j2 k σ_{11} + a2 l σ_{11} + f2 l σ_{11} + h2 l σ_{11} + a l2 σ_{11} + c l2 σ_{11} + h l2 σ_{11} +
k l2 σ_{11} + b n2 σ_{11} + d n2 σ_{11} + g n2 σ_{11} + l n2 σ_{11} + f g2 σ_{12} + e2 i σ_{12} + g2 i σ_{12} + e i2 σ_{12} +
c2 j σ_{12} + i2 j σ_{12} + b k2 σ_{12} + e k2 σ_{12} + j k2 σ_{12} + a2 m σ_{12} + c2 m σ_{12} + i2 m σ_{12} + k2 m σ_{12} +
a m2 σ_{12} + f m2 σ_{12} + i m2 σ_{12} + b2 n σ_{12} + e2 n σ_{12} + g2 n σ_{12} + m2 n σ_{12} + f h2 σ_{13} + f2 i σ_{13} +
h2 i σ_{13} + d2 j σ_{13} + e j2 σ_{13} + j j2 σ_{13} + b l2 σ_{13} + e l2 σ_{13} + j l2 σ_{13} + b2 m σ_{13} + d2 m σ_{13} +
j2 m σ_{13} + l2 m σ_{13} + a2 n σ_{13} + f2 n σ_{13} + h2 n σ_{13} + a n2 σ_{13} + f n2 σ_{13} + i n2 σ_{13} + n n2 σ_{13},
{σ_{0}, σ_{1}, σ_{2}, σ_{3}, σ_{4}, σ_{5}, σ_{6}, σ_{7}, σ_{8}, σ_{9}, σ_{10}, σ_{11}, σ_{12}, σ_{13}}]**

These are the coefficients of the product:

```
(a a2 + b b2) σ0 + (a2 b + a b2) σ1 +
 (a2 c + a c2 + c c2 + b2 d + b e2 + d e2) σ2 + (b2 c + a2 d + a d2 + c d2 + b f2 + d f2) σ3 +
 (b c2 + a2 e + c2 e + a e2 + b2 f + e2 f) σ4 + (b d2 + b2 e + d2 e + a2 f + a f2 + f f2) σ5 +
 (c2 d + c e2 + a2 g + c2 g + a g2 + c g2 + b2 h + e2 h + g2 h + b i2 + d i2 +
    g i2 + e2 k + g2 k + d k2 + g k2 + c2 l + i2 l + k2 l + c m2 + h m2 + k m2) σ6 +
 (d d2 + c f2 + b2 g + d2 g + a2 h + f2 h + a h2 + c h2 + h h2 + b j2 + d j2 + g j2 +
    f2 k + h2 k + d2 l + j2 l + d l2 + g l2 + l l2 + c n2 + h n2 + k n2) σ7 +
 (e e2 + c2 f + b g2 + e g2 + a2 i + c2 i + a i2 + f i2 + i i2 + b2 j + e2 j + g2 j +
    f k2 + i k2 + e2 m + g2 m + e m2 + j m2 + m m2 + c2 n + i2 n + k2 n) σ8 +
 (d2 f + e f2 + b h2 + e h2 + b2 i + d2 i + a2 j + f2 j + h2 j + a j2 + f j2 + i j2 +
    f l2 + i l2 + f2 m + h2 m + d2 n + j2 n + l2 n + e n2 + j n2 + m n2) σ9 +
 (e2 g + d g2 + g g2 + c2 h + c i2 + h i2 + a2 k + c2 k + i2 k + a k2 + c k2 +
    h k2 + k k2 + b2 l + e2 l + g2 l + b m2 + d m2 + g m2 + l m2) σ10 +
 (f2 g + d2 h + d h2 + g h2 + c j2 + h j2 + b2 k + d2 k + j2 k + a2 l + f2 l +
    h2 l + a l2 + c l2 + h l2 + k l2 + b n2 + d n2 + g n2 + l n2) σ11 +
 (f g2 + e2 i + g2 i + e i2 + c2 j + i2 j + b k2 + e k2 + j k2 + a2 m + c2 m +
    i2 m + k2 m + a m2 + f m2 + i m2 + b2 n + e2 n + g2 n + m2 n) σ12 +
 (f h2 + f2 i + h2 i + d2 j + e j2 + j j2 + b l2 + e l2 + j l2 + b2 m + d2 m +
    j2 m + l2 m + a2 n + f2 n + h2 n + a n2 + f n2 + i n2 + n n2) σ13
```

We then turn the linear first-degree sum into a matrix multiplication form:

```
CoefficientArrays[
 {(a a2 + b b2) == 0 , (a2 b + a b2) == 0, (a2 c + a c2 + c c2 + b2 d + b e2 + d e2) == 0,
  (b2 c + a2 d + a d2 + c d2 + b f2 + d f2) == 0, (b c2 + a2 e + c2 e + a e2 + b2 f + e2 f) == 0 ,
  (b d2 + b2 e + d2 e + a2 f + a f2 + f f2) == 0 , (c2 d + c e2 + a2 g + c2 g + a g2 + c g2 + b2 h + e2 h +
     g2 h + b i2 + d i2 + g i2 + e2 k + g2 k + d k2 + g k2 + c2 l + i2 l + k2 l + c m2 + h m2 + k m2) == 0,
  (d d2 + c f2 + b2 g + d2 g + a2 h + f2 h + a h2 + c h2 + h h2 + b j2 + d j2 + g j2 +
     f2 k + h2 k + d2 l + j2 l + d l2 + g l2 + l l2 + c n2 + h n2 + k n2) == 0,
  (e e2 + c2 f + b g2 + e g2 + a2 i + c2 i + a i2 + f i2 + i i2 + b2 j + e2 j + g2 j +
     f k2 + i k2 + e2 m + g2 m + e m2 + j m2 + m m2 + c2 n + i2 n + k2 n) == 0,
  (d2 f + e f2 + b h2 + e h2 + b2 i + d2 i + a2 j + f2 j + h2 j + a j2 + f j2 + i j2 +
     f l2 + i l2 + f2 m + h2 m + d2 n + j2 n + l2 n + e n2 + j n2 + m n2) == 0 ,
  (e2 g + d g2 + g g2 + c2 h + c i2 + h i2 + a2 k + c2 k + i2 k + a k2 + c k2 + h k2 +
     k k2 + b2 l + e2 l + g2 l + b m2 + d m2 + g m2 + l m2) == 0,
  (f2 g + d2 h + d h2 + g h2 + c j2 + h j2 + b2 k + d2 k + j2 k + a2 l + f2 l + h2 l +
     a l2 + c l2 + h l2 + k l2 + b n2 + d n2 + g n2 + l n2) == 0 ,
  (f g2 + e2 i + g2 i + e i2 + c2 j + i2 j + b k2 + e k2 + j k2 + a2 m + c2 m + i2 m +
     k2 m + a m2 + f m2 + i m2 + b2 n + e2 n + g2 n + m2 n) == 0,
  (f h2 + f2 i + h2 i + d2 j + e j2 + j j2 + b l2 + e l2 + j l2 + b2 m + d2 m + j2 m +
     l2 m + a2 n + f2 n + h2 n + a n2 + f n2 + i n2 + n n2) == 0},
 {a2, b2, c2, d2, e2, f2, g2, h2, i2, j2, k2, l2, m2, n2}]
{SparseArray[<0>, {14}], SparseArray[<84>, {14, 14}]}
```

```
coeff = Normal[%][[2]];
coeff // MatrixForm
```

a	b	0	0	0	0	0	0	0	0	0	0	0	0
b	a	0	0	0	0	0	0	0	0	0	0	0	0
c	d	a+c	0	b+d	0	0	0	0	0	0	0	0	0
d	c	0	a+c	0	b+d	0	0	0	0	0	0	0	0
e	f	b+e	0	a+f	0	0	0	0	0	0	0	0	0
f	e	0	b+e	0	a+f	0	0	0	0	0	0	0	0
g	h	d+g+l	0	c+h+k	0	a+c+h+k	0	b+d+g+l	0	d+g+l	0	c+h+k	0
h	g	0	d+g+l	0	c+h+k	0	a+c+h+k	0	b+d+g+l	0	d+g+l	0	c+h+k
i	j	f+i+n	0	e+j+m	0	b+e+j+m	0	a+f+i+n	0	f+i+n	0	e+j+m	0
j	i	0	f+i+n	0	e+j+m	0	b+e+j+m	0	a+f+i+n	0	f+i+n	0	e+j+m
k	l	h+k	0	g+l	0	d+g+l	0	c+h+k	0	a+c+h+k	0	b+d+g+l	0
l	k	0	h+k	0	g+l	0	d+g+l	0	c+h+k	0	a+c+h+k	0	b+d+g+l
m	n	j+m	0	i+n	0	f+i+n	0	e+j+m	0	b+e+j+m	0	a+f+i+n	0
n	m	0	j+m	0	i+n	0	f+i+n	0	e+j+m	0	b+e+j+m	0	a+f+i+n

Let's code this result into a *Mathematica* function:

```
(* ⊙ product in 14Sets algebra *)
Notation[ x_⊙y_  ⟹  vectorProduct[x_, y_] ];
Notation[ (a_ x_)⊙(b_ y_)  ⟹  a_*b_ vectorProduct[x_, y_] ];
vectorProduct[x0_, y0_] :=
 Module[{x = x0, y = y0, cfsx, cfsy, a, b, c, d, e, f, g, h, i, j,
    k, l, m, n, a2, b2, c2, d2, e2, f2, g2, h2, i2, j2, k2, l2, m2, n2},
  cfsx = Coefficient[x, {σ0, σ1, σ2, σ3, σ4, σ5, σ6, σ7, σ8, σ9, σ10, σ11, σ12, σ13}];
  cfsy = Coefficient[y, {σ0, σ1, σ2, σ3, σ4, σ5, σ6, σ7, σ8, σ9, σ10, σ11, σ12, σ13}];

  a = cfsx[[1]];
  b = cfsx[[2]];
  c = cfsx[[3]];
  d = cfsx[[4]];
  e = cfsx[[5]];
  f = cfsx[[6]];
  g = cfsx[[7]];
  h = cfsx[[8]];
  i = cfsx[[9]];
  j = cfsx[[10]];
  k = cfsx[[11]];
  l = cfsx[[12]];
  m = cfsx[[13]];
  n = cfsx[[14]];

  a2 = cfsy[[1]];
  b2 = cfsy[[2]];
  c2 = cfsy[[3]];
  d2 = cfsy[[4]];
  e2 = cfsy[[5]];
  f2 = cfsy[[6]];
  g2 = cfsy[[7]];
  h2 = cfsy[[8]];
  i2 = cfsy[[9]];
  j2 = cfsy[[10]];
  k2 = cfsy[[11]];
  l2 = cfsy[[12]];
  m2 = cfsy[[13]];
  n2 = cfsy[[14]];
```

```
Simplify[(a a2 + b b2) σ0 + (a2 b + a b2) σ1 +
   (a2 c + a c2 + c c2 + b2 d + b e2 + d e2) σ2 + (b2 c + a2 d + a d2 + c d2 + b f2 + d f2) σ3 +
   (b c2 + a2 e + c2 e + a e2 + b2 f + e2 f) σ4 + (b d2 + b2 e + d2 e + a2 f + a f2 + f f2) σ5 +
   (c2 d + c e2 + a2 g + c2 g + a g2 + c g2 + b2 h + e2 h + g2 h + b i2 + d i2 +
      g i2 + e2 k + g2 k + d k2 + g k2 + c2 l + i2 l + k2 l + c m2 + h m2 + k m2) σ6 +
   (d d2 + c f2 + b2 g + d2 g + a2 h + f2 h + a h2 + c h2 + h h2 + b j2 + d j2 + g j2 +
      f2 k + h2 k + d2 l + j2 l + d l2 + g l2 + l l2 + c n2 + h n2 + k n2) σ7 +
   (e e2 + c2 f + b g2 + e g2 + a2 i + c2 i + a i2 + f i2 + i i2 + b2 j + e2 j + g2 j +
      f k2 + i k2 + e2 m + g2 m + e m2 + j m2 + m m2 + c2 n + i2 n + k2 n) σ8 +
   (d2 f + e f2 + b h2 + e h2 + b2 i + d2 i + a2 j + f2 j + h2 j + a j2 + f j2 + i j2 +
      f l2 + i l2 + f2 m + h2 m + d2 n + j2 n + l2 n + e n2 + j n2 + m n2) σ9 +
   (e2 g + d g2 + g g2 + c2 h + c i2 + h i2 + a2 k + c2 k + i2 k + a k2 + c k2 + h k2 +
      k k2 + b2 l + e2 l + g2 l + b m2 + d m2 + g m2 + l m2) σ10 +
   (f2 g + d2 h + d h2 + g h2 + c j2 + h j2 + b2 k + d2 k + j2 k + a2 l + f2 l + h2 l +
      a l2 + c l2 + h l2 + k l2 + b n2 + d n2 + g n2 + l n2) σ11 +
   (f g2 + e2 i + g2 i + e i2 + c2 j + i2 j + b k2 + e k2 + j k2 + a2 m + c2 m + i2 m +
      k2 m + a m2 + f m2 + i m2 + b2 n + e2 n + g2 n + m2 n) σ12 +
   (f h2 + f2 i + h2 i + d2 j + e j2 + j j2 + b l2 + e l2 + j l2 + b2 m + d2 m + j2 m +
      l2 m + a2 n + f2 n + h2 n + a n2 + f n2 + i n2 + n n2) σ13]
]
```

3. Matrix Representation

Using what was just computed above the matrix representation form as follows:

$$\begin{pmatrix} a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & d & a+c & 0 & b+d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & c & 0 & a+c & 0 & b+d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e & f & b+e & 0 & a+f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f & e & 0 & b+e & 0 & a+f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g & h & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l & 0 & d+g+l & 0 & c+h+k & 0 \\ h & g & 0 & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l & 0 & d+g+l & 0 & c+h+k \\ i & j & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n & 0 & f+i+n & 0 & e+j+m & 0 \\ j & i & 0 & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n & 0 & f+i+n & 0 & e+j+m \\ k & l & h+k & 0 & g+l & 0 & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l & 0 \\ l & k & 0 & h+k & 0 & g+l & 0 & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l \\ m & n & j+m & 0 & i+n & 0 & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n & 0 \\ n & m & 0 & j+m & 0 & i+n & 0 & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n \end{pmatrix} \begin{pmatrix} \text{a2} \\ \text{b2} \\ \text{c2} \\ \text{d2} \\ \text{e2} \\ \text{f2} \\ \text{g2} \\ \text{h2} \\ \text{i2} \\ \text{j2} \\ \text{k2} \\ \text{l2} \\ \text{m2} \\ \text{n2} \end{pmatrix} \sigma$$

(EQ 3.2)

where $\sigma = [\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}]$. And let's test make sure the results are the same:

```
Collect[Expand[(coeff.{a2, b2, c2, d2, e2, f2, g2, h2, i2, j2, k2, l2, m2, n2}).
     {σ0, σ1, σ2, σ3, σ4, σ5, σ6, σ7, σ8, σ9, σ10, σ11, σ12, σ13}],
  {σ0, σ1, σ2, σ3, σ4, σ5, σ6, σ7, σ8, σ9, σ10, σ11, σ12, σ13}] ===
 Collect[v1v2, {σ0, σ1, σ2, σ3, σ4, σ5, σ6, σ7, σ8, σ9, σ10, σ11, σ12, σ13}]
True
```

Corollary 3.1: *Matrix representation (EQ 3.2) applies to both multiplication and addition of 14-Sets vectors.*

Remark 3.1*: Stochastic Matrices (see section 4) are closed under the multiplication of (EQ3.2) representation but not addition.*

4. Stochastic Matrix

This matrix representation is a Left Stochastic Matrix if:

$$a+b+c+d+e+f+g+h+i+j+k+l+m+n = 1 \quad \text{(EQ 4.1)}$$

and all these coefficients are assumed greater than or equal to 0. Therefore the column sums are all 1.

Let's make functions for columns and row sums:

```
sumColumn[mat_] := Table[Total[mat[[All, w]]], {w, 1, 14}]
sumRow[mat_] := Table[Total[mat[[w, All]]], {w, 1, 14}]

(* make random 14-Sets Stochastic matrices with all colums adding to 1 *)
rand14SetsStoch[] := Module[{rands, cfs},
  rands = Join[Sort[ RandomReal[{0, 1}, 13], Less], {1}];
  cfs = Join[{rands[[1]]}, Table[rands[[w]] - rands[[w - 1]], {w, 2, 14}]]
 ]
```

General column sum:

```
sumColumn[coeff] // MatrixForm
```

$$\begin{pmatrix} a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \end{pmatrix}$$

Transpose the coefficient matrix:

```
sumRow[Transpose[coeff]] // MatrixForm
```

$$\begin{pmatrix} a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \\ a+b+c+d+e+f+g+h+i+j+k+l+m+n \end{pmatrix}$$

Definition 4.1: *14-Sets representation matrix with all columns add to 1 are called 14-Sets Stochastic Matrix.*

Corollary 4.1: 1*4-Sets matrix representation's column sums are constant and are the sum of all coefficients.*

There are at least two non-trivial eigenvalues with value '1', no assumptions on the coefficients:

```
Eigenvalues[coeff /. {n → 1 - (a + b + c + d + e + f + g + h + i + j + k + l + m)}]
```

$$\Big\{1,\ 1,\ a-b,\ a-b,\ a-b,\ a+b,\ a+b,\ a+b,$$
$$\frac{1}{2}\left(2a+c+f-\sqrt{4b^2+c^2+4bd+4be+4de-2cf+f^2}\right),$$
$$\frac{1}{2}\left(2a+c+f-\sqrt{4b^2+c^2+4bd+4be+4de-2cf+f^2}\right),$$
$$\frac{1}{2}\left(2a+c+f+\sqrt{4b^2+c^2+4bd+4be+4de-2cf+f^2}\right),$$
$$\frac{1}{2}\left(2a+c+f+\sqrt{4b^2+c^2+4bd+4be+4de-2cf+f^2}\right),$$
$$1-2b-2d-2e-2g-2j-2l-2m,\ 1-2b-2d-2e-2g-2j-2l-2m\Big\}$$

Remark 4.1: *The pair 1 eigenvalues are not* {*1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1*}*, for some reason Mathematica does not issue this trivial Eigenvectors and issues one less 1 eigenvalue, therefore in general there are 3 1-valued eigenvalues.*

Let's calculate the determinant:

```
detStoch = Det[coeff /. {n → 1 - (a + b + c + d + e + f + g + h + i + j + k + l + m)}]
```

$$\left(a^2-b^2\right)\left(a^2-b^2+a\,c-b\,d-b\,e-d\,e+a\,f+c\,f\right)^2 \left(a^2-2\,a^2\,b-b^2+2\,b^3-2\,a^2\,d+2\,b^2\,d-2\,a^2\,e+2\,b^2\,e-2\,a^2\,g+2\,b^2\,g-2\,a^2\,j+2\,b^2\,j-2\,a^2\,l+2\,b^2\,l-2\,a^2\,m+2\,b^2\,m\right)^2$$

Unlike the general case of EQ 7.1, the determinant's Zero for the Stochastic version is only depending on the first 6 elements, and the remaining 8 elements cannot play any role due to the 1-valued sum of the probabilities:

```
Solve[detStoch == 0, a]
```

$$\left\{\{a\to -b\},\{a\to -b\},\{a\to -b\},\{a\to b\},\{a\to b\},\{a\to b\},\left\{a\to \frac{1}{2}\left(-c-f-\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\},\left\{a\to \frac{1}{2}\left(-c-f-\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\},\left\{a\to \frac{1}{2}\left(-c-f+\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\},\left\{a\to \frac{1}{2}\left(-c-f+\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\}\right\}$$

Lemma 4.1: *First 6 coefficients determine if the 14-Sets Stochastic matrix has an inverse. The other 8 coefficients play no role.*

Let's make random positive coefficients between 0 and 1 which sum up to 1:

```
rands = Join[Sort[ RandomReal[{0, 1}, 13], Less], {1}];
cfs = Join[{rands[[1]]}, Table[rands[[w]] - rands[[w - 1]], {w, 2, 14}]]
Total[cfs]
```

```
{0.0329978, 0.283497, 0.00679916, 0.154915, 0.0678861, 0.0377381, 0.0691335,
 0.101474, 0.0728054, 0.0214534, 0.0152676, 0.0462279, 0.0474148, 0.04239}
```

```
1.
```

Compute the matrix representation:

```
Ncoeff = MatrixRep14Sets[cfs];
Style[Ncoeff // MatrixForm, FontSize → 7]
```

0.0329978	0.283497	0	0	0	0	0	0	0	0	0	0	0	0
0.283497	0.0329978	0	0	0	0	0	0	0	0	0	0	0	0
0.00679916	0.154915	0.0397969	0	0.438412	0	0	0	0	0	0	0	0	0
0.154915	0.00679916	0	0.0397969	0	0.438412	0	0	0	0	0	0	0	0
0.0678861	0.0377381	0.351383	0	0.0707359	0	0	0	0	0	0	0	0	0
0.0377381	0.0678861	0	0.351383	0	0.0707359	0	0	0	0	0	0	0	0
0.0691335	0.101474	0.270276	0	0.123541	0	0.156539	0	0.553773	0	0.270276	0	0.123541	0
0.101474	0.0691335	0	0.270276	0	0.123541	0	0.156539	0	0.553773	0	0.270276	0	0.123541
0.0728054	0.0214534	0.152933	0	0.136754	0	0.420251	0	0.185931	0	0.152933	0	0.136754	0
0.0214534	0.0728054	0	0.152933	0	0.136754	0	0.420251	0	0.185931	0	0.152933	0	0.136754
0.0152676	0.0462279	0.116742	0	0.115361	0	0.270276	0	0.123541	0	0.156539	0	0.553773	0
0.0462279	0.0152676	0	0.116742	0	0.115361	0	0.270276	0	0.123541	0	0.156539	0	0.553773
0.0474148	0.04239	0.0688681	0	0.115195	0	0.152933	0	0.136754	0	0.420251	0	0.185931	0
0.04239	0.0474148	0	0.0688681	0	0.115195	0	0.152933	0	0.136754	0	0.420251	0	0.185931

Transpose this matrix to get a Right Stochastic Matrix, since *Mathematica* does not compute left-eigenvalues:

```
transNcoeff = Transpose[Ncoeff];
Style[transNcoeff // MatrixForm, FontSize → 7]
```

0.0329978	0.283497	0.00679916	0.154915	0.0678861	0.0377381	0.0691335	0.101474	0.0728054	0.0214534	0.0152676	0.0462279	0.0474148	0.04239
0.283497	0.0329978	0.154915	0.00679916	0.0377381	0.0678861	0.101474	0.0691335	0.0214534	0.0728054	0.0462279	0.0152676	0.04239	0.0474148
0	0	0.0397969	0	0.351383	0	0.270276	0	0.152933	0	0.116742	0	0.0688681	0
0	0	0	0.0397969	0	0.351383	0	0.270276	0	0.152933	0	0.116742	0	0.0688681
0	0	0.438412	0	0.0707359	0	0.123541	0	0.136754	0	0.115361	0	0.115195	0
0	0	0	0.438412	0	0.0707359	0	0.123541	0	0.136754	0	0.115361	0	0.115195
0	0	0	0	0	0	0.156539	0	0.420251	0	0.270276	0	0.152933	0
0	0	0	0	0	0	0	0.156539	0	0.420251	0	0.270276	0	0.152933
0	0	0	0	0	0	0.553773	0	0.185931	0	0.123541	0	0.136754	0
0	0	0	0	0	0	0	0.553773	0	0.185931	0	0.123541	0	0.136754
0	0	0	0	0	0	0.270276	0	0.152933	0	0.156539	0	0.420251	0
0	0	0	0	0	0	0	0.270276	0	0.152933	0	0.156539	0	0.420251
0	0	0	0	0	0	0.123541	0	0.136754	0	0.553773	0	0.185931	0
0	0	0	0	0	0	0	0.123541	0	0.136754	0	0.553773	0	0.185931

Sum up the rows make sure they add up to 1 and calculate the determinant:

```
sumRow[transNcoeff]
Det[Ncoeff]
```

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

-7.9892×10^{-11}

Remark 4.2: *Due to the nature of combinatorics multiplication of a 14 dimensional determinant, and entries between 0 and 1, the determinants in order of* 10^{-11} *to* 10^{-16} *are not 0, they are non-zero determinants.*

5. Powers of 14-Sets Stochastic Matrices

```
(* use a large power, 16 should do, the Limit did not work in Mathematica *)
infPower = Chop[MatrixPower[transNcoeff, 100]];
Style[infPower // MatrixForm, FontSize → 10]
```

0	0	0	0	0	0	0.121885	0.148933	0.103147	0.126036	0.121885	0.148933	0.103147	0.126036
0	0	0	0	0	0	0.148933	0.121885	0.126036	0.103147	0.148933	0.121885	0.126036	0.103147
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182

Note that the power matrix is also Right Stochastic, however for large powers its determinant approaches 0:

```
sumRow[infPower]
Det[infPower]
```

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

0.

This infinite power matrix is obviously Idempotent:

```
Style[Chop[infPower.infPower] // MatrixForm, FontSize → 10]
```

0	0	0	0	0	0	0.121885	0.148933	0.103147	0.126036	0.121885	0.148933	0.103147	0.126036
0	0	0	0	0	0	0.148933	0.121885	0.126036	0.103147	0.148933	0.121885	0.126036	0.103147
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182
0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182	0
0	0	0	0	0	0	0	0.270818	0	0.229182	0	0.270818	0	0.229182

Let’s calculate the eigenvalues and eigenvectors:

```
Eigenvalues[infPower]
eigenv = Chop[Eigenvectors[infPower]]
```

$\{1., 1., 4.45671\times10^{-17}, 1.30687\times10^{-17}, -8.65344\times10^{-18}, -8.65344\times10^{-18}, 0., 0., 0., 0., 0., 0., 0., 0.\}$

```
{{0.176462, 0.21562, 0.392082, 0, 0.392082,
  0, 0.392082, 0, 0.392082, 0, 0.392082, 0, 0.392082, 0},
 {-0.210174, -0.169634, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512,
  -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859},
 {-0.248588, -0.124294, 0.295199, -0.482485, 0.295199, -0.482485, -0.177999,
  0.280815, 0.228096, -0.331622, -0.0372807, -0.0588287, 0.026294, 0.0693067},
 {0.898695, 0.359478, 0.0680094, 0.0412316, 0.0680094, 0.0412316, 0.0595218,
  0.00170567, 0.115174, 0.0168121, -0.00661464, -0.0352167, -0.177693, 0.0227868},
 {0.552263, 0.297365, -0.346495, 0.252642, -0.346495, 0.252642, 0.25542,
  0.0517482, -0.138013, -0.0162059, -0.0802833, 0.0349794, -0.0689406, -0.0862774},
 {0, 0.0416355, 0.143577, -0.157279, 0.143577, -0.157279, -0.0245231,
  -0.138085, 0.0162562, 0.0488765, 0.073074, 0.0739753, -0.0736272, 0.0268794},
 {1., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 1., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 1., 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1., 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, -0.841016, 0, 0.312353, 0, 0.312353, 0, 0.312353, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Two 1-valued eigenvalues are computer and their corresponding eigenvectors:

```
infPower.eigenv[[1]]
 eigenv[[1]]
```

```
{0.176462, 0.21562, 0.392082, 0., 0.392082, 0.,
 0.392082, 0., 0.392082, 0., 0.392082, 0., 0.392082, 0.}
```

```
{0.176462, 0.21562, 0.392082, 0, 0.392082,
 0, 0.392082, 0, 0.392082, 0, 0.392082, 0, 0.392082, 0}
```

```
infPower.eigenv[[2]]
 eigenv[[2]]
```

```
{-0.210174, -0.169634, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512,
 -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859}
```

```
{-0.210174, -0.169634, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512,
 -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859}
```

In Bra and Ket notations above commutations just showed:

1. Infinite power of a 14-Sets Stochastic matrix is Stochastic
2. 14-Sets Stochastic matrix has all 1 Eigenvectors with 1-valued eigenvalue

infPower $\left| 1_{14} \right>$

```
{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}
```

3. 14-Sets Stochastic matrix has 2 more 1-valued eigenvectors

```
infPower | eigenv[[1]] >
```

```
{0.176462, 0.21562, 0.392082, 0., 0.392082, 0.,
 0.392082, 0., 0.392082, 0., 0.392082, 0., 0.392082, 0.}
```

```
eigenv[[1]]
```

```
{0.176462, 0.21562, 0.392082, 0, 0.392082,
 0, 0.392082, 0, 0.392082, 0, 0.392082, 0, 0.392082, 0}
```

```
infPower | eigenv[[2]] >
```

```
{-0.210174, -0.169634, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512,
 -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859}
```

```
eigenv[[2]]
```

```
{-0.210174, -0.169634, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512,
 -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859, 0.0130512, -0.392859}
```

Summarize in a Theorem, note that the conditions for the coefficients could be relaxed:

Theorem 5.1: *Infinite power of a 14-Sets Stochastic matrix with all positive entries and non-zero first 6 coefficients, has a limit which is Stochastic, with three 1-valued Eigenvalues with one Eigenvector* {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}*. Moreover all 6 coefficients for the infinite power limit are 0. Consequently the determinant of the infinite power limit is 0 (no assumption for the determinant of the original matrix).*

Corollary 5.1*: Given the first 6 non-zero coefficients, the characteristic polynomial of infinite power is* $\lambda^{12}(\lambda - 1)^2$.

Make a random set of coefficients with the first 6 coefficients non-zero and make sure the columns sum up to 1:

```
rands = Join[Sort[ RandomReal[{0, 1}, 13], Less], {1}];
cfs = Join[{rands[[1]]}, Table[rands[[w]] - rands[[w - 1]], {w, 2, 14}]];
Total[cfs]
Ncoeff2 = MatrixRep14Sets[cfs];
Style[Chop[Ncoeff2] // MatrixForm, FontSize → 9]
transNcoeff2 = Transpose[Ncoeff2];
```

```
1.
```

0.0242291	0.0493424	0	0	0	0	0	0	0	0	0	0
0.0493424	0.0242291	0	0	0	0	0	0	0	0	0	0
0.0680045	0.0215485	0.0922335	0	0.0708909	0	0	0	0	0	0	0
0.0215485	0.0680045	0	0.0922335	0	0.0708909	0	0	0	0	0	0
0.0854014	0.0505312	0.134744	0	0.0747603	0	0	0	0	0	0	0
0.0505312	0.0854014	0	0.134744	0	0.0747603	0	0	0	0	0	0
0.00696923	0.111031	0.0361729	0	0.248439	0	0.272668	0	0.0855153	0	0.0361729	0
0.111031	0.00696923	0	0.0361729	0	0.248439	0	0.272668	0	0.0855153	0	0.036
0.179087	0.118048	0.370594	0	0.271223	0	0.320565	0	0.394823	0	0.370594	0
0.118048	0.179087	0	0.370594	0	0.271223	0	0.320565	0	0.394823	0	0.37
0.0694041	0.00765518	0.180435	0	0.0146244	0	0.0361729	0	0.248439	0	0.272668	0
0.00765518	0.0694041	0	0.180435	0	0.0146244	0	0.0361729	0	0.248439	0	0.27
0.0677732	0.140975	0.185821	0	0.320062	0	0.370594	0	0.271223	0	0.320565	0
0.140975	0.0677732	0	0.185821	0	0.320062	0	0.370594	0	0.271223	0	0.32

Compute the infinite power:

```
infPower2 = Chop[MatrixPower[transNcoeff2, 200]];
```

```
Style[Chop[Transpose[infPower2]] // MatrixForm, FontSize → 9]
```

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0.0835766	0.0793101	0.162887	0	0.162887	0	0.162887	0	0.162887	0	0.162887	0	0.16
0.0793101	0.0835766	0	0.162887	0	0.162887	0	0.162887	0	0.162887	0	0.162887	0
0.172972	0.164142	0.337113	0	0.337113	0	0.337113	0	0.337113	0	0.337113	0	0.33
0.164142	0.172972	0	0.337113	0	0.337113	0	0.337113	0	0.337113	0	0.337113	0
0.0835766	0.0793101	0.162887	0	0.162887	0	0.162887	0	0.162887	0	0.162887	0	0.16
0.0793101	0.0835766	0	0.162887	0	0.162887	0	0.162887	0	0.162887	0	0.162887	0
0.172972	0.164142	0.337113	0	0.337113	0	0.337113	0	0.337113	0	0.337113	0	0.33
0.164142	0.172972	0	0.337113	0	0.337113	0	0.337113	0	0.337113	0	0.337113	0

Compute the characteristic polynomial:

```
Simplify[Chop[CharacteristicPolynomial[infPower2, λ]]]
```

$\lambda^{12}\left(1.-2.\,\lambda+\lambda^{2}\right)$

6. Roots of +1 and -1

Lets reach an element of 14-Sets algebra to power 2:

v1 = a σ_0 + b σ_1 + c σ_2 + d σ_3 + e σ_4 + f σ_5 + g σ_6 + h σ_7 + i σ_8 + j σ_9 + k σ_{10} + l σ_{11} + m σ_{12} + n σ_{13};
v1power = v1 ⊙ v1

$$\left(a^2+b^2\right)\sigma_0+2\,a\,b\,\sigma_1+\left(2\,a\,c+c^2+b\,d+b\,e+d\,e\right)\sigma_2+\left(b\,(c+f)+d\,(2\,a+c+f)\right)\sigma_3+\left(b\,(c+f)+e\,(2\,a+c+f)\right)\sigma_4+\left(d\,e+b\,(d+e)+f\,(2\,a+f)\right)\sigma_5+\left(2\,a\,g+b\,h+e\,h+g\,h+b\,i+d\,i+g\,i+d\,k+e\,k+2\,g\,k+i\,l+k\,l+h\,m+k\,m+c\,(d+e+2\,g+l+m)\right)\sigma_6+\left(d^2+b\,g+2\,a\,h+f\,h+h^2+b\,j+g\,j+f\,k+h\,k+g\,l+j\,l+l^2+d\,(g+j+2\,l)+h\,n+k\,n+c\,(f+h+n)\right)\sigma_7+\left(e^2+b\,g+2\,a\,i+f\,i+i^2+b\,j+g\,j+f\,k+i\,k+g\,m+j\,m+m^2+e\,(g+j+2\,m)+i\,n+k\,n+c\,(f+i+n)\right)\sigma_8+\left(b\,h+b\,i+2\,a\,j+2\,f\,j+h\,j+i\,j+f\,l+i\,l+f\,m+h\,m+2\,j\,n+l\,n+m\,n+e\,(f+h+n)+d\,(f+i+n)\right)\sigma_9+\left(g^2+c\,h+c\,i+h\,i+2\,a\,k+2\,c\,k+h\,k+i\,k+k^2+b\,l+g\,l+e\,(g+l)+b\,m+g\,m+l\,m+d\,(g+m)\right)\sigma_{10}+\left(g\,h+c\,j+h\,j+b\,k+j\,k+2\,a\,l+c\,l+2\,h\,l+k\,l+f\,(g+l)+b\,n+g\,n+l\,n+d\,(2\,h+k+n)\right)\sigma_{11}+\left(g\,i+c\,j+i\,j+b\,k+j\,k+2\,a\,m+c\,m+2\,i\,m+k\,m+f\,(g+m)+b\,n+g\,n+m\,n+e\,(2\,i+k+n)\right)\sigma_{12}+\left(d\,j+e\,j+j^2+b\,l+e\,l+j\,l+b\,m+d\,m+j\,m+l\,m+2\,a\,n+i\,n+n^2+h\,(i+n)+f\,(h+i+2\,n)\right)\sigma_{13}$$

Because of the first term $\left(a^2+b^2\right)$ there are only Complex solutions for -1 roots, therefore no Real roots.

So far the author has found 7 roots of +1:

roots1 =
Solve [(Coefficient[v1power, {σ_0, σ_1, σ_2, σ_3, σ_4, σ_5, σ_6, σ_7, σ_8, σ_9, σ_{10}, σ_{11}, σ_{12}, σ_{13}}] /.
{d → 0, e → 0, g → 0, h → 0, i → 0, j → 0, k → 0, l → 0, m → 0, n → 0}) ==
{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {a, b, c, f}]

{{a → -1, b → 0, c → 0, f → 2}, {a → -1, b → 0, c → 2, f → 0},
{a → 0, b → -1, c → 0, f → 0}, {a → 0, b → 1, c → 0, f → 0}, {a → 1, b → 0, c → -2, f → 0},
{a → 1, b → 0, c → 0, f → -2}, {a → -1, b → 0, c → 0, f → 0}, {a → 1, b → 0, c → 0, f → 0}}

(σ_1) ⊙ (σ_1)

σ_0

($-\sigma_0$ + 2 * σ_2) ⊙ ($-\sigma_0$ + 2 * σ_2)

σ_0

(σ_0 - 2 * σ_5) ⊙ (σ_0 - 2 * σ_5)

σ_0

(σ_0 - 2 * σ_7) ⊙ (σ_0 - 2 * σ_7)

σ_0

```
(σ0 - 2 * σ8) ⊙ (σ0 - 2 * σ8)
```

σ_0

```
(σ0 - 2 * σ10) ⊙ (σ0 - 2 * σ10)
```

σ_0

```
(σ0 - 2 * σ13) ⊙ (σ0 - 2 * σ13)
```

σ_0

7. Inverse Elements

EQ 3.2 renders a 14x14 matrix representation for v1 as left-hand multiplicand in vector product v1v2 . In order for this product to allow for inverses the determinant of the representing matrix must be non-zero. Following *Mathematica* code calculates the hairy determinant formula and its zeros:

```
det = Det[coeff]
```

```
(a^2 - b^2) (a^2 - b^2 + a c - b d - b e - d e + a f + c f)^2
 ((e + j + m) (- (a + c + h + k) (-b c + a e - b h + a j - b k + a m) +
       (d + g + l) ((d + g + l) (e + j + m) - (c + h + k) (f + i + n)) +
       (b + d + g + l) (- (d + g + l) (b + e + j + m) + (a + c + h + k) (f + i + n))) -
    (f + i + n) ((b + d + g + l) (-a d + b f - a g + b i - a l + b n) +
       (c + h + k) ((d + g + l) (e + j + m) - (c + h + k) (f + i + n)) -
       (a + c + h + k) ((b + d + g + l) (e + j + m) - (c + h + k) (a + f + i + n))) +
    (a + f + i + n) ((d + g + l) (-a d + b f - a g + b i - a l + b n) -
       (c + h + k) (- (d + g + l) (b + e + j + m) + (a + c + h + k) (f + i + n)) +
       (a + c + h + k) (- (b + d + g + l) (b + e + j + m) + (a + c + h + k) (a + f + i + n))) -
    (b + e + j + m) (- (c + h + k) (-b c + a e - b h + a j - b k + a m) +
       (d + g + l) ((b + d + g + l) (e + j + m) - (c + h + k) (a + f + i + n)) +
       (b + d + g + l) (- (b + d + g + l) (b + e + j + m) + (a + c + h + k) (a + f + i + n))))^2
```

General criteria for zeros:

```
(* Mathematica Warning: Solve::svars:
  "Equations may not give solutions for all \"solve\" variables. "*)
```

```
Solve[det == 0, {a, b, c, d, e, f, g, h, i, j, k, l, n}]
```

$$\left\{\{b\to -a\},\ \{b\to a\},\ \left\{f\to \frac{-a^2+b^2-a\,c+b\,d+b\,e+d\,e}{a+c}\right\},\ \{n\to -a-b-c-d-e-f-g-h-i-j-k-l-m\},\ \{n\to -a+b-c+d+e-f+g-h-i+j-k+l+m\},\ \{c\to -a,\ d\to -b\},\ \{c\to -a,\ e\to -b\}\right\}$$

Solve for variable ‘a’ in terms of the others :

```
Solve[det == 0, a]
```

$$\Big\{\{a\to -b\},\ \{a\to -b\},\ \{a\to -b\},\ \{a\to b\},\ \{a\to b\},\ \{a\to b\},$$
$$\left\{a\to \tfrac{1}{2}\left(-c-f-\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\},$$
$$\left\{a\to \tfrac{1}{2}\left(-c-f-\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\},$$
$$\left\{a\to \tfrac{1}{2}\left(-c-f+\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\},$$
$$\left\{a\to \tfrac{1}{2}\left(-c-f+\sqrt{4\,b^2+c^2+4\,b\,d+4\,b\,e+4\,d\,e-2\,c\,f+f^2}\right)\right\},$$
$$\{a\to -b-c-d-e-f-g-h-i-j-k-l-m-n\},$$
$$\{a\to -b-c-d-e-f-g-h-i-j-k-l-m-n\},$$
$$\{a\to b-c+d+e-f+g-h-i+j-k+l+m-n\},\ \{a\to b-c+d+e-f+g-h-i+j-k+l+m-n\}\Big\}$$

EQ 7.1

$\{a\to -b\}$ means if $a = -b$ then the determinant has zero solution. Therefore EQ 7.1 issues exact criteria for when the determinant is 0.

Let’s repeat the same calculations for v2 in product v1v2 :

```
CoefficientArrays[
 {(a a2 + b b2) == 0 , (a2 b + a b2) == 0, (a2 c + a c2 + c c2 + b2 d + b e2 + d e2) == 0,
  (b2 c + a2 d + a d2 + c d2 + b f2 + d f2) == 0, (b c2 + a2 e + c2 e + a e2 + b2 f + e2 f) == 0 ,
  (b d2 + b2 e + d2 e + a2 f + a f2 + f f2) == 0 , (c2 d + c e2 + a2 g + c2 g + a g2 + c g2 + b2 h + e2 h +
     g2 h + b i2 + d i2 + g i2 + e2 k + g2 k + d k2 + g k2 + c2 l + i2 l + k2 l + c m2 + h m2 + k m2) == 0,
  (d d2 + c f2 + b2 g + d2 g + a2 h + f2 h + a h2 + c h2 + h h2 + b j2 + d j2 + g j2 +
     f2 k + h2 k + d2 l + j2 l + d l2 + g l2 + l l2 + c n2 + h n2 + k n2) == 0,
  (e e2 + c2 f + b g2 + e g2 + a2 i + c2 i + a i2 + f i2 + i i2 + b2 j + e2 j + g2 j +
     f k2 + i k2 + e2 m + g2 m + e m2 + j m2 + m m2 + c2 n + i2 n + k2 n) == 0,
  (d2 f + e f2 + b h2 + e h2 + b2 i + d2 i + a2 j + f2 j + h2 j + a j2 + f j2 + i j2 +
     f l2 + i l2 + f2 m + h2 m + d2 n + j2 n + l2 n + e n2 + j n2 + m n2) == 0 ,
  (e2 g + d g2 + g g2 + c2 h + c i2 + h i2 + a2 k + c2 k + i2 k + a k2 + c k2 + h k2 +
     k k2 + b2 l + e2 l + g2 l + b m2 + d m2 + g m2 + l m2) == 0,
  (f2 g + d2 h + d h2 + g h2 + c j2 + h j2 + b2 k + d2 k + j2 k + a2 l + f2 l + h2 l +
     a l2 + c l2 + h l2 + k l2 + b n2 + d n2 + g n2 + l n2) == 0 ,
  (f g2 + e2 i + g2 i + e i2 + c2 j + i2 j + b k2 + e k2 + j k2 + a2 m + c2 m + i2 m +
     k2 m + a m2 + f m2 + i m2 + b2 n + e2 n + g2 n + m2 n) == 0,
  (f h2 + f2 i + h2 i + d2 j + e j2 + j j2 + b l2 + e l2 + j l2 + b2 m + d2 m + j2 m + l2 m + a2 n +
     f2 n + h2 n + a n2 + f n2 + i n2 + n n2) == 0}, {a, b, c, d, e, f, g, h, i, j, k, l, m, n}]
{SparseArray[<0>, {14}], SparseArray[<84>, {14, 14}]}
```

```
coeffRIGHT = Normal[%][[2]];
coeffRIGHT // MatrixForm
```

a2	b2	0	0	0	0	0	0	0	0	0	0	0	0
b2	a2	0	0	0	0	0	0	0	0	0	0	0	0
c2	e2	a2 + c2	b2 + e2	0	0	0	0	0	0	0	0	0	0
d2	f2	b2 + d2	a2 + f2	0	0	0	0	0	0	0	0	0	0
e2	c2	0	0	a2 + c2	b2 + e2	0	0	0	0	0	0	0	0
f2	d2	0	0	b2 + d2	a2 + f2	0	0	0	0	0	0	0	0
g2	i2	e2 + g2 + m2	c2 + i2 + k2	0	0	a2 + c2 + i2 + k2	b2 + e2 + g2 + m2	0	0	e2 + g2 + m2	c2 + i2 + k2	0	0
h2	j2	f2 + h2 + n2	d2 + j2 + l2	0	0	b2 + d2 + j2 + l2	a2 + f2 + h2 + n2	0	0	f2 + h2 + n2	d2 + j2 + l2	0	0
i2	g2	0	0	e2 + g2 + m2	c2 + i2 + k2	0	0	a2 + c2 + i2 + k2	b2 + e2 + g2 + m2	0	0	e2 + g2 + m2	c2 + i2 + k2
j2	h2	0	0	f2 + h2 + n2	d2 + j2 + l2	0	0	b2 + d2 + j2 + l2	a2 + f2 + h2 + n2	0	0	f2 + h2 + n2	d2 + j2 + l2
k2	m2	i2 + k2	g2 + m2	0	0	e2 + g2 + m2	c2 + i2 + k2	0	0	a2 + c2 + i2 + k2	b2 + e2 + g2 + m2	0	0
l2	n2	j2 + l2	h2 + n2	0	0	f2 + h2 + n2	d2 + j2 + l2	0	0	b2 + d2 + j2 + l2	a2 + f2 + h2 + n2	0	0
m2	k2	0	0	i2 + k2	g2 + m2	0	0	e2 + g2 + m2	c2 + i2 + k2	0	0	a2 + c2 + i2 + k2	b2 + e2 + g2 + m2
n2	l2	0	0	j2 + l2	h2 + n2	0	0	f2 + h2 + n2	d2 + j2 + l2	0	0	b2 + d2 + j2 + l2	a2 + f2 + h2 + n2

In summary:

$$\begin{pmatrix}
a2 & b2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b2 & a2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c2 & e2 & a2+c2 & b2+e2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d2 & f2 & b2+d2 & a2+f2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e2 & c2 & 0 & 0 & a2+c2 & b2+e2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f2 & d2 & 0 & 0 & b2+d2 & a2+f2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g2 & i2 & e2+g2+m2 & c2+i2+k2 & 0 & 0 & a2+c2+i2+k2 & b2+e2+g2+m2 & 0 & 0 & e2+g2+m2 & c2+i2+k2 & 0 & 0 \\
h2 & j2 & f2+h2+n2 & d2+j2+l2 & 0 & 0 & b2+d2+j2+l2 & a2+f2+h2+n2 & 0 & 0 & f2+h2+n2 & d2+j2+l2 & 0 & 0 \\
i2 & g2 & 0 & 0 & e2+g2+m2 & c2+i2+k2 & 0 & 0 & a2+c2+i2+k2 & b2+e2+g2+m2 & 0 & 0 & e2+g2+m2 & c2+i2+k2 \\
j2 & h2 & 0 & 0 & f2+h2+n2 & d2+j2+l2 & 0 & 0 & b2+d2+j2+l2 & a2+f2+h2+n2 & 0 & 0 & f2+h2+n2 & d2+j2+l2 \\
k2 & m2 & i2+k2 & g2+m2 & 0 & 0 & e2+g2+m2 & c2+i2+k2 & 0 & 0 & a2+c2+i2+k2 & b2+e2+g2+m2 & 0 & 0 \\
l2 & n2 & j2+l2 & h2+n2 & 0 & 0 & f2+h2+n2 & d2+j2+l2 & 0 & 0 & b2+d2+j2+l2 & a2+f2+h2+n2 & 0 & 0 \\
m2 & k2 & 0 & 0 & i2+k2 & g2+m2 & 0 & 0 & e2+g2+m2 & c2+i2+k2 & 0 & 0 & a2+c2+i2+k2 & b2+e2+g2+m2 \\
n2 & l2 & 0 & 0 & j2+l2 & h2+n2 & 0 & 0 & f2+h2+n2 & d2+j2+l2 & 0 & 0 & b2+d2+j2+l2 & a2+f2+h2+n2
\end{pmatrix}
\begin{pmatrix} a1 \\ b1 \\ c1 \\ d1 \\ e1 \\ f1 \\ g1 \\ h1 \\ i1 \\ j1 \\ k1 \\ l1 \\ m1 \\ n1 \end{pmatrix} \sigma$$

EQ 7.2

```
detRIGHT = Det[coeffRIGHT]
```

$$\left(a2^2 - b2^2\right) \left(a2^2 - b2^2 + a2\, c2 - b2\, d2 - b2\, e2 - d2\, e2 + a2\, f2 + c2\, f2\right)^2$$
$$((d2 + j2 + l2)\ (-(a2 + c2 + i2 + k2)\ (-b2\, c2 + a2\, d2 - b2\, i2 + a2\, j2 - b2\, k2 + a2\, l2) +$$
$$(e2 + g2 + m2)\ ((d2 + j2 + l2)\ (e2 + g2 + m2) - (c2 + i2 + k2)\ (f2 + h2 + n2)) + (b2 + e2 +$$
$$g2 + m2)\ (-(b2 + d2 + j2 + l2)\ (e2 + g2 + m2) + (a2 + c2 + i2 + k2)\ (f2 + h2 + n2))) -$$
$$(f2 + h2 + n2)\ ((b2 + e2 + g2 + m2)\ (-a2\, e2 + b2\, f2 - a2\, g2 + b2\, h2 - a2\, m2 + b2\, n2) +$$
$$(c2 + i2 + k2)\ ((d2 + j2 + l2)\ (e2 + g2 + m2) - (c2 + i2 + k2)\ (f2 + h2 + n2)) - (a2 + c2 +$$
$$i2 + k2)\ ((d2 + j2 + l2)\ (b2 + e2 + g2 + m2) - (c2 + i2 + k2)\ (a2 + f2 + h2 + n2))) +$$
$$(a2 + f2 + h2 + n2)\ ((e2 + g2 + m2)\ (-a2\, e2 + b2\, f2 - a2\, g2 + b2\, h2 - a2\, m2 + b2\, n2) -$$
$$(c2 + i2 + k2)\ (-(b2 + d2 + j2 + l2)\ (e2 + g2 + m2) + (a2 + c2 + i2 + k2)\ (f2 + h2 + n2)) +$$
$$(a2 + c2 + i2 + k2)$$
$$(-(b2 + d2 + j2 + l2)\ (b2 + e2 + g2 + m2) + (a2 + c2 + i2 + k2)\ (a2 + f2 + h2 + n2))) -$$
$$(b2 + d2 + j2 + l2)\ (-(c2 + i2 + k2)\ (-b2\, c2 + a2\, d2 - b2\, i2 + a2\, j2 - b2\, k2 + a2\, l2) +$$
$$(e2 + g2 + m2)\ ((d2 + j2 + l2)\ (b2 + e2 + g2 + m2) - (c2 + i2 + k2)\ (a2 + f2 + h2 + n2)) +$$
$$(b2 + e2 + g2 + m2)$$
$$(-(b2 + d2 + j2 + l2)\ (b2 + e2 + g2 + m2) + (a2 + c2 + i2 + k2)\ (a2 + f2 + h2 + n2))))^2$$

```
Solve[detRIGHT == 0, {a2, b2, c2, d2, e2, f2, g2, h2, i2, j2, k2, l2, n2}]
```

$$\left\{\{b2 \to -a2\},\ \{b2 \to a2\},\ \left\{f2 \to \frac{-a2^2 + b2^2 - a2\, c2 + b2\, d2 + b2\, e2 + d2\, e2}{a2 + c2}\right\},\right.$$
$$\{n2 \to -a2 - b2 - c2 - d2 - e2 - f2 - g2 - h2 - i2 - j2 - k2 - l2 - m2\},$$
$$\{n2 \to -a2 + b2 - c2 + d2 + e2 - f2 + g2 - h2 - i2 + j2 - k2 + l2 + m2\},$$
$$\left.\{c2 \to -a2,\ d2 \to -b2\},\ \{c2 \to -a2,\ e2 \to -b2\}\right\}$$

```
Solve[detRIGHT == 0, a2]
```

$$\left\{\{a2 \to -b2\},\ \{a2 \to -b2\},\ \{a2 \to -b2\},\ \{a2 \to b2\},\ \{a2 \to b2\},\ \{a2 \to b2\},\right.$$
$$\left\{a2 \to \frac{1}{2}\left(-c2 - f2 - \sqrt{\left(4\, b2^2 + c2^2 + 4\, b2\, d2 + 4\, b2\, e2 + 4\, d2\, e2 - 2\, c2\, f2 + f2^2\right)}\right)\right\},$$
$$\left\{a2 \to \frac{1}{2}\left(-c2 - f2 - \sqrt{\left(4\, b2^2 + c2^2 + 4\, b2\, d2 + 4\, b2\, e2 + 4\, d2\, e2 - 2\, c2\, f2 + f2^2\right)}\right)\right\},$$
$$\left\{a2 \to \frac{1}{2}\left(-c2 - f2 + \sqrt{\left(4\, b2^2 + c2^2 + 4\, b2\, d2 + 4\, b2\, e2 + 4\, d2\, e2 - 2\, c2\, f2 + f2^2\right)}\right)\right\},$$
$$\left\{a2 \to \frac{1}{2}\left(-c2 - f2 + \sqrt{\left(4\, b2^2 + c2^2 + 4\, b2\, d2 + 4\, b2\, e2 + 4\, d2\, e2 - 2\, c2\, f2 + f2^2\right)}\right)\right\},$$
$$\{a2 \to -b2 - c2 - d2 - e2 - f2 - g2 - h2 - i2 - j2 - k2 - l2 - m2 - n2\},$$
$$\{a2 \to -b2 - c2 - d2 - e2 - f2 - g2 - h2 - i2 - j2 - k2 - l2 - m2 - n2\},$$
$$\{a2 \to b2 - c2 + d2 + e2 - f2 + g2 - h2 - i2 + j2 - k2 + l2 + m2 - n2\},$$
$$\left.\{a2 \to b2 - c2 + d2 + e2 - f2 + g2 - h2 - i2 + j2 - k2 + l2 + m2 - n2\}\right\}$$

EQ 7.3

From EQ 7.1 and 7.2 we can conclude:

Lemma 7.1: *The conditions allowing for the inverse element, either from left or right is the same.*

1 in this algebra has representation of $\mathbb{I}_{14\times14}$ or the identity matrix:

```
one = coeff /. {a -> 1, b → 0, c → 0, d → 0, e → 0,
    f → 0, g → 0, h → 0, i → 0, j → 0, k → 0, l → 0, m → 0, n → 0};
one // MatrixForm
```

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The products $v1v1^{-1}$ and $v1^{-1}v1$ must give unique inverses for both right and left since the determinate is non-zero (EQ 7.1, 7.3), therefore :

Lemma 7.2: *The left and right inverses are identical (assuming EQ 7.1, 7.3).*

In general violation of EQ 7.1, 7.3 could cause no inverse or non-unique inverse.

And since the matrix multiplication of these matrix representations are associative therefore this algebra is also associative, let’s call this algebra 14-Sets:

Theorem 7.1: 14-Sets algebra is associative, given constraints of EQ 7.1, 7.3 then this algebra forms a group.

In general 14-Sets algebra is not commutative.

8. Random Coefficients Theorem

Theorem 8.1: Elements of 14-Sets algebra with (almost-incompressible) almost-random coefficients have inverse.

Proof: We model the concept of randomness by Kolmogorov complexity (Appendix A). By Theorem 1 and constraints of EQ 7.1, 7.3 an irreversible element has to have a pattern to its coefficients i.e. constraints EQ 7.1, 7.3 and therefore could not be as random as possible since its Kolmogorov complexity will be considerably less than the length of strings of its coefficients, recall Theorem A.2 and $C(x) \geqslant x$ if x (concatenated coefficients) were incompressible . Therefore all random or incompressible vectors have inverse.

Remark 8.1 : *What this theorem indicates is that the randomness in the universe is the backbone for all the algebras we understand and compute with! It is the reverse of what we have learned at school and against our contemporary understanding.*

Keep running the few lines below and you see the random matrix representations always have non-zero determinant:

```
v = MatrixRep14Sets[RandomReal[{5, -5}, 14]];
Style[v // MatrixForm, FontSize → 7]
Det[v]
```

-3.30809	-3.4264	0	0	0	0	0	0	0	0	0	0	0	0
-3.4264	-3.30809	0	0	0	0	0	0	0	0	0	0	0	0
-0.144406	4.31252	-3.45249	0	0.886118	0	0	0	0	0	0	0	0	0
4.31252	-0.144406	0	-3.45249	0	0.886118	0	0	0	0	0	0	0	0
-1.53178	-0.557539	-4.95818	0	-3.86563	0	0	0	0	0	0	0	0	0
-0.557539	-1.53178	0	-4.95818	0	-3.86563	0	0	0	0	0	0	0	0
-2.44593	1.39232	1.9833	0	3.42784	0	0.119753	0	-1.4431	0	1.9833	0	3.42784	0
1.39232	-2.44593	0	1.9833	0	3.42784	0	0.119753	0	-1.4431	0	1.9833	0	3.42784
2.26295	4.76527	-1.32898	0	0.753287	0	-2.67312	0	-4.63707	0	-1.32898	0	0.753287	0
4.76527	2.26295	0	-1.32898	0	0.753287	0	-2.67312	0	-4.63707	0	-1.32898	0	0.753287
2.17993	0.116716	3.57225	0	-2.32922	0	1.9833	0	3.42784	0	0.119753	0	-1.4431	0
0.116716	2.17993	0	3.57225	0	-2.32922	0	1.9833	0	3.42784	0	0.119753	0	-1.4431
-2.4802	-3.03439	2.28507	0	-0.771444	0	-1.32898	0	0.753287	0	-2.67312	0	-4.63707	0
-3.03439	-2.4802	0	2.28507	0	-0.771444	0	-1.32898	0	0.753287	0	-2.67312	0	-4.63707

-154.889

9. Meaning of Coefficients

Probabilities

One of the most natural interpretations of the 14-Sets Stochastic matrix representation is of course that of probabilities of states summing up to 1. This is a well studied and understood interpretation: *i* , *j* entries contains the probability of going from the state *j* to state *i*. In case of the 14-Sets matrix representations, one possible related interpretation: $m_{i+1,\, j+1}$ = probability or frequency of a given Kuratowski operator σ_j to switch to another σ_i, we show the latter by $\sigma_j \longrightarrow \sigma_i$. In general let's use the notations:

$$\sigma_i \underset{n}{\overset{p}{\longrightarrow}} \sigma_j$$

where P is the probability of going/switching from σ_i to σ_j after n steps, ≠ in place of P means non-zero probability.

Example 9.1: Probabilities along the columns add to 1:

```
Style[Ncoeff // MatrixForm, FontSize → 7]
```

0.0329978	0.283497	0	0	0	0	0	0	0	0	0	0	0	0
0.283497	0.0329978	0	0	0	0	0	0	0	0	0	0	0	0
0.00679916	0.154915	0.0397969	0	0.438412	0	0	0	0	0	0	0	0	0
0.154915	0.00679916	0	0.0397969	0	0.438412	0	0	0	0	0	0	0	0
0.0678861	0.0377381	0.351383	0	0.0707359	0	0	0	0	0	0	0	0	0
0.0377381	0.0678861	0	0.351383	0	0.0707359	0	0	0	0	0	0	0	0
0.0691335	0.101474	0.270276	0	0.123541	0	0.156539	0	0.553773	0	0.270276	0	0.123541	0
0.101474	0.0691335	0	0.270276	0	0.123541	0	0.156539	0	0.553773	0	0.270276	0	0.123541
0.0728054	0.0214534	0.152933	0	0.136754	0	0.420251	0	0.185931	0	0.152933	0	0.136754	0
0.0214534	0.0728054	0	0.152933	0	0.136754	0	0.420251	0	0.185931	0	0.152933	0	0.136754
0.0152676	0.0462279	0.116742	0	0.115361	0	0.270276	0	0.123541	0	0.156539	0	0.553773	0
0.0462279	0.0152676	0	0.116742	0	0.115361	0	0.270276	0	0.123541	0	0.156539	0	0.553773
0.0474148	0.04239	0.0688681	0	0.115195	0	0.152933	0	0.136754	0	0.420251	0	0.185931	0
0.04239	0.0474148	0	0.0688681	0	0.115195	0	0.152933	0	0.136754	0	0.420251	0	0.185931

Therefore the following computation explain such a probabilistic Subjective Form:

```
< 1_14 | Ncoeff
```

```
{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}
```

Imagine a machine which has large array of sensors sensing the space around an area (object). Each sensor can sense the Kuratowski's operators in action:

σ_0 sensor senses the individual points in space or on a surface
σ_1 sensor senses its location has no points from a specified area
σ_2 sensor senses the boundary of an object e.g. edge of a table, skin of an apple
σ_5 sensor senses an interior of some object
...

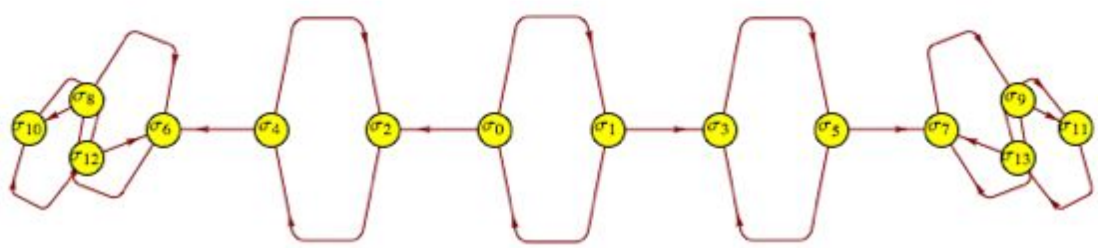
The sensor is in an infinite-loop, continually sensing the space and objects, but each time triggered it might sense the action of a different Kuratowski's operator e.g. it could be sensing σ_0 this moment and next moment σ_5. This could be due to the motion in space or objects changing shapes or the space itself changing configuration.

Let's code a function that does the Adjacency Graph of a Stochastic 14-Sets vector:

```
(* Self-Loops are removed *)
graph14Sets [cfs0_, scale0_] :=
 Module[{cfs = cfs0, matREP, matREP2, matREP3, matREP4, g, gPower, scale = scale0 },

  matREP = MatrixRep14Sets[cfs];
  matREP2 = Table[Boole[matREP[[u]][[w]] ≠ 0], {u, 1, 14}, {w, 1, 14}];

  g = GraphPlot[Transpose[matREP2], DirectedEdges → {True, "ArrowheadsSize" → 0.01},
    MultiedgeStyle → 2, SelfLoopStyle → False, VertexRenderingFunction →
     ({EdgeForm[Black], Yellow, Disk[#1, scale * 0.18], Black, Text[σ(#2-1), #1]} &)];

  matREP3 = Chop[N[MatrixPower[matREP, 200]], 10^-9];
  matREP4 = Table[Boole[matREP3[[u]][[w]] ≠ 0], {u, 1, 14}, {w, 1, 14}];

  gPower = GraphPlot[Transpose[matREP4],
    DirectedEdges → {True, "ArrowheadsSize" → 0.01}, MultiedgeStyle → 1,
    SelfLoopStyle → False, VertexRenderingFunction → ({EdgeForm[Black],
        Yellow, Disk[#1, scale * 0.15 / 2], Black, Text[σ(#2-1), #1]} &)];

  (* matREP = matrix representation, matREP2 the boolean,
  matREP3 infinite power, matREP4 boolean of the infinite power,
  g graph of the first stochastic matrix,
  gPower the graph of the infinite power stochastic matrix *)
  {matREP, matREP2, matREP3, matREP4, g, gPower}
 ]
```

Example 9.2: Start with a Stochastic 14-Sets vector $v = \left(\frac{1}{3}\right)\sigma_0 + \left(\frac{1}{3}\right)\sigma_1 + \left(\frac{1}{3}\right)\sigma_2$, the following is the Adjacency Graph of this vector:

```
(* recall self-loops are removed *)
res = graph14Sets [{1 / 3, 1 / 3, 1 / 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 1];
res[[5]]
```



Look at its matrix representation (call it m) below and by graph above you can see there is $m_{3,1} = \frac{1}{3}$ probability to go from σ_0 to σ_2 , $m_{4,4} = \frac{2}{3}$ probability to go from σ_3 to σ_3 i.e. stays in σ_3 state or next trigger of the sensor will be again the action of σ_3, there are no arrow from σ_6 to σ_4 i.e. probability $m_{5,7} = 0$, or if in state σ_6 no way to go to σ_4 or if the sensor is sensing the action of σ_6 next trigger cannot be σ_4, while there is an arrow from σ_4 to σ_6 i.e. probability $m_{7,5} = \frac{1}{3}$:

```
Style[res[[1]] // MatrixForm, FontSize → 7]
```

$$\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3}
\end{pmatrix}$$

For v to stay as v i.e. sensors triggered for the second time yet again the sensors sensed $\left(\frac{1}{3}\right)$ frequency for $\sigma_0 \longrightarrow \sigma_0$ and $\left(\frac{1}{3}\right)$ frequency for $\sigma_0 \longrightarrow \sigma_1$ and $\left(\frac{1}{3}\right)$ frequency for $\sigma_0 \longrightarrow \sigma_2$, the corresponding vector to the last statement is v^2 (matrix power), and for next set of triggers to get the same frequencies corresponding 14-Sets vector is $v^3$and so on.

Remark 9.1: $(v^n)^T = \left(v^T\right)^n$ *, therefore we need not to transpose v.*

Therefore to have the same frequencies for the Kuratowski's operators, the corresponding vector (after infinite amount of triggers):

$$v^\infty = \lim_{n \longrightarrow \infty} v^n \qquad \text{EQ 9.1}$$

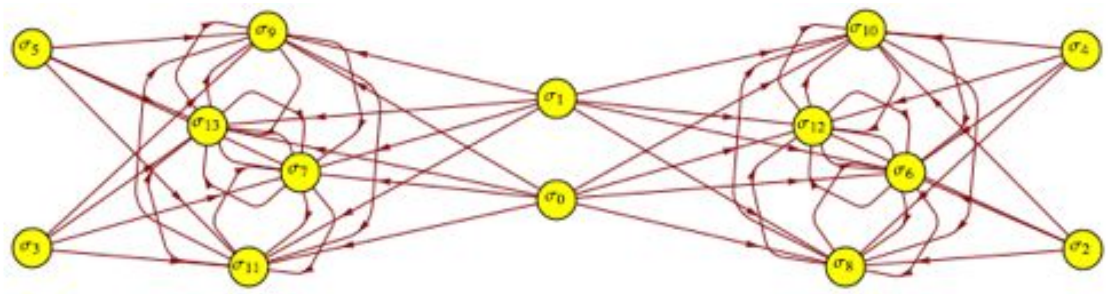
By v^∞ it is meant the stable state of such probabilities staying fixed:

$$v^\infty . v = v^\infty . v^\infty = v^\infty \qquad \text{EQ 9.2}$$

where $(v^\infty)_{i+1,\, j+1}$ = probability to go from σ_j to σ_i in infinite steps .

The following is the Adjacency Graph for v^∞ for the above v:

```
res[[6]]
```



```
Style[res[[3]] // MatrixForm, FontSize → 7]
```

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.222222 & 0.111111 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 \\
0.111111 & 0.222222 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 \\
0.111111 & 0.0555556 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 \\
0.0555556 & 0.111111 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 \\
0.222222 & 0.111111 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 \\
0.111111 & 0.222222 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 & 0 & 0.333333 \\
0.111111 & 0.0555556 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 \\
0.0555556 & 0.111111 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667 & 0 & 0.166667
\end{pmatrix}$$

This graph shows us:

$\sigma_0 \xrightarrow[\infty]{0.0555556} \sigma_9$ while $\sigma_9 \xrightarrow[\infty]{0} \sigma_0$ as an example by in general:

$\sigma_i \xrightarrow[\infty]{\neq 0} \sigma_j$, $i = 0$ to 2, $j = 7$ to 13

while

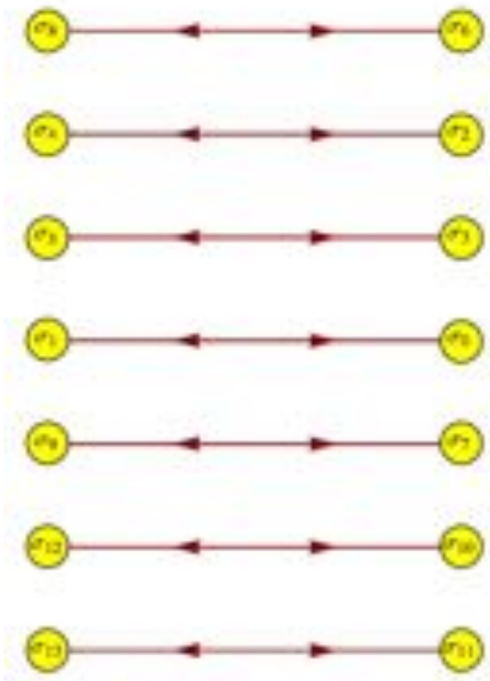
$\sigma_i \xrightarrow[\infty]{0} \sigma_j$, $i = 0$ to 13, $j = 0$ to 5

In other words the probabilities from σ_0 , σ_1 to σ_5 , after infinite steps, got 'drained' and re-distributed amongst σ_6 to σ_{13} .

$$\lim_{n \longrightarrow \infty} \left(\left(\frac{1}{3}\right)\sigma_0 + \left(\frac{1}{3}\right)\sigma_1 + \left(\frac{1}{3}\right)\sigma_2\right)^n = 0.222222\,\sigma_6 + 0.111111\,\sigma_7 + 0.111111\,\sigma_8 + 0.055556\,\sigma_9 + 0.222222\,\sigma_{10} + 0.111111\,\sigma_{11} + 0.111111\,\sigma_{12} + 0.055556\,\sigma_{13}$$

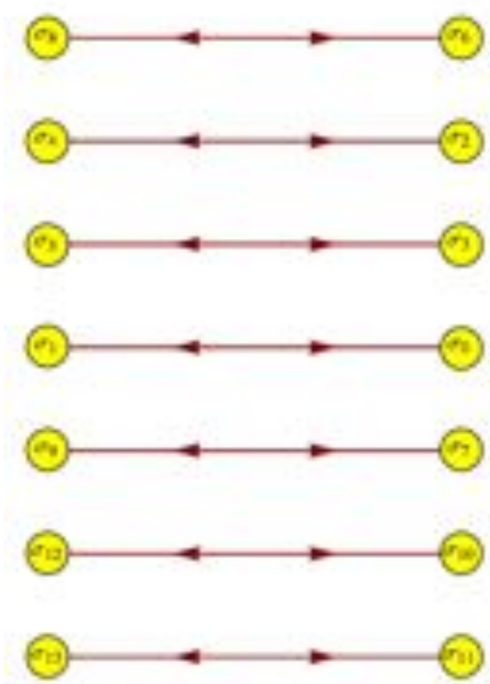
Example 9.3: Start with a Stochastic 14-Sets vector $v = \left(\frac{1}{2}\right)\sigma_0 + \left(\frac{1}{2}\right)\sigma_1$, corresponding to general discrete topological spaces, the following is the Adjacency Graph of this vector:

```
(* recall self-loops are removed *)
(* set MultiedgeStyle→0 *)
res = graph14Sets [{1 / 2, 1 / 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 1];
res[[5]]
```



Infinite power produces the same! There is no ‘draining’ as in the other case.

```
res[[6]]
```



$$\underset{n \to \infty}{\text{Lim}} \left(\left(\frac{1}{2}\right)\sigma_0 + \left(\frac{1}{2}\right)\sigma_1\right)^n = \left(\frac{1}{2}\right)\sigma_0 + \left(\frac{1}{2}\right)\sigma_1$$

These meanings are called **Subjective Form** [3] : **How** a 14-Sets vector goes from one state to another, or **How** points from one space to another, or **How** projects from one realm to another, or **How** copies from location to another and so on.

Preserved Linear Form

We can go further and explain other Subjective Forms:

$$< \underset{14}{1} \left| \textbf{Ncoeff} \right| \ {}^{0}_{14}\mathbf{x} >$$

$1.\ x_0 + 1.\ x_1 + 1.\ x_2 + 1.\ x_3 + 1.\ x_4 + 1.\ x_5 + 1.\ x_6 + 1.\ x_7 + 1.\ x_8 + 1.\ x_9 + 1.\ x_{10} + 1.\ x_{11} + 1.\ x_{12} + 1.\ x_{13}$

The coefficients for the Kuratowski operators preserve a linear form or preserve a hyperplane perpendicular to {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}.

Preserved Quadratic Form

Replacing x by a square x^2 :

$$< \underset{14}{1} \left| \textbf{Ncoeff} \right| \ {}^{0}_{14}\mathbf{x}^2 >$$

$1.\ {x^2}_0 + 1.\ {x^2}_1 + 1.\ {x^2}_2 + 1.\ {x^2}_3 + 1.\ {x^2}_4 + 1.\ {x^2}_5 + 1.\ {x^2}_6 + 1.\ {x^2}_7 + 1.\ {x^2}_8 + 1.\ {x^2}_9 + 1.\ {x^2}_{10} + 1.\ {x^2}_{11} + 1.\ {x^2}_{12} + 1.\ {x^2}_{13}$

9.1 Source and Sinks

In investigating the stable vectors of EQ 9.1:

$$v^\infty = \underset{n \to \infty}{\text{Lim}}\ v^n$$

Random such vectors reach stability by draining (zeroing) their first 6 coefficients!

Make a 14-Sets stochastic vector with random coefficients:

```
cfs = rand14SetsStoch[]
Total[cfs]
```

{0.0306064, 0.0405948, 0.128805, 0.0457791, 0.0398854, 0.00137605, 0.101888, 0.148462, 0.0635421, 0.149689, 0.100425, 0.00260541, 0.00371443, 0.142628}

1.

Look at the first column where the coefficients reside, all random:

```
res = graph14Sets[cfs, 1];
(* v the 14-Sets vector with cfs coefficients *)
v = res[[1]];
Style[v // MatrixForm, FontSize → 7]
```

0.0306064	0.0405948	0	0	0	0	0	0	0	0	0	0	0	0
0.0405948	0.0306064	0	0	0	0	0	0	0	0	0	0	0	0
0.128805	0.0457791	0.159411	0	0.0863738	0	0	0	0	0	0	0	0	0
0.0457791	0.128805	0	0.159411	0	0.0863738	0	0	0	0	0	0	0	0
0.0398854	0.00137605	0.0804802	0	0.0319824	0	0	0	0	0	0	0	0	0
0.00137605	0.0398854	0	0.0804802	0	0.0319824	0	0	0	0	0	0	0	0
0.101888	0.148462	0.150273	0	0.377691	0	0.408298	0	0.190868	0	0.150273	0	0.377691	0
0.148462	0.101888	0	0.150273	0	0.377691	0	0.408298	0	0.190868	0	0.150273	0	0.377691
0.0635421	0.149689	0.207546	0	0.193288	0	0.233883	0	0.238153	0	0.207546	0	0.193288	0
0.149689	0.0635421	0	0.207546	0	0.193288	0	0.233883	0	0.238153	0	0.207546	0	0.193288
0.100425	0.00260541	0.248887	0	0.104494	0	0.150273	0	0.377691	0	0.408298	0	0.190868	0
0.00260541	0.100425	0	0.248887	0	0.104494	0	0.150273	0	0.377691	0	0.408298	0	0.190868
0.00371443	0.142628	0.153403	0	0.20617	0	0.207546	0	0.193288	0	0.233883	0	0.238153	0
0.142628	0.00371443	0	0.153403	0	0.20617	0	0.207546	0	0.193288	0	0.233883	0	0.238153

Stochastic requirement checked for all columns:

```
Total[v]
```

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

Square the vector v and notice the first 6 coefficients in column 1 drop in value:

```
v2 = Chop[N[MatrixPower[v, 2]], 10^-9];
Style[v2 // MatrixForm, FontSize → 7]
sumColumn[v2]
```

0.00258469	0.00248492	0	0	0	0	0	0	0	0	0	0	0	0
0.00248492	0.00258469	0	0	0	0	0	0	0	0	0	0	0	0
0.0297786	0.0140465	0.0323633	0	0.0165314	0	0	0	0	0	0	0	0	0
0.0140465	0.0297786	0	0.0323633	0	0.0165314	0	0	0	0	0	0	0	0
0.0129185	0.00538957	0.0154034	0	0.00797426	0	0	0	0	0	0	0	0	0
0.00538957	0.0129185	0	0.0154034	0	0.00797426	0	0	0	0	0	0	0	0
0.113788	0.159527	0.250662	0	0.309733	0	0.312318	0	0.253147	0	0.250662	0	0.309733	0
0.159527	0.113788	0	0.250662	0	0.309733	0	0.312318	0	0.253147	0	0.250662	0	0.309733
0.102987	0.115409	0.214522	0	0.220014	0	0.222499	0	0.217106	0	0.214522	0	0.220014	0
0.115409	0.102987	0	0.214522	0	0.220014	0	0.222499	0	0.217106	0	0.214522	0	0.220014
0.120428	0.122827	0.279955	0	0.236615	0	0.250662	0	0.309733	0	0.312318	0	0.253147	0
0.122827	0.120428	0	0.279955	0	0.236615	0	0.250662	0	0.309733	0	0.312318	0	0.253147
0.0916867	0.106145	0.207095	0	0.209132	0	0.214522	0	0.220014	0	0.222499	0	0.217106	0
0.106145	0.0916867	0	0.207095	0	0.209132	0	0.214522	0	0.220014	0	0.222499	0	0.217106

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

Reach to the power 4, serious drop in the value of the first 6 coefficients:

```
v4 = Chop[N[MatrixPower[v, 4]], 10^-9];
Style[v4 // MatrixForm, FontSize → 7]
sumColumn[v4]
```

0.0000128554	0.0000128455	0	0	0	0	0	0	0	0	0	0	0	0
0.0000128455	0.0000128554	0	0	0	0	0	0	0	0	0	0	0	0
0.00128917	0.00065399	0.00130202	0	0.000666836	0	0	0	0	0	0	0	0	0
0.00065399	0.00128917	0	0.00130202	0	0.000666836	0	0	0	0	0	0	0	0
0.000608489	0.000305373	0.000621335	0	0.000318228	0	0	0	0	0	0	0	0	0
0.000305373	0.000608489	0	0.000621335	0	0.000318228	0	0	0	0	0	0	0	0
0.13235	0.148588	0.279793	0	0.28313	0	0.283143	0	0.279806	0	0.279793	0	0.28313	0
0.148588	0.13235	0	0.279793	0	0.28313	0	0.283143	0	0.279806	0	0.279793	0	0.28313
0.103467	0.115006	0.218298	0	0.218753	0	0.218766	0	0.218311	0	0.218298	0	0.218753	0
0.115006	0.103467	0	0.218298	0	0.218753	0	0.218766	0	0.218311	0	0.218298	0	0.218753
0.133253	0.146789	0.281841	0	0.279139	0	0.279793	0	0.28313	0	0.283143	0	0.279806	0
0.146789	0.133253	0	0.281841	0	0.279139	0	0.279793	0	0.28313	0	0.283143	0	0.279806
0.103139	0.114525	0.218145	0	0.217992	0	0.218298	0	0.218753	0	0.218766	0	0.218311	0
0.114525	0.103139	0	0.218145	0	0.217992	0	0.218298	0	0.218753	0	0.218766	0	0.218311

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

Reach to the power 8, almost all 6 coefficients 0:

```
v8 = Chop[N[MatrixPower[v, 8]], 10^-9];
Style[v8 // MatrixForm, FontSize → 7]
sumColumn[v8]
```

0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.10926×10^{-6}	1.08011×10^{-6}	2.10959×10^{-6}	0	1.08044×10^{-6}	0	0	0	0	0	0	0	0	0
1.08011×10^{-6}	2.10926×10^{-6}	0	2.10959×10^{-6}	0	1.08044×10^{-6}	0	0	0	0	0	0	0	0
1.00639×10^{-6}	5.15267×10^{-7}	1.00672×10^{-6}	0	5.15598×10^{-7}	0	0	0	0	0	0	0	0	0
5.15267×10^{-7}	1.00639×10^{-6}	0	1.00672×10^{-6}	0	5.15598×10^{-7}	0	0	0	0	0	0	0	0
0.133446	0.14802	0.281463	0	0.281473	0	0.281473	0	0.281463	0	0.281463	0	0.281473	0
0.14802	0.133446	0	0.281463	0	0.281473	0	0.281473	0	0.281463	0	0.281463	0	0.28147
0.10361	0.114922	0.218531	0	0.218533	0	0.218533	0	0.218531	0	0.218531	0	0.218533	0
0.114922	0.10361	0	0.218531	0	0.218533	0	0.218533	0	0.218531	0	0.218531	0	0.21853
0.13345	0.148016	0.281471	0	0.281462	0	0.281463	0	0.281473	0	0.281473	0	0.281463	0
0.148016	0.13345	0	0.281471	0	0.281462	0	0.281463	0	0.281473	0	0.281473	0	0.28146
0.10361	0.114921	0.218532	0	0.218531	0	0.218531	0	0.218533	0	0.218533	0	0.218531	0
0.114921	0.10361	0	0.218532	0	0.218531	0	0.218531	0	0.218533	0	0.218533	0	0.21853

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

Reach to the power 16, very close the infinite power limit, all 6 first coefficients 0 valued:

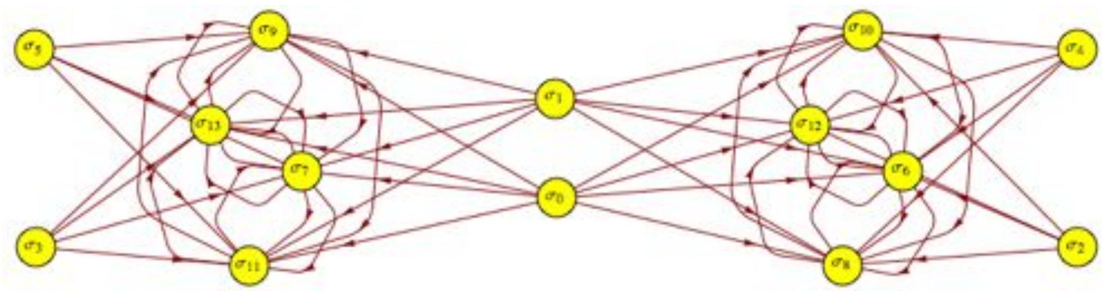
```
v16 = Chop[N[MatrixPower[v, 16]], 10^-9];
Style[v16 // MatrixForm, FontSize → 7]
sumColumn[v16]
```

0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.133449	0.148019	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0
0.148019	0.133449	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468
0.10361	0.114922	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0
0.114922	0.10361	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532
0.133449	0.148019	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0
0.148019	0.133449	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468	0	0.281468
0.10361	0.114922	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0
0.114922	0.10361	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532	0	0.218532

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

Note that though the first 6 coefficients were drained to 0, the remaining 8 got larger at each stage due to the stochastic requirement that each column must always add to 1.

```
res[[6]]
```



More and more such trials indicated that is a pattern for such random vectors. But further examination showed that always the remaining non-zero 8 coefficients (See first column) are grouped as two identical 4 Real numbers!

It is as though these two groups of 4 Reals serve as two sinks for the source which comprised of the first (non-zero) 6 coefficients.

This 'current' of probability distributions drains the 6 coefficient in speedy fashion i.e. v^{16} is a very close approximation to v^{∞}. Multiplication after multiplication the first 6 coefficients diminish and the last 8 increase, and split into the said 2 groups of 4 identical coefficients (still sum up to 1).

Remark 9.1.1: *An odd thought occurred to the author that in this case v^{∞} is not a space we are accustomed to i.e. with missing Complement and Closure operators, and since the limit approaches very fast therefore there has to be mechanism to reset v^{∞} back to original v. This gave birth to the idea of a Topological Oscillator or Topological Pump.*

Counterexample 9.1.1: Following gives a counterexample i.e. some 0s in the first 6 coefficients, where the last 8 coefficients are not bundled in two the 2 identical groups:

```
cfs =.;
cfs = {1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0};
cfs = cfs / Total[cfs]
Total[cfs]
```

$\left\{\frac{1}{3}, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\right\}$

1

```
cyl = graph14Sets[cfs, 1];
Style[cyl[[1]] // MatrixForm, FontSize → 9]
Style[cyl[[3]] // MatrixForm, FontSize → 9]
```

$$\begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3}
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.25 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\
0.25 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\
0.25 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0.25 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\
0.25 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0.25 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\
0 & 0.25 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\
0.25 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5
\end{pmatrix}$$

10. Subjective Form: SO (7)

In previous explanation for Subjective Form $< 1_{14} \,\big|\, \text{Ncoeff} \,\big|\, {}^{0}_{14}\, x^2 >$ we assumed that x^2 is any Real variable. What if we assumed x^2 has to be a square in the 14-Sets algebra, thus limiting the domain for x, for that matter we note that there are only 7 such idempotent squares residing on the diagonal of the Cayley table:

$\boldsymbol{\sigma_0^2}$

σ_0

σ_2^2

σ_2

σ_5^2

σ_5

σ_7^2

σ_7

σ_8^2

σ_8

σ_{10}^2

σ_{10}

σ_{13}^2

σ_{13}

And replace x^2 :

$\,^{0}_{14}\,x^2$

$\{x^2{}_0, x^2{}_1, x^2{}_2, x^2{}_3, x^2{}_4, x^2{}_5, x^2{}_6, x^2{}_7, x^2{}_8, x^2{}_9, x^2{}_{10}, x^2{}_{11}, x^2{}_{12}, x^2{}_{13}\}$

By the said squares in 14-Sets algebra:

$$\text{squares} = \left\{\frac{\sigma_0^2}{2}, \frac{\sigma_1^2}{2}, \sigma_2^2, \frac{\sigma_3^2}{3}, \frac{\sigma_4^2}{3}, \sigma_5^2, \frac{\sigma_6^2}{2}, \frac{\sigma_7^2}{3}, \frac{\sigma_8^2}{3}, \frac{\sigma_9^2}{2}, \frac{\sigma_{10}^2}{2}, \frac{\sigma_{11}^2}{3}, \frac{\sigma_{12}^2}{3}, \frac{\sigma_{13}^2}{2}\right\}$$

The following quadratic form is obtained:

$< \,_{14}^{1} \,|\, \text{Ncoeff} \,|\, \text{squares} >$

$1.\ \sigma_0 + 1.\ \sigma_2 + 1.\ \sigma_5 + 1.\ \sigma_7 + 1.\ \sigma_8 + 1.\ \sigma_{10} + 1.\ \sigma_{13}$

Note that each term is a square (*Mathematica* simplifies). Or:

$$\sigma_0{}^2 + \sigma_2{}^2 + \sigma_5{}^2 + \sigma_7{}^2 + \sigma_8{}^2 + \sigma_{10}{}^2 + \sigma_{13}{}^2$$

Preservation of this quadratic form is a rotation in 7 dimensions however the rotation is non-linear. We can map this non-linear rotation to a linear one in SO (7).

11. Subjective Form: O (7 - 1, 1), O (14 - 1, 1) Lorenz Group

As in the case of SO (7), given 7 squares in 14-Sets algebra, generalized Lorenz Group can form explanations for the Subjective
Form:

$$\text{squares2} = \left\{-\frac{\sigma_0^2}{2}, -\frac{\sigma_1^2}{2}, -\sigma_2^2, -\frac{\sigma_3^2}{3}, -\frac{\sigma_4^2}{3}, -\sigma_5^2, -\frac{\sigma_6^2}{2}, -\frac{\sigma_7^2}{3}, -\frac{\sigma_8^2}{3}, \frac{\sigma_9^2}{2}, -\frac{\sigma_{10}^2}{2}, -\frac{\sigma_{11}^2}{3}, -\frac{\sigma_{12}^2}{3}, \frac{\sigma_{13}^2}{2}\right\}$$

$$< \underset{14}{1} \,\Big|\, \mathbf{Ncoeff} \,\Big|\, \mathbf{squares2} >$$

$$-1.\ \sigma_0 - 1.\ \sigma_2 - 1.\ \sigma_5 - 1.\ \sigma_7 - 1.\ \sigma_8 - 1.\ \sigma_{10} + 1.\ \sigma_{13}$$

Note that each term is a square (*Mathematica* simplifies). Or:

$$-\sigma_0{}^2 - \sigma_2{}^2 - \sigma_5{}^2 - \sigma_7{}^2 - \sigma_8{}^2 - \sigma_{10}{}^2 + \sigma_{13}{}^2$$

Similar computation would result to O (14 - 1, 1) as well.

12. Subjective Form: Aff (14 - 1, ℝ)

[2] Stochastic matrices in general belong to the Lie Group $M(n, \mathbb{R})$ or in our case $M(14, \mathbb{R})$ which is isomorphic to the Affine group $n - 1$ or $14 - 1$ dimensions (consisting of the general linear group and translations i.e. Geometric contraction, expansion, dilation, reflection, rotation, shear, similarity transformations, spiral similarities, and translation are all affine transformations, as are their combinations. In general, an affine transformation is a composition of rotations, translations, dilations, and shears.)

This follows from the result earlier [2] that $M(14, \mathbb{R})$ consists of linear transformations in $\mathrm{GL}(14, \mathbb{R})$ which are restricted to the transformations in the hyperplane perpendicular to $\left\langle \underset{14}{1} \right|$ i.e.

$$< \underset{14}{1} \,\Big|\, \text{Ncoeff} \,\Big|\, {}^{0}_{14}x > \; = 1.\ x^2{}_0 + 1.\ x^2{}_1 + 1.\ x^2{}_2 + 1.\ x^2{}_3 + 1.\ x^2{}_4 + 1.\ x^2{}_5 + 1.\ x^2{}_6 + 1.\ x^2{}_7 + 1.\ x^2{}_8 + 1.\ x^2{}_9 + 1.\ x^2{}_{10} + 1.\ x^2{}_{11} + 1.\ x^2{}_{12} + 1.\ x^2{}_{13}$$

The actual isomorphism can be implemented by a coordinate transformation, which rotates x_{13} into the vector $\left\langle 1 \right|_{14}$ after which $14^2 - 14$ linear transformations which were previously in the hyperplane now become linear transformations on the subspace $x_0\, x_1 \ldots x_{12}$, leaving the x_{13} axis invariant.

13. Construct: Octonions

By simply looking at the top left corner of the Kuratowski Monoid's Cayley table, we find $\mathbb{Z}_2$:

```
M = kuraTable[];
M // MatrixForm
```

$$\begin{pmatrix}
\sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_1 & \sigma_0 & \sigma_4 & \sigma_5 & \sigma_2 & \sigma_3 & \sigma_8 & \sigma_9 & \sigma_6 & \sigma_7 & \sigma_{12} & \sigma_{13} & \sigma_{10} & \sigma_{11} \\
\sigma_2 & \sigma_3 & \sigma_2 & \sigma_3 & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_3 & \sigma_2 & \sigma_6 & \sigma_7 & \sigma_2 & \sigma_3 & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_4 & \sigma_5 & \sigma_4 & \sigma_5 & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_5 & \sigma_4 & \sigma_8 & \sigma_9 & \sigma_4 & \sigma_5 & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} \\
\sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_7 & \sigma_6 & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} \\
\sigma_9 & \sigma_8 & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_{11} & \sigma_{10} & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_{13} & \sigma_{12} & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13}
\end{pmatrix}$$

Corollary 13.1: *$\mathbb{Z}_2$ is an invariant of all topological spaces! a subgroup of 14-Sets algebra.*

This simple fact is actually an important discovery since no matter what Kuratowski operators we choose on no matter what topological space, we shall always uncover $\mathbb{Z}_2$ as an algebra to compute with or construct larger algebras. In many constructions of Octonions $\mathbb{Z}_2$ is used but no one cares to explain where it originates from. This overlooked fact is worthy of great concern.

Equipped with $\mathbb{Z}_2$ and the following ingredients obtained from the topological spaces and 14-Sets algebra, we can build the Octonions:

1. Topological spaces
2. $\mathbb{Z}_2$ which is found as an invariant of any topological space (see Theorem 1.1 as well)
3. Product of topological spaces (which is again a topological space), $\mathbb{Z}_2$ being a discrete topology therefor we can construct ${\mathbb{Z}_2}^3$and ${\mathbb{Z}_2}^3 \times {\mathbb{Z}_2}^3$
4. Identification or gluing for topological spaces by identifying points to be the same (glued)
5. Reflections as members of Subjective Form Aff (14 - 1, $\mathbb{R}$)

$\mathbb{Z}_2$:

	1	e1
1	1	e1
e1	e1	1

$\mathbb{Z}_2{}^3 = \{1, e1, e2, e3, e4, e5, e6, e7\}$ with the following Cayley table:

	1	e1	e2	e3	e4	e5	e6	e7
1	1	e1	e2	e3	e4	e5	e6	e7
e1	e1	1	e3	e2	e5	e4	e7	e6
e2	e2	e3	1	e1	e6	e7	e4	e5
e3	e3	e2	e1	1	e7	e6	e5	e4
e4	e4	e5	e6	e7	1	e1	e2	e3
e5	e5	e4	e7	e6	e1	1	e3	e2
e6	e6	e7	e4	e5	e2	e3	1	e1
e7	e7	e6	e5	e4	e3	e2	e1	1

This table is actually a subset or discrete subspace of discrete topological space $(\mathbb{Z}_2{}^3 \times \mathbb{Z}_2{}^3) \times \mathbb{Z}_2{}^3$.

Remark 13.1: *We can take the latter and the other product spaces and map them to other spaces e.g.* $(G_2 \times G_2) \times G_2$. *Basically we took the Cayley table and mapped it into an arbitrary topological space.*

We then add the Twist to this table i.e. multiply some elements of the table by -1. We model this Twist as follows:

$$(\mathbb{Z}_2{}^3 \times \mathbb{Z}_2{}^3) \times (\mathbb{Z}_2{}^3 \times \mathrm{Aff}(14-1, \mathbb{R})) \qquad \text{EQ 13.1}$$

which the entire product itself is a topological space, however we select a finite discrete subspace comprised of either Identity matrix or Reflection transformation matrix.

Remark 13.1: *Instead of the Cartesian products above, in particular the product of Aff(14-1,* $\mathbb{R}$*) perhaps the topological identification would be more fitting and intuitive. Basically we are gluing the entries of the table to* $\mathbb{Z}_2{}^3 \times \mathbb{Z}_2{}^3$ *and glue to the Twist to the entry which is an element of* $\mathbb{Z}_2{}^3$*.*

A sample point in this finite discrete topological space looks like:

(e1, e1, e1$*$e1, Ref) , algebraic expression extracted $(e1 * e1)_{Twist} = e1 * e1 * Det(Ref) = -1$

where $(e1 * e1)_{Twist}$ is the product obtained by a twist i.e. $i * i = -1$, indeed Ref is a reflection matrix with determinant -1:

```
r =.;

(* r ={r0,r1,r2,r3,r4,r5,r6,r7,r8,r9,r10,r11,r12,r13}
  perpendicular to the plane of reflection *)
(* Assume r is unit *)
rcoors = {}^0_{14} r;
Element[rcoors, Reals];
ref = Simplify[ReflectionMatrix[rcoors]];
Ref = ref /. {Abs[r0]^2 + Abs[r1]^2 + Abs[r2]^2 + Abs[r3]^2 + Abs[r4]^2 + Abs[r5]^2 + Abs[r6]^2 +
        Abs[r7]^2 + Abs[r8]^2 + Abs[r9]^2 + Abs[r10]^2 + Abs[r11]^2 + Abs[r12]^2 + Abs[r13]^2 → 1,
      1 / Abs[r0]^2 + Abs[r1]^2 + Abs[r2]^2 + Abs[r3]^2 + Abs[r4]^2 + Abs[r5]^2 + Abs[r6]^2 +
        Abs[r7]^2 + Abs[r8]^2 + Abs[r9]^2 + Abs[r10]^2 + Abs[r11]^2 + Abs[r12]^2 + Abs[r13]^2 → 1,
      Conjugate[r0] → r0,
      Conjugate[r1] → r1,
      Conjugate[r2] → r2,
      Conjugate[r3] → r3,
      Conjugate[r4] → r4,
      Conjugate[r5] → r5,
      Conjugate[r6] → r6,
      Conjugate[r7] → r7,
      Conjugate[r8] → r8,
      Conjugate[r9] → r9,
      Conjugate[r10] → r10,
      Conjugate[r11] → r11,
      Conjugate[r12] → r12,
      Conjugate[r13] → r13,
      Abs[r0] → r0,
      Abs[r1] → r1,
      Abs[r2] → r2,
      Abs[r3] → r3,
      Abs[r4] → r4,
      Abs[r5] → r5,
      Abs[r6] → r6,
      Abs[r7] → r7,
      Abs[r8] → r8,
      Abs[r9] → r9,
      Abs[r10] → r10,
      Abs[r11] → r11,
      Abs[r12] → r12,
      Abs[r13] → r13};


Style[Ref // MatrixForm, FontSize → 8]
```

$$\begin{pmatrix}
1-2r_0^2 & -2r_0r_1 & -2r_0r_2 & -2r_0r_3 & -2r_0r_4 & -2r_0r_5 & -2r_0r_6 & -2r_0r_7 & -2r_0r_8 & -2r_0r_9 & -2r_0r_{10} & -2r_0r_{11} & -2r_0r_{12} & -2r_0r_{13} \\
-2r_0r_1 & 1-2r_1^2 & -2r_1r_2 & -2r_1r_3 & -2r_1r_4 & -2r_1r_5 & -2r_1r_6 & -2r_1r_7 & -2r_1r_8 & -2r_1r_9 & -2r_1r_{10} & -2r_1r_{11} & -2r_1r_{12} & -2r_1r_{13} \\
-2r_0r_2 & -2r_1r_2 & 1-2r_2^2 & -2r_2r_3 & -2r_2r_4 & -2r_2r_5 & -2r_2r_6 & -2r_2r_7 & -2r_2r_8 & -2r_2r_9 & -2r_2r_{10} & -2r_2r_{11} & -2r_2r_{12} & -2r_2r_{13} \\
-2r_0r_3 & -2r_1r_3 & -2r_2r_3 & 1-2r_3^2 & -2r_3r_4 & -2r_3r_5 & -2r_3r_6 & -2r_3r_7 & -2r_3r_8 & -2r_3r_9 & -2r_3r_{10} & -2r_3r_{11} & -2r_3r_{12} & -2r_3r_{13} \\
-2r_0r_4 & -2r_1r_4 & -2r_2r_4 & -2r_3r_4 & 1-2r_4^2 & -2r_4r_5 & -2r_4r_6 & -2r_4r_7 & -2r_4r_8 & -2r_4r_9 & -2r_4r_{10} & -2r_4r_{11} & -2r_4r_{12} & -2r_4r_{13} \\
-2r_0r_5 & -2r_1r_5 & -2r_2r_5 & -2r_3r_5 & -2r_4r_5 & 1-2r_5^2 & -2r_5r_6 & -2r_5r_7 & -2r_5r_8 & -2r_5r_9 & -2r_5r_{10} & -2r_5r_{11} & -2r_5r_{12} & -2r_5r_{13} \\
-2r_0r_6 & -2r_1r_6 & -2r_2r_6 & -2r_3r_6 & -2r_4r_6 & -2r_5r_6 & 1-2r_6^2 & -2r_6r_7 & -2r_6r_8 & -2r_6r_9 & -2r_6r_{10} & -2r_6r_{11} & -2r_6r_{12} & -2r_6r_{13} \\
-2r_0r_7 & -2r_1r_7 & -2r_2r_7 & -2r_3r_7 & -2r_4r_7 & -2r_5r_7 & -2r_6r_7 & 1-2r_7^2 & -2r_7r_8 & -2r_7r_9 & -2r_7r_{10} & -2r_7r_{11} & -2r_7r_{12} & -2r_7r_{13} \\
-2r_0r_8 & -2r_1r_8 & -2r_2r_8 & -2r_3r_8 & -2r_4r_8 & -2r_5r_8 & -2r_6r_8 & -2r_7r_8 & 1-2r_8^2 & -2r_8r_9 & -2r_8r_{10} & -2r_8r_{11} & -2r_8r_{12} & -2r_8r_{13} \\
-2r_0r_9 & -2r_1r_9 & -2r_2r_9 & -2r_3r_9 & -2r_4r_9 & -2r_5r_9 & -2r_6r_9 & -2r_7r_9 & -2r_8r_9 & 1-2r_9^2 & -2r_9r_{10} & -2r_9r_{11} & -2r_9r_{12} & -2r_9r_{13} \\
-2r_0r_{10} & -2r_1r_{10} & -2r_2r_{10} & -2r_3r_{10} & -2r_4r_{10} & -2r_5r_{10} & -2r_6r_{10} & -2r_7r_{10} & -2r_8r_{10} & -2r_9r_{10} & 1-2r_{10}^2 & -2r_{10}r_{11} & -2r_{10}r_{12} & -2r_{10}r_{13} \\
-2r_0r_{11} & -2r_1r_{11} & -2r_2r_{11} & -2r_3r_{11} & -2r_4r_{11} & -2r_5r_{11} & -2r_6r_{11} & -2r_7r_{11} & -2r_8r_{11} & -2r_9r_{11} & -2r_{10}r_{11} & 1-2r_{11}^2 & -2r_{11}r_{12} & -2r_{11}r_{13} \\
-2r_0r_{12} & -2r_1r_{12} & -2r_2r_{12} & -2r_3r_{12} & -2r_4r_{12} & -2r_5r_{12} & -2r_6r_{12} & -2r_7r_{12} & -2r_8r_{12} & -2r_9r_{12} & -2r_{10}r_{12} & -2r_{11}r_{12} & 1-2r_{12}^2 & -2r_{12}r_{13} \\
-2r_0r_{13} & -2r_1r_{13} & -2r_2r_{13} & -2r_3r_{13} & -2r_4r_{13} & -2r_5r_{13} & -2r_6r_{13} & -2r_7r_{13} & -2r_8r_{13} & -2r_9r_{13} & -2r_{10}r_{13} & -2r_{11}r_{13} & -2r_{12}r_{13} & 1-2r_{13}^2
\end{pmatrix}$$

rcoors was the coordinates of:

$\mathbf{{}^{0}_{14}r}$

$\{r_0, r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}\}$

which is also the normal vector to the plane of reflection in 14-D space.

Therefore if we apply the Reflection transformation i.e. multiply its matrix multiplication to r, we should get -r :

```
Simplify[Ref.rcoors] /.
```
$$\{2 r_0^2 + 2 r_1^2 + 2 r_2^2 + 2 r_3^2 + 2 r_4^2 + 2 r_5^2 + 2 r_6^2 + 2 r_7^2 + 2 r_8^2 + 2 r_9^2 + 2 r_{10}^2 + 2 r_{11}^2 + 2 r_{12}^2 + 2 r_{13}^2 \to 2\}$$

$\{-r_0, -r_1, -r_2, -r_3, -r_4, -r_5, -r_6, -r_7, -r_8, -r_9, -r_{10}, -r_{11}, -r_{12}, -r_{13}\}$

Compute the determinant of the Reflection matrix and you see its value being -1:

```
Det [Ref] /.
```
$$\{-2 r_0^2 - 2 r_1^2 - 2 r_2^2 - 2 r_3^2 - 2 r_4^2 - 2 r_5^2 - 2 r_6^2 - 2 r_7^2 - 2 r_8^2 - 2 r_9^2 - 2 r_{10}^2 - 2 r_{11}^2 - 2 r_{12}^2 - 2 r_{13}^2 \to -2\}$$

-1

Twist

Where is the Twist in all these?

Imagine the Cayley table was placed in front of a mirror and at each entry of the table we placed a ribbon or the equivalent of a theoretical physicist standing up.

Stretch the ribbon and place it just touching the mirror, it is as though you stretched it all the way to the entry inside the mirror (reflection) but twisted it a half-turn.

Or stretch the right arm of the theoretical physicist at the entry in the Cayley table to the right arm of his reflection inside the mirror, and do the same for his left arm, you introduce a twist (criss-cross of the arms).

Warning: *Do not attempt this experiment at home, the person in the image is a highly trained theoretical physicist with many years of engrossing training.*

If the fellow and his reflection shook hands, then there is a twist like this:

That is why all the fuss was made for the discrete topological subspace specifications as above, we can map them into other familiar topological spaces, but the twist stays the same nonetheless! topologically speaking.

And this twist can accurately be modelled as the determinant of the said reflection matrix.

The final result will be two copies of the table below:

	1	e1	e2	e3	e4	e5	e6	e7
1	1	e1	e2	e3	e4	e5	e6	e7
e1	e1	1	e3	e2	e5	e4	e7	e6
e2	e2	e3	1	e1	e6	e7	e4	e5
e3	e3	e2	e1	1	e7	e6	e5	e4
e4	e4	e5	e6	e7	1	e1	e2	e3
e5	e5	e4	e7	e6	e1	1	e3	e2
e6	e6	e7	e4	e5	e2	e3	1	e1
e7	e7	e6	e5	e4	e3	e2	e1	1

The two copies are connected (Identified) by twisting something at the entries e.g. a guy standing up or a ribbon, but we can model those things by Reflection matrices and multiply the determinant which is -1 if the matrix is of type Reflection +1 if identity matrix:

	1	e1	e2	e3	e4	e5	e6	e7
1	1	e1	e2	e3	e4	e5	e6	e7
e1	e1	-1	e3	-e2	e5	-e4	-e7	e6
e2	e2	-e3	-1	e1	e6	e7	-e4	-e5
e3	e3	e2	-e1	-1	e7	-e6	e5	-e4
e4	e4	-e5	-e6	-e7	-1	e1	e2	e3
e5	e5	e4	-e7	e6	-e1	-1	-e3	e2
e6	e6	e7	e4	-e5	-e2	e3	-1	-e1
e7	e7	-e6	e5	e4	-e3	-e2	e1	-1

Remark 13.2: *Instead of using Aff(14-1, $\mathbb{R}$) we could use a Braid group which is proven to be linear i.e. it has matrix representation in GL(14, $\mathbb{R}$) .*

14. Commutator

In general the Commutator is non-zero:

$\boldsymbol{\sigma_4\,\sigma_6 - \sigma_6\,\sigma_4}$

$\sigma_8 - \sigma_{10}$

```
commute = Table[SameQ[σ_i σ_j - σ_j σ_i, 0], {i, 0, 13}, {j, 0, 13}];
commute // MatrixForm
```

```
True True  True  True  True  True  True  True  True  True  True  True  True
True True  False False False False False False False False False False False F
True False True  False False False True  False False False True  False False F
True False False True  False False False True  False False False True  False F
True False False False True  False False False True  False False False True  F
True False False False False True  False False False True  False False False
True False True  False False False True  False False False True  False False F
True False False True  False False False True  False False False True  False F
True False False False True  False False False True  False False False True  F
True False False False False True  False False False True  False False False
True False True  False False False True  False False False True  False False F
True False False True  False False False True  False False False True  False F
True False False False True  False False False True  False False False True  F
True False False False False True  False False False True  False False False
```

Therefore the algebra is also non-commutative.

Compute the Commutator:

```
Distribute[v1 ** v2] - Distribute[v2 ** v1]
```

$(a\,\sigma_0) ** (a2\,\sigma_0) + (a\,\sigma_0) ** (b2\,\sigma_1) + (a\,\sigma_0) ** (c2\,\sigma_2) + (a\,\sigma_0) ** (d2\,\sigma_3) + (a\,\sigma_0) ** (e2\,\sigma_4) +$
$(a\,\sigma_0) ** (f2\,\sigma_5) + (a\,\sigma_0) ** (g2\,\sigma_6) + (a\,\sigma_0) ** (h2\,\sigma_7) + (a\,\sigma_0) ** (i2\,\sigma_8) + (a\,\sigma_0) ** (j2\,\sigma_9) +$
$(a\,\sigma_0) ** (k2\,\sigma_{10}) + (a\,\sigma_0) ** (l2\,\sigma_{11}) + (a\,\sigma_0) ** (m2\,\sigma_{12}) + (a\,\sigma_0) ** (n2\,\sigma_{13}) -$
$(a2\,\sigma_0) ** (a\,\sigma_0) - (a2\,\sigma_0) ** (b\,\sigma_1) - (a2\,\sigma_0) ** (c\,\sigma_2) - (a2\,\sigma_0) ** (d\,\sigma_3) - (a2\,\sigma_0) ** (e\,\sigma_4) -$
$(a2\,\sigma_0) ** (f\,\sigma_5) - (a2\,\sigma_0) ** (g\,\sigma_6) - (a2\,\sigma_0) ** (h\,\sigma_7) - (a2\,\sigma_0) ** (i\,\sigma_8) - (a2\,\sigma_0) ** (j\,\sigma_9) -$
$(a2\,\sigma_0) ** (k\,\sigma_{10}) - (a2\,\sigma_0) ** (l\,\sigma_{11}) - (a2\,\sigma_0) ** (m\,\sigma_{12}) - (a2\,\sigma_0) ** (n\,\sigma_{13}) +$
$(b\,\sigma_1) ** (a2\,\sigma_0) + (b\,\sigma_1) ** (b2\,\sigma_1) + (b\,\sigma_1) ** (c2\,\sigma_2) + (b\,\sigma_1) ** (d2\,\sigma_3) + (b\,\sigma_1) ** (e2\,\sigma_4) +$
$(b\,\sigma_1) ** (f2\,\sigma_5) + (b\,\sigma_1) ** (g2\,\sigma_6) + (b\,\sigma_1) ** (h2\,\sigma_7) + (b\,\sigma_1) ** (i2\,\sigma_8) + (b\,\sigma_1) ** (j2\,\sigma_9) +$
$(b\,\sigma_1) ** (k2\,\sigma_{10}) + (b\,\sigma_1) ** (l2\,\sigma_{11}) + (b\,\sigma_1) ** (m2\,\sigma_{12}) + (b\,\sigma_1) ** (n2\,\sigma_{13}) -$
$(b2\,\sigma_1) ** (a\,\sigma_0) - (b2\,\sigma_1) ** (b\,\sigma_1) - (b2\,\sigma_1) ** (c\,\sigma_2) - (b2\,\sigma_1) ** (d\,\sigma_3) - (b2\,\sigma_1) ** (e\,\sigma_4) -$
$(b2\,\sigma_1) ** (f\,\sigma_5) - (b2\,\sigma_1) ** (g\,\sigma_6) - (b2\,\sigma_1) ** (h\,\sigma_7) - (b2\,\sigma_1) ** (i\,\sigma_8) - (b2\,\sigma_1) ** (j\,\sigma_9) -$
$(b2\,\sigma_1) ** (k\,\sigma_{10}) - (b2\,\sigma_1) ** (l\,\sigma_{11}) - (b2\,\sigma_1) ** (m\,\sigma_{12}) - (b2\,\sigma_1) ** (n\,\sigma_{13}) +$
$(c\,\sigma_2) ** (a2\,\sigma_0) + (c\,\sigma_2) ** (b2\,\sigma_1) + (c\,\sigma_2) ** (c2\,\sigma_2) + (c\,\sigma_2) ** (d2\,\sigma_3) + (c\,\sigma_2) ** (e2\,\sigma_4) +$
$(c\,\sigma_2) ** (f2\,\sigma_5) + (c\,\sigma_2) ** (g2\,\sigma_6) + (c\,\sigma_2) ** (h2\,\sigma_7) + (c\,\sigma_2) ** (i2\,\sigma_8) + (c\,\sigma_2) ** (j2\,\sigma_9) +$
$(c\,\sigma_2) ** (k2\,\sigma_{10}) + (c\,\sigma_2) ** (l2\,\sigma_{11}) + (c\,\sigma_2) ** (m2\,\sigma_{12}) + (c\,\sigma_2) ** (n2\,\sigma_{13}) -$
$(c2\,\sigma_2) ** (a\,\sigma_0) - (c2\,\sigma_2) ** (b\,\sigma_1) - (c2\,\sigma_2) ** (c\,\sigma_2) - (c2\,\sigma_2) ** (d\,\sigma_3) - (c2\,\sigma_2) ** (e\,\sigma_4) -$
$(c2\,\sigma_2) ** (f\,\sigma_5) - (c2\,\sigma_2) ** (g\,\sigma_6) - (c2\,\sigma_2) ** (h\,\sigma_7) - (c2\,\sigma_2) ** (i\,\sigma_8) - (c2\,\sigma_2) ** (j\,\sigma_9) -$
$(c2\,\sigma_2) ** (k\,\sigma_{10}) - (c2\,\sigma_2) ** (l\,\sigma_{11}) - (c2\,\sigma_2) ** (m\,\sigma_{12}) - (c2\,\sigma_2) ** (n\,\sigma_{13}) +$
$(d\,\sigma_3) ** (a2\,\sigma_0) + (d\,\sigma_3) ** (b2\,\sigma_1) + (d\,\sigma_3) ** (c2\,\sigma_2) + (d\,\sigma_3) ** (d2\,\sigma_3) + (d\,\sigma_3) ** (e2\,\sigma_4) +$
$(d\,\sigma_3) ** (f2\,\sigma_5) + (d\,\sigma_3) ** (g2\,\sigma_6) + (d\,\sigma_3) ** (h2\,\sigma_7) + (d\,\sigma_3) ** (i2\,\sigma_8) + (d\,\sigma_3) ** (j2\,\sigma_9) +$

$(d\ \sigma_3) ** (k2\ \sigma_{10}) + (d\ \sigma_3) ** (l2\ \sigma_{11}) + (d\ \sigma_3) ** (m2\ \sigma_{12}) + (d\ \sigma_3) ** (n2\ \sigma_{13}) -$
$(d2\ \sigma_3) ** (a\ \sigma_0) - (d2\ \sigma_3) ** (b\ \sigma_1) - (d2\ \sigma_3) ** (c\ \sigma_2) - (d2\ \sigma_3) ** (d\ \sigma_3) - (d2\ \sigma_3) ** (e\ \sigma_4) -$
$(d2\ \sigma_3) ** (f\ \sigma_5) - (d2\ \sigma_3) ** (g\ \sigma_6) - (d2\ \sigma_3) ** (h\ \sigma_7) - (d2\ \sigma_3) ** (i\ \sigma_8) - (d2\ \sigma_3) ** (j\ \sigma_9) -$
$(d2\ \sigma_3) ** (k\ \sigma_{10}) - (d2\ \sigma_3) ** (l\ \sigma_{11}) - (d2\ \sigma_3) ** (m\ \sigma_{12}) - (d2\ \sigma_3) ** (n\ \sigma_{13}) +$
$(e\ \sigma_4) ** (a2\ \sigma_0) + (e\ \sigma_4) ** (b2\ \sigma_1) + (e\ \sigma_4) ** (c2\ \sigma_2) + (e\ \sigma_4) ** (d2\ \sigma_3) + (e\ \sigma_4) ** (e2\ \sigma_4) +$
$(e\ \sigma_4) ** (f2\ \sigma_5) + (e\ \sigma_4) ** (g2\ \sigma_6) + (e\ \sigma_4) ** (h2\ \sigma_7) + (e\ \sigma_4) ** (i2\ \sigma_8) + (e\ \sigma_4) ** (j2\ \sigma_9) +$
$(e\ \sigma_4) ** (k2\ \sigma_{10}) + (e\ \sigma_4) ** (l2\ \sigma_{11}) + (e\ \sigma_4) ** (m2\ \sigma_{12}) + (e\ \sigma_4) ** (n2\ \sigma_{13}) -$
$(e2\ \sigma_4) ** (a\ \sigma_0) - (e2\ \sigma_4) ** (b\ \sigma_1) - (e2\ \sigma_4) ** (c\ \sigma_2) - (e2\ \sigma_4) ** (d\ \sigma_3) - (e2\ \sigma_4) ** (e\ \sigma_4) -$
$(e2\ \sigma_4) ** (f\ \sigma_5) - (e2\ \sigma_4) ** (g\ \sigma_6) - (e2\ \sigma_4) ** (h\ \sigma_7) - (e2\ \sigma_4) ** (i\ \sigma_8) - (e2\ \sigma_4) ** (j\ \sigma_9) -$
$(e2\ \sigma_4) ** (k\ \sigma_{10}) - (e2\ \sigma_4) ** (l\ \sigma_{11}) - (e2\ \sigma_4) ** (m\ \sigma_{12}) - (e2\ \sigma_4) ** (n\ \sigma_{13}) +$
$(f\ \sigma_5) ** (a2\ \sigma_0) + (f\ \sigma_5) ** (b2\ \sigma_1) + (f\ \sigma_5) ** (c2\ \sigma_2) + (f\ \sigma_5) ** (d2\ \sigma_3) + (f\ \sigma_5) ** (e2\ \sigma_4) +$
$(f\ \sigma_5) ** (f2\ \sigma_5) + (f\ \sigma_5) ** (g2\ \sigma_6) + (f\ \sigma_5) ** (h2\ \sigma_7) + (f\ \sigma_5) ** (i2\ \sigma_8) + (f\ \sigma_5) ** (j2\ \sigma_9) +$
$(f\ \sigma_5) ** (k2\ \sigma_{10}) + (f\ \sigma_5) ** (l2\ \sigma_{11}) + (f\ \sigma_5) ** (m2\ \sigma_{12}) + (f\ \sigma_5) ** (n2\ \sigma_{13}) -$
$(f2\ \sigma_5) ** (a\ \sigma_0) - (f2\ \sigma_5) ** (b\ \sigma_1) - (f2\ \sigma_5) ** (c\ \sigma_2) - (f2\ \sigma_5) ** (d\ \sigma_3) - (f2\ \sigma_5) ** (e\ \sigma_4) -$
$(f2\ \sigma_5) ** (f\ \sigma_5) - (f2\ \sigma_5) ** (g\ \sigma_6) - (f2\ \sigma_5) ** (h\ \sigma_7) - (f2\ \sigma_5) ** (i\ \sigma_8) - (f2\ \sigma_5) ** (j\ \sigma_9) -$
$(f2\ \sigma_5) ** (k\ \sigma_{10}) - (f2\ \sigma_5) ** (l\ \sigma_{11}) - (f2\ \sigma_5) ** (m\ \sigma_{12}) - (f2\ \sigma_5) ** (n\ \sigma_{13}) +$
$(g\ \sigma_6) ** (a2\ \sigma_0) + (g\ \sigma_6) ** (b2\ \sigma_1) + (g\ \sigma_6) ** (c2\ \sigma_2) + (g\ \sigma_6) ** (d2\ \sigma_3) + (g\ \sigma_6) ** (e2\ \sigma_4) +$
$(g\ \sigma_6) ** (f2\ \sigma_5) + (g\ \sigma_6) ** (g2\ \sigma_6) + (g\ \sigma_6) ** (h2\ \sigma_7) + (g\ \sigma_6) ** (i2\ \sigma_8) + (g\ \sigma_6) ** (j2\ \sigma_9) +$
$(g\ \sigma_6) ** (k2\ \sigma_{10}) + (g\ \sigma_6) ** (l2\ \sigma_{11}) + (g\ \sigma_6) ** (m2\ \sigma_{12}) + (g\ \sigma_6) ** (n2\ \sigma_{13}) -$
$(g2\ \sigma_6) ** (a\ \sigma_0) - (g2\ \sigma_6) ** (b\ \sigma_1) - (g2\ \sigma_6) ** (c\ \sigma_2) - (g2\ \sigma_6) ** (d\ \sigma_3) - (g2\ \sigma_6) ** (e\ \sigma_4) -$
$(g2\ \sigma_6) ** (f\ \sigma_5) - (g2\ \sigma_6) ** (g\ \sigma_6) - (g2\ \sigma_6) ** (h\ \sigma_7) - (g2\ \sigma_6) ** (i\ \sigma_8) - (g2\ \sigma_6) ** (j\ \sigma_9) -$
$(g2\ \sigma_6) ** (k\ \sigma_{10}) - (g2\ \sigma_6) ** (l\ \sigma_{11}) - (g2\ \sigma_6) ** (m\ \sigma_{12}) - (g2\ \sigma_6) ** (n\ \sigma_{13}) +$
$(h\ \sigma_7) ** (a2\ \sigma_0) + (h\ \sigma_7) ** (b2\ \sigma_1) + (h\ \sigma_7) ** (c2\ \sigma_2) + (h\ \sigma_7) ** (d2\ \sigma_3) + (h\ \sigma_7) ** (e2\ \sigma_4) +$
$(h\ \sigma_7) ** (f2\ \sigma_5) + (h\ \sigma_7) ** (g2\ \sigma_6) + (h\ \sigma_7) ** (h2\ \sigma_7) + (h\ \sigma_7) ** (i2\ \sigma_8) + (h\ \sigma_7) ** (j2\ \sigma_9) +$
$(h\ \sigma_7) ** (k2\ \sigma_{10}) + (h\ \sigma_7) ** (l2\ \sigma_{11}) + (h\ \sigma_7) ** (m2\ \sigma_{12}) + (h\ \sigma_7) ** (n2\ \sigma_{13}) -$
$(h2\ \sigma_7) ** (a\ \sigma_0) - (h2\ \sigma_7) ** (b\ \sigma_1) - (h2\ \sigma_7) ** (c\ \sigma_2) - (h2\ \sigma_7) ** (d\ \sigma_3) - (h2\ \sigma_7) ** (e\ \sigma_4) -$
$(h2\ \sigma_7) ** (f\ \sigma_5) - (h2\ \sigma_7) ** (g\ \sigma_6) - (h2\ \sigma_7) ** (h\ \sigma_7) - (h2\ \sigma_7) ** (i\ \sigma_8) - (h2\ \sigma_7) ** (j\ \sigma_9) -$
$(h2\ \sigma_7) ** (k\ \sigma_{10}) - (h2\ \sigma_7) ** (l\ \sigma_{11}) - (h2\ \sigma_7) ** (m\ \sigma_{12}) - (h2\ \sigma_7) ** (n\ \sigma_{13}) +$
$(i\ \sigma_8) ** (a2\ \sigma_0) + (i\ \sigma_8) ** (b2\ \sigma_1) + (i\ \sigma_8) ** (c2\ \sigma_2) + (i\ \sigma_8) ** (d2\ \sigma_3) + (i\ \sigma_8) ** (e2\ \sigma_4) +$
$(i\ \sigma_8) ** (f2\ \sigma_5) + (i\ \sigma_8) ** (g2\ \sigma_6) + (i\ \sigma_8) ** (h2\ \sigma_7) + (i\ \sigma_8) ** (i2\ \sigma_8) + (i\ \sigma_8) ** (j2\ \sigma_9) +$
$(i\ \sigma_8) ** (k2\ \sigma_{10}) + (i\ \sigma_8) ** (l2\ \sigma_{11}) + (i\ \sigma_8) ** (m2\ \sigma_{12}) + (i\ \sigma_8) ** (n2\ \sigma_{13}) -$
$(i2\ \sigma_8) ** (a\ \sigma_0) - (i2\ \sigma_8) ** (b\ \sigma_1) - (i2\ \sigma_8) ** (c\ \sigma_2) - (i2\ \sigma_8) ** (d\ \sigma_3) - (i2\ \sigma_8) ** (e\ \sigma_4) -$
$(i2\ \sigma_8) ** (f\ \sigma_5) - (i2\ \sigma_8) ** (g\ \sigma_6) - (i2\ \sigma_8) ** (h\ \sigma_7) - (i2\ \sigma_8) ** (i\ \sigma_8) - (i2\ \sigma_8) ** (j\ \sigma_9) -$
$(i2\ \sigma_8) ** (k\ \sigma_{10}) - (i2\ \sigma_8) ** (l\ \sigma_{11}) - (i2\ \sigma_8) ** (m\ \sigma_{12}) - (i2\ \sigma_8) ** (n\ \sigma_{13}) +$
$(j\ \sigma_9) ** (a2\ \sigma_0) + (j\ \sigma_9) ** (b2\ \sigma_1) + (j\ \sigma_9) ** (c2\ \sigma_2) + (j\ \sigma_9) ** (d2\ \sigma_3) + (j\ \sigma_9) ** (e2\ \sigma_4) +$
$(j\ \sigma_9) ** (f2\ \sigma_5) + (j\ \sigma_9) ** (g2\ \sigma_6) + (j\ \sigma_9) ** (h2\ \sigma_7) + (j\ \sigma_9) ** (i2\ \sigma_8) + (j\ \sigma_9) ** (j2\ \sigma_9) +$
$(j\ \sigma_9) ** (k2\ \sigma_{10}) + (j\ \sigma_9) ** (l2\ \sigma_{11}) + (j\ \sigma_9) ** (m2\ \sigma_{12}) + (j\ \sigma_9) ** (n2\ \sigma_{13}) -$
$(j2\ \sigma_9) ** (a\ \sigma_0) - (j2\ \sigma_9) ** (b\ \sigma_1) - (j2\ \sigma_9) ** (c\ \sigma_2) - (j2\ \sigma_9) ** (d\ \sigma_3) - (j2\ \sigma_9) ** (e\ \sigma_4) -$
$(j2\ \sigma_9) ** (f\ \sigma_5) - (j2\ \sigma_9) ** (g\ \sigma_6) - (j2\ \sigma_9) ** (h\ \sigma_7) - (j2\ \sigma_9) ** (i\ \sigma_8) - (j2\ \sigma_9) ** (j\ \sigma_9) -$
$(j2\ \sigma_9) ** (k\ \sigma_{10}) - (j2\ \sigma_9) ** (l\ \sigma_{11}) - (j2\ \sigma_9) ** (m\ \sigma_{12}) - (j2\ \sigma_9) ** (n\ \sigma_{13}) +$
$(k\ \sigma_{10}) ** (a2\ \sigma_0) + (k\ \sigma_{10}) ** (b2\ \sigma_1) + (k\ \sigma_{10}) ** (c2\ \sigma_2) + (k\ \sigma_{10}) ** (d2\ \sigma_3) +$
$(k\ \sigma_{10}) ** (e2\ \sigma_4) + (k\ \sigma_{10}) ** (f2\ \sigma_5) + (k\ \sigma_{10}) ** (g2\ \sigma_6) + (k\ \sigma_{10}) ** (h2\ \sigma_7) +$
$(k\ \sigma_{10}) ** (i2\ \sigma_8) + (k\ \sigma_{10}) ** (j2\ \sigma_9) + (k\ \sigma_{10}) ** (k2\ \sigma_{10}) + (k\ \sigma_{10}) ** (l2\ \sigma_{11}) +$
$(k\ \sigma_{10}) ** (m2\ \sigma_{12}) + (k\ \sigma_{10}) ** (n2\ \sigma_{13}) - (k2\ \sigma_{10}) ** (a\ \sigma_0) - (k2\ \sigma_{10}) ** (b\ \sigma_1) -$
$(k2\ \sigma_{10}) ** (c\ \sigma_2) - (k2\ \sigma_{10}) ** (d\ \sigma_3) - (k2\ \sigma_{10}) ** (e\ \sigma_4) - (k2\ \sigma_{10}) ** (f\ \sigma_5) -$
$(k2\ \sigma_{10}) ** (g\ \sigma_6) - (k2\ \sigma_{10}) ** (h\ \sigma_7) - (k2\ \sigma_{10}) ** (i\ \sigma_8) - (k2\ \sigma_{10}) ** (j\ \sigma_9) -$
$(k2\ \sigma_{10}) ** (k\ \sigma_{10}) - (k2\ \sigma_{10}) ** (l\ \sigma_{11}) - (k2\ \sigma_{10}) ** (m\ \sigma_{12}) - (k2\ \sigma_{10}) ** (n\ \sigma_{13}) +$
$(l\ \sigma_{11}) ** (a2\ \sigma_0) + (l\ \sigma_{11}) ** (b2\ \sigma_1) + (l\ \sigma_{11}) ** (c2\ \sigma_2) + (l\ \sigma_{11}) ** (d2\ \sigma_3) +$
$(l\ \sigma_{11}) ** (e2\ \sigma_4) + (l\ \sigma_{11}) ** (f2\ \sigma_5) + (l\ \sigma_{11}) ** (g2\ \sigma_6) + (l\ \sigma_{11}) ** (h2\ \sigma_7) +$
$(l\ \sigma_{11}) ** (i2\ \sigma_8) + (l\ \sigma_{11}) ** (j2\ \sigma_9) + (l\ \sigma_{11}) ** (k2\ \sigma_{10}) + (l\ \sigma_{11}) ** (l2\ \sigma_{11}) +$
$(l\ \sigma_{11}) ** (m2\ \sigma_{12}) + (l\ \sigma_{11}) ** (n2\ \sigma_{13}) - (l2\ \sigma_{11}) ** (a\ \sigma_0) - (l2\ \sigma_{11}) ** (b\ \sigma_1) -$
$(l2\ \sigma_{11}) ** (c\ \sigma_2) - (l2\ \sigma_{11}) ** (d\ \sigma_3) - (l2\ \sigma_{11}) ** (e\ \sigma_4) - (l2\ \sigma_{11}) ** (f\ \sigma_5) -$
$(l2\ \sigma_{11}) ** (g\ \sigma_6) - (l2\ \sigma_{11}) ** (h\ \sigma_7) - (l2\ \sigma_{11}) ** (i\ \sigma_8) - (l2\ \sigma_{11}) ** (j\ \sigma_9) -$
$(l2\ \sigma_{11}) ** (k\ \sigma_{10}) - (l2\ \sigma_{11}) ** (l\ \sigma_{11}) - (l2\ \sigma_{11}) ** (m\ \sigma_{12}) - (l2\ \sigma_{11}) ** (n\ \sigma_{13}) +$
$(m\ \sigma_{12}) ** (a2\ \sigma_0) + (m\ \sigma_{12}) ** (b2\ \sigma_1) + (m\ \sigma_{12}) ** (c2\ \sigma_2) + (m\ \sigma_{12}) ** (d2\ \sigma_3) +$

(m σ_{12}) ** (e2 σ_{4}) + (m σ_{12}) ** (f2 σ_{5}) + (m σ_{12}) ** (g2 σ_{6}) + (m σ_{12}) ** (h2 σ_{7}) +
(m σ_{12}) ** (i2 σ_{8}) + (m σ_{12}) ** (j2 σ_{9}) + (m σ_{12}) ** (k2 σ_{10}) + (m σ_{12}) ** (l2 σ_{11}) +
(m σ_{12}) ** (m2 σ_{12}) + (m σ_{12}) ** (n2 σ_{13}) - (m2 σ_{12}) ** (a σ_{0}) - (m2 σ_{12}) ** (b σ_{1}) -
(m2 σ_{12}) ** (c σ_{2}) - (m2 σ_{12}) ** (d σ_{3}) - (m2 σ_{12}) ** (e σ_{4}) - (m2 σ_{12}) ** (f σ_{5}) -
(m2 σ_{12}) ** (g σ_{6}) - (m2 σ_{12}) ** (h σ_{7}) - (m2 σ_{12}) ** (i σ_{8}) - (m2 σ_{12}) ** (j σ_{9}) -
(m2 σ_{12}) ** (k σ_{10}) - (m2 σ_{12}) ** (l σ_{11}) - (m2 σ_{12}) ** (m σ_{12}) - (m2 σ_{12}) ** (n σ_{13}) +
(n σ_{13}) ** (a2 σ_{0}) + (n σ_{13}) ** (b2 σ_{1}) + (n σ_{13}) ** (c2 σ_{2}) + (n σ_{13}) ** (d2 σ_{3}) +
(n σ_{13}) ** (e2 σ_{4}) + (n σ_{13}) ** (f2 σ_{5}) + (n σ_{13}) ** (g2 σ_{6}) + (n σ_{13}) ** (h2 σ_{7}) +
(n σ_{13}) ** (i2 σ_{8}) + (n σ_{13}) ** (j2 σ_{9}) + (n σ_{13}) ** (k2 σ_{10}) + (n σ_{13}) ** (l2 σ_{11}) +
(n σ_{13}) ** (m2 σ_{12}) + (n σ_{13}) ** (n2 σ_{13}) - (n2 σ_{13}) ** (a σ_{0}) - (n2 σ_{13}) ** (b σ_{1}) -
(n2 σ_{13}) ** (c σ_{2}) - (n2 σ_{13}) ** (d σ_{3}) - (n2 σ_{13}) ** (e σ_{4}) - (n2 σ_{13}) ** (f σ_{5}) -
(n2 σ_{13}) ** (g σ_{6}) - (n2 σ_{13}) ** (h σ_{7}) - (n2 σ_{13}) ** (i σ_{8}) - (n2 σ_{13}) ** (j σ_{9}) -
(n2 σ_{13}) ** (k σ_{10}) - (n2 σ_{13}) ** (l σ_{11}) - (n2 σ_{13}) ** (m σ_{12}) - (n2 σ_{13}) ** (n σ_{13})

We need to run the output of the Distribute [ ] one more time to have the kuraTable figure out the products:

(a σ_{0}) ** (a2 σ_{0}) + (a σ_{0}) ** (b2 σ_{1}) + (a σ_{0}) ** (c2 σ_{2}) + (a σ_{0}) ** (d2 σ_{3}) + (a σ_{0}) ** (e2 σ_{4}) +
(a σ_{0}) ** (f2 σ_{5}) + (a σ_{0}) ** (g2 σ_{6}) + (a σ_{0}) ** (h2 σ_{7}) + (a σ_{0}) ** (i2 σ_{8}) + (a σ_{0}) ** (j2 σ_{9}) +
(a σ_{0}) ** (k2 σ_{10}) + (a σ_{0}) ** (l2 σ_{11}) + (a σ_{0}) ** (m2 σ_{12}) + (a σ_{0}) ** (n2 σ_{13}) -
(a2 σ_{0}) ** (a σ_{0}) - (a2 σ_{0}) ** (b σ_{1}) - (a2 σ_{0}) ** (c σ_{2}) - (a2 σ_{0}) ** (d σ_{3}) - (a2 σ_{0}) ** (e σ_{4}) -
(a2 σ_{0}) ** (f σ_{5}) - (a2 σ_{0}) ** (g σ_{6}) - (a2 σ_{0}) ** (h σ_{7}) - (a2 σ_{0}) ** (i σ_{8}) - (a2 σ_{0}) ** (j σ_{9}) -
(a2 σ_{0}) ** (k σ_{10}) - (a2 σ_{0}) ** (l σ_{11}) - (a2 σ_{0}) ** (m σ_{12}) - (a2 σ_{0}) ** (n σ_{13}) +
(b σ_{1}) ** (a2 σ_{0}) + (b σ_{1}) ** (b2 σ_{1}) + (b σ_{1}) ** (c2 σ_{2}) + (b σ_{1}) ** (d2 σ_{3}) + (b σ_{1}) ** (e2 σ_{4}) +
(b σ_{1}) ** (f2 σ_{5}) + (b σ_{1}) ** (g2 σ_{6}) + (b σ_{1}) ** (h2 σ_{7}) + (b σ_{1}) ** (i2 σ_{8}) + (b σ_{1}) ** (j2 σ_{9}) +
(b σ_{1}) ** (k2 σ_{10}) + (b σ_{1}) ** (l2 σ_{11}) + (b σ_{1}) ** (m2 σ_{12}) + (b σ_{1}) ** (n2 σ_{13}) -
(b2 σ_{1}) ** (a σ_{0}) - (b2 σ_{1}) ** (b σ_{1}) - (b2 σ_{1}) ** (c σ_{2}) - (b2 σ_{1}) ** (d σ_{3}) - (b2 σ_{1}) ** (e σ_{4}) -
(b2 σ_{1}) ** (f σ_{5}) - (b2 σ_{1}) ** (g σ_{6}) - (b2 σ_{1}) ** (h σ_{7}) - (b2 σ_{1}) ** (i σ_{8}) - (b2 σ_{1}) ** (j σ_{9}) -
(b2 σ_{1}) ** (k σ_{10}) - (b2 σ_{1}) ** (l σ_{11}) - (b2 σ_{1}) ** (m σ_{12}) - (b2 σ_{1}) ** (n σ_{13}) +
(c σ_{2}) ** (a2 σ_{0}) + (c σ_{2}) ** (b2 σ_{1}) + (c σ_{2}) ** (c2 σ_{2}) + (c σ_{2}) ** (d2 σ_{3}) + (c σ_{2}) ** (e2 σ_{4}) +
(c σ_{2}) ** (f2 σ_{5}) + (c σ_{2}) ** (g2 σ_{6}) + (c σ_{2}) ** (h2 σ_{7}) + (c σ_{2}) ** (i2 σ_{8}) + (c σ_{2}) ** (j2 σ_{9}) +
(c σ_{2}) ** (k2 σ_{10}) + (c σ_{2}) ** (l2 σ_{11}) + (c σ_{2}) ** (m2 σ_{12}) + (c σ_{2}) ** (n2 σ_{13}) -
(c2 σ_{2}) ** (a σ_{0}) - (c2 σ_{2}) ** (b σ_{1}) - (c2 σ_{2}) ** (c σ_{2}) - (c2 σ_{2}) ** (d σ_{3}) - (c2 σ_{2}) ** (e σ_{4}) -
(c2 σ_{2}) ** (f σ_{5}) - (c2 σ_{2}) ** (g σ_{6}) - (c2 σ_{2}) ** (h σ_{7}) - (c2 σ_{2}) ** (i σ_{8}) - (c2 σ_{2}) ** (j σ_{9}) -
(c2 σ_{2}) ** (k σ_{10}) - (c2 σ_{2}) ** (l σ_{11}) - (c2 σ_{2}) ** (m σ_{12}) - (c2 σ_{2}) ** (n σ_{13}) +
(d σ_{3}) ** (a2 σ_{0}) + (d σ_{3}) ** (b2 σ_{1}) + (d σ_{3}) ** (c2 σ_{2}) + (d σ_{3}) ** (d2 σ_{3}) + (d σ_{3}) ** (e2 σ_{4}) +
(d σ_{3}) ** (f2 σ_{5}) + (d σ_{3}) ** (g2 σ_{6}) + (d σ_{3}) ** (h2 σ_{7}) + (d σ_{3}) ** (i2 σ_{8}) + (d σ_{3}) ** (j2 σ_{9}) +
(d σ_{3}) ** (k2 σ_{10}) + (d σ_{3}) ** (l2 σ_{11}) + (d σ_{3}) ** (m2 σ_{12}) + (d σ_{3}) ** (n2 σ_{13}) -
(d2 σ_{3}) ** (a σ_{0}) - (d2 σ_{3}) ** (b σ_{1}) - (d2 σ_{3}) ** (c σ_{2}) - (d2 σ_{3}) ** (d σ_{3}) - (d2 σ_{3}) ** (e σ_{4}) -
(d2 σ_{3}) ** (f σ_{5}) - (d2 σ_{3}) ** (g σ_{6}) - (d2 σ_{3}) ** (h σ_{7}) - (d2 σ_{3}) ** (i σ_{8}) - (d2 σ_{3}) ** (j σ_{9}) -
(d2 σ_{3}) ** (k σ_{10}) - (d2 σ_{3}) ** (l σ_{11}) - (d2 σ_{3}) ** (m σ_{12}) - (d2 σ_{3}) ** (n σ_{13}) +
(e σ_{4}) ** (a2 σ_{0}) + (e σ_{4}) ** (b2 σ_{1}) + (e σ_{4}) ** (c2 σ_{2}) + (e σ_{4}) ** (d2 σ_{3}) + (e σ_{4}) ** (e2 σ_{4}) +
(e σ_{4}) ** (f2 σ_{5}) + (e σ_{4}) ** (g2 σ_{6}) + (e σ_{4}) ** (h2 σ_{7}) + (e σ_{4}) ** (i2 σ_{8}) + (e σ_{4}) ** (j2 σ_{9}) +
(e σ_{4}) ** (k2 σ_{10}) + (e σ_{4}) ** (l2 σ_{11}) + (e σ_{4}) ** (m2 σ_{12}) + (e σ_{4}) ** (n2 σ_{13}) -
(e2 σ_{4}) ** (a σ_{0}) - (e2 σ_{4}) ** (b σ_{1}) - (e2 σ_{4}) ** (c σ_{2}) - (e2 σ_{4}) ** (d σ_{3}) - (e2 σ_{4}) ** (e σ_{4}) -
(e2 σ_{4}) ** (f σ_{5}) - (e2 σ_{4}) ** (g σ_{6}) - (e2 σ_{4}) ** (h σ_{7}) - (e2 σ_{4}) ** (i σ_{8}) - (e2 σ_{4}) ** (j σ_{9}) -
(e2 σ_{4}) ** (k σ_{10}) - (e2 σ_{4}) ** (l σ_{11}) - (e2 σ_{4}) ** (m σ_{12}) - (e2 σ_{4}) ** (n σ_{13}) +
(f σ_{5}) ** (a2 σ_{0}) + (f σ_{5}) ** (b2 σ_{1}) + (f σ_{5}) ** (c2 σ_{2}) + (f σ_{5}) ** (d2 σ_{3}) + (f σ_{5}) ** (e2 σ_{4}) +
(f σ_{5}) ** (f2 σ_{5}) + (f σ_{5}) ** (g2 σ_{6}) + (f σ_{5}) ** (h2 σ_{7}) + (f σ_{5}) ** (i2 σ_{8}) + (f σ_{5}) ** (j2 σ_{9}) +
(f σ_{5}) ** (k2 σ_{10}) + (f σ_{5}) ** (l2 σ_{11}) + (f σ_{5}) ** (m2 σ_{12}) + (f σ_{5}) ** (n2 σ_{13}) -
(f2 σ_{5}) ** (a σ_{0}) - (f2 σ_{5}) ** (b σ_{1}) - (f2 σ_{5}) ** (c σ_{2}) - (f2 σ_{5}) ** (d σ_{3}) - (f2 σ_{5}) ** (e σ_{4}) -
(f2 σ_{5}) ** (f σ_{5}) - (f2 σ_{5}) ** (g σ_{6}) - (f2 σ_{5}) ** (h σ_{7}) - (f2 σ_{5}) ** (i σ_{8}) - (f2 σ_{5}) ** (j σ_{9}) -
(f2 σ_{5}) ** (k σ_{10}) - (f2 σ_{5}) ** (l σ_{11}) - (f2 σ_{5}) ** (m σ_{12}) - (f2 σ_{5}) ** (n σ_{13}) +

$(g \sigma_6) ** (a2 \sigma_0) + (g \sigma_6) ** (b2 \sigma_1) + (g \sigma_6) ** (c2 \sigma_2) + (g \sigma_6) ** (d2 \sigma_3) + (g \sigma_6) ** (e2 \sigma_4) +$
$(g \sigma_6) ** (f2 \sigma_5) + (g \sigma_6) ** (g2 \sigma_6) + (g \sigma_6) ** (h2 \sigma_7) + (g \sigma_6) ** (i2 \sigma_8) + (g \sigma_6) ** (j2 \sigma_9) +$
$(g \sigma_6) ** (k2 \sigma_{10}) + (g \sigma_6) ** (l2 \sigma_{11}) + (g \sigma_6) ** (m2 \sigma_{12}) + (g \sigma_6) ** (n2 \sigma_{13}) -$
$(g2 \sigma_6) ** (a \sigma_0) - (g2 \sigma_6) ** (b \sigma_1) - (g2 \sigma_6) ** (c \sigma_2) - (g2 \sigma_6) ** (d \sigma_3) - (g2 \sigma_6) ** (e \sigma_4) -$
$(g2 \sigma_6) ** (f \sigma_5) - (g2 \sigma_6) ** (g \sigma_6) - (g2 \sigma_6) ** (h \sigma_7) - (g2 \sigma_6) ** (i \sigma_8) - (g2 \sigma_6) ** (j \sigma_9) -$
$(g2 \sigma_6) ** (k \sigma_{10}) - (g2 \sigma_6) ** (l \sigma_{11}) - (g2 \sigma_6) ** (m \sigma_{12}) - (g2 \sigma_6) ** (n \sigma_{13}) +$
$(h \sigma_7) ** (a2 \sigma_0) + (h \sigma_7) ** (b2 \sigma_1) + (h \sigma_7) ** (c2 \sigma_2) + (h \sigma_7) ** (d2 \sigma_3) + (h \sigma_7) ** (e2 \sigma_4) +$
$(h \sigma_7) ** (f2 \sigma_5) + (h \sigma_7) ** (g2 \sigma_6) + (h \sigma_7) ** (h2 \sigma_7) + (h \sigma_7) ** (i2 \sigma_8) + (h \sigma_7) ** (j2 \sigma_9) +$
$(h \sigma_7) ** (k2 \sigma_{10}) + (h \sigma_7) ** (l2 \sigma_{11}) + (h \sigma_7) ** (m2 \sigma_{12}) + (h \sigma_7) ** (n2 \sigma_{13}) -$
$(h2 \sigma_7) ** (a \sigma_0) - (h2 \sigma_7) ** (b \sigma_1) - (h2 \sigma_7) ** (c \sigma_2) - (h2 \sigma_7) ** (d \sigma_3) - (h2 \sigma_7) ** (e \sigma_4) -$
$(h2 \sigma_7) ** (f \sigma_5) - (h2 \sigma_7) ** (g \sigma_6) - (h2 \sigma_7) ** (h \sigma_7) - (h2 \sigma_7) ** (i \sigma_8) - (h2 \sigma_7) ** (j \sigma_9) -$
$(h2 \sigma_7) ** (k \sigma_{10}) - (h2 \sigma_7) ** (l \sigma_{11}) - (h2 \sigma_7) ** (m \sigma_{12}) - (h2 \sigma_7) ** (n \sigma_{13}) +$
$(i \sigma_8) ** (a2 \sigma_0) + (i \sigma_8) ** (b2 \sigma_1) + (i \sigma_8) ** (c2 \sigma_2) + (i \sigma_8) ** (d2 \sigma_3) + (i \sigma_8) ** (e2 \sigma_4) +$
$(i \sigma_8) ** (f2 \sigma_5) + (i \sigma_8) ** (g2 \sigma_6) + (i \sigma_8) ** (h2 \sigma_7) + (i \sigma_8) ** (i2 \sigma_8) + (i \sigma_8) ** (j2 \sigma_9) +$
$(i \sigma_8) ** (k2 \sigma_{10}) + (i \sigma_8) ** (l2 \sigma_{11}) + (i \sigma_8) ** (m2 \sigma_{12}) + (i \sigma_8) ** (n2 \sigma_{13}) -$
$(i2 \sigma_8) ** (a \sigma_0) - (i2 \sigma_8) ** (b \sigma_1) - (i2 \sigma_8) ** (c \sigma_2) - (i2 \sigma_8) ** (d \sigma_3) - (i2 \sigma_8) ** (e \sigma_4) -$
$(i2 \sigma_8) ** (f \sigma_5) - (i2 \sigma_8) ** (g \sigma_6) - (i2 \sigma_8) ** (h \sigma_7) - (i2 \sigma_8) ** (i \sigma_8) - (i2 \sigma_8) ** (j \sigma_9) -$
$(i2 \sigma_8) ** (k \sigma_{10}) - (i2 \sigma_8) ** (l \sigma_{11}) - (i2 \sigma_8) ** (m \sigma_{12}) - (i2 \sigma_8) ** (n \sigma_{13}) +$
$(j \sigma_9) ** (a2 \sigma_0) + (j \sigma_9) ** (b2 \sigma_1) + (j \sigma_9) ** (c2 \sigma_2) + (j \sigma_9) ** (d2 \sigma_3) + (j \sigma_9) ** (e2 \sigma_4) +$
$(j \sigma_9) ** (f2 \sigma_5) + (j \sigma_9) ** (g2 \sigma_6) + (j \sigma_9) ** (h2 \sigma_7) + (j \sigma_9) ** (i2 \sigma_8) + (j \sigma_9) ** (j2 \sigma_9) +$
$(j \sigma_9) ** (k2 \sigma_{10}) + (j \sigma_9) ** (l2 \sigma_{11}) + (j \sigma_9) ** (m2 \sigma_{12}) + (j \sigma_9) ** (n2 \sigma_{13}) -$
$(j2 \sigma_9) ** (a \sigma_0) - (j2 \sigma_9) ** (b \sigma_1) - (j2 \sigma_9) ** (c \sigma_2) - (j2 \sigma_9) ** (d \sigma_3) - (j2 \sigma_9) ** (e \sigma_4) -$
$(j2 \sigma_9) ** (f \sigma_5) - (j2 \sigma_9) ** (g \sigma_6) - (j2 \sigma_9) ** (h \sigma_7) - (j2 \sigma_9) ** (i \sigma_8) - (j2 \sigma_9) ** (j \sigma_9) -$
$(j2 \sigma_9) ** (k \sigma_{10}) - (j2 \sigma_9) ** (l \sigma_{11}) - (j2 \sigma_9) ** (m \sigma_{12}) - (j2 \sigma_9) ** (n \sigma_{13}) +$
$(k \sigma_{10}) ** (a2 \sigma_0) + (k \sigma_{10}) ** (b2 \sigma_1) + (k \sigma_{10}) ** (c2 \sigma_2) + (k \sigma_{10}) ** (d2 \sigma_3) +$
$(k \sigma_{10}) ** (e2 \sigma_4) + (k \sigma_{10}) ** (f2 \sigma_5) + (k \sigma_{10}) ** (g2 \sigma_6) + (k \sigma_{10}) ** (h2 \sigma_7) +$
$(k \sigma_{10}) ** (i2 \sigma_8) + (k \sigma_{10}) ** (j2 \sigma_9) + (k \sigma_{10}) ** (k2 \sigma_{10}) + (k \sigma_{10}) ** (l2 \sigma_{11}) +$
$(k \sigma_{10}) ** (m2 \sigma_{12}) + (k \sigma_{10}) ** (n2 \sigma_{13}) - (k2 \sigma_{10}) ** (a \sigma_0) - (k2 \sigma_{10}) ** (b \sigma_1) -$
$(k2 \sigma_{10}) ** (c \sigma_2) - (k2 \sigma_{10}) ** (d \sigma_3) - (k2 \sigma_{10}) ** (e \sigma_4) - (k2 \sigma_{10}) ** (f \sigma_5) -$
$(k2 \sigma_{10}) ** (g \sigma_6) - (k2 \sigma_{10}) ** (h \sigma_7) - (k2 \sigma_{10}) ** (i \sigma_8) - (k2 \sigma_{10}) ** (j \sigma_9) -$
$(k2 \sigma_{10}) ** (k \sigma_{10}) - (k2 \sigma_{10}) ** (l \sigma_{11}) - (k2 \sigma_{10}) ** (m \sigma_{12}) - (k2 \sigma_{10}) ** (n \sigma_{13}) +$
$(l \sigma_{11}) ** (a2 \sigma_0) + (l \sigma_{11}) ** (b2 \sigma_1) + (l \sigma_{11}) ** (c2 \sigma_2) + (l \sigma_{11}) ** (d2 \sigma_3) +$
$(l \sigma_{11}) ** (e2 \sigma_4) + (l \sigma_{11}) ** (f2 \sigma_5) + (l \sigma_{11}) ** (g2 \sigma_6) + (l \sigma_{11}) ** (h2 \sigma_7) +$
$(l \sigma_{11}) ** (i2 \sigma_8) + (l \sigma_{11}) ** (j2 \sigma_9) + (l \sigma_{11}) ** (k2 \sigma_{10}) + (l \sigma_{11}) ** (l2 \sigma_{11}) +$
$(l \sigma_{11}) ** (m2 \sigma_{12}) + (l \sigma_{11}) ** (n2 \sigma_{13}) - (l2 \sigma_{11}) ** (a \sigma_0) - (l2 \sigma_{11}) ** (b \sigma_1) -$
$(l2 \sigma_{11}) ** (c \sigma_2) - (l2 \sigma_{11}) ** (d \sigma_3) - (l2 \sigma_{11}) ** (e \sigma_4) - (l2 \sigma_{11}) ** (f \sigma_5) -$
$(l2 \sigma_{11}) ** (g \sigma_6) - (l2 \sigma_{11}) ** (h \sigma_7) - (l2 \sigma_{11}) ** (i \sigma_8) - (l2 \sigma_{11}) ** (j \sigma_9) -$
$(l2 \sigma_{11}) ** (k \sigma_{10}) - (l2 \sigma_{11}) ** (l \sigma_{11}) - (l2 \sigma_{11}) ** (m \sigma_{12}) - (l2 \sigma_{11}) ** (n \sigma_{13}) +$
$(m \sigma_{12}) ** (a2 \sigma_0) + (m \sigma_{12}) ** (b2 \sigma_1) + (m \sigma_{12}) ** (c2 \sigma_2) + (m \sigma_{12}) ** (d2 \sigma_3) +$
$(m \sigma_{12}) ** (e2 \sigma_4) + (m \sigma_{12}) ** (f2 \sigma_5) + (m \sigma_{12}) ** (g2 \sigma_6) + (m \sigma_{12}) ** (h2 \sigma_7) +$
$(m \sigma_{12}) ** (i2 \sigma_8) + (m \sigma_{12}) ** (j2 \sigma_9) + (m \sigma_{12}) ** (k2 \sigma_{10}) + (m \sigma_{12}) ** (l2 \sigma_{11}) +$
$(m \sigma_{12}) ** (m2 \sigma_{12}) + (m \sigma_{12}) ** (n2 \sigma_{13}) - (m2 \sigma_{12}) ** (a \sigma_0) - (m2 \sigma_{12}) ** (b \sigma_1) -$
$(m2 \sigma_{12}) ** (c \sigma_2) - (m2 \sigma_{12}) ** (d \sigma_3) - (m2 \sigma_{12}) ** (e \sigma_4) - (m2 \sigma_{12}) ** (f \sigma_5) -$
$(m2 \sigma_{12}) ** (g \sigma_6) - (m2 \sigma_{12}) ** (h \sigma_7) - (m2 \sigma_{12}) ** (i \sigma_8) - (m2 \sigma_{12}) ** (j \sigma_9) -$
$(m2 \sigma_{12}) ** (k \sigma_{10}) - (m2 \sigma_{12}) ** (l \sigma_{11}) - (m2 \sigma_{12}) ** (m \sigma_{12}) - (m2 \sigma_{12}) ** (n \sigma_{13}) +$
$(n \sigma_{13}) ** (a2 \sigma_0) + (n \sigma_{13}) ** (b2 \sigma_1) + (n \sigma_{13}) ** (c2 \sigma_2) + (n \sigma_{13}) ** (d2 \sigma_3) +$
$(n \sigma_{13}) ** (e2 \sigma_4) + (n \sigma_{13}) ** (f2 \sigma_5) + (n \sigma_{13}) ** (g2 \sigma_6) + (n \sigma_{13}) ** (h2 \sigma_7) +$
$(n \sigma_{13}) ** (i2 \sigma_8) + (n \sigma_{13}) ** (j2 \sigma_9) + (n \sigma_{13}) ** (k2 \sigma_{10}) + (n \sigma_{13}) ** (l2 \sigma_{11}) +$
$(n \sigma_{13}) ** (m2 \sigma_{12}) + (n \sigma_{13}) ** (n2 \sigma_{13}) - (n2 \sigma_{13}) ** (a \sigma_0) - (n2 \sigma_{13}) ** (b \sigma_1) -$
$(n2 \sigma_{13}) ** (c \sigma_2) - (n2 \sigma_{13}) ** (d \sigma_3) - (n2 \sigma_{13}) ** (e \sigma_4) - (n2 \sigma_{13}) ** (f \sigma_5) -$
$(n2 \sigma_{13}) ** (g \sigma_6) - (n2 \sigma_{13}) ** (h \sigma_7) - (n2 \sigma_{13}) ** (i \sigma_8) - (n2 \sigma_{13}) ** (j \sigma_9) -$
$(n2 \sigma_{13}) ** (k \sigma_{10}) - (n2 \sigma_{13}) ** (l \sigma_{11}) - (n2 \sigma_{13}) ** (m \sigma_{12}) - (n2 \sigma_{13}) ** (n \sigma_{13})$

And we get:

$$
\begin{aligned}
& b2\ d\ \sigma_{2} - b\ d2\ \sigma_{2} - b2\ e\ \sigma_{2} - d2\ e\ \sigma_{2} + b\ e2\ \sigma_{2} + d\ e2\ \sigma_{2} + b2\ c\ \sigma_{3} - b\ c2\ \sigma_{3} - c2\ d\ \sigma_{3} + c\ d2\ \sigma_{3} - \\
& b2\ f\ \sigma_{3} - d2\ f\ \sigma_{3} + b\ f2\ \sigma_{3} + d\ f2\ \sigma_{3} - b2\ c\ \sigma_{4} + b\ c2\ \sigma_{4} + c2\ e\ \sigma_{4} - c\ e2\ \sigma_{4} + b2\ f\ \sigma_{4} + e2\ f\ \sigma_{4} - \\
& b\ f2\ \sigma_{4} - e\ f2\ \sigma_{4} - b2\ d\ \sigma_{5} + b\ d2\ \sigma_{5} + b2\ e\ \sigma_{5} + d2\ e\ \sigma_{5} - b\ e2\ \sigma_{5} - d\ e2\ \sigma_{5} + c2\ d\ \sigma_{6} - c\ d2\ \sigma_{6} - \\
& c2\ e\ \sigma_{6} + c\ e2\ \sigma_{6} + b2\ h\ \sigma_{6} + e2\ h\ \sigma_{6} + g2\ h\ \sigma_{6} - b\ h2\ \sigma_{6} - e\ h2\ \sigma_{6} - g\ h2\ \sigma_{6} - b2\ i\ \sigma_{6} - d2\ i\ \sigma_{6} - \\
& g2\ i\ \sigma_{6} + b\ i2\ \sigma_{6} + d\ i2\ \sigma_{6} + g\ i2\ \sigma_{6} - d2\ k\ \sigma_{6} + e2\ k\ \sigma_{6} + d\ k2\ \sigma_{6} - e\ k2\ \sigma_{6} + c2\ l\ \sigma_{6} + i2\ l\ \sigma_{6} + \\
& k2\ l\ \sigma_{6} - c\ l2\ \sigma_{6} - i\ l2\ \sigma_{6} - k\ l2\ \sigma_{6} - c2\ m\ \sigma_{6} - h2\ m\ \sigma_{6} - k2\ m\ \sigma_{6} + c\ m2\ \sigma_{6} + h\ m2\ \sigma_{6} + \\
& k\ m2\ \sigma_{6} - c2\ f\ \sigma_{7} + c\ f2\ \sigma_{7} + b2\ g\ \sigma_{7} + d2\ g\ \sigma_{7} - b\ g2\ \sigma_{7} - d\ g2\ \sigma_{7} - c2\ h\ \sigma_{7} + f2\ h\ \sigma_{7} + c\ h2\ \sigma_{7} - \\
& f\ h2\ \sigma_{7} - b2\ j\ \sigma_{7} - d2\ j\ \sigma_{7} - g2\ j\ \sigma_{7} + b\ j2\ \sigma_{7} + d\ j2\ \sigma_{7} + g\ j2\ \sigma_{7} + f2\ k\ \sigma_{7} + h2\ k\ \sigma_{7} - f\ k2\ \sigma_{7} - \\
& h\ k2\ \sigma_{7} - g2\ l\ \sigma_{7} + j2\ l\ \sigma_{7} + g\ l2\ \sigma_{7} - j\ l2\ \sigma_{7} - c2\ n\ \sigma_{7} - h2\ n\ \sigma_{7} - k2\ n\ \sigma_{7} + c\ n2\ \sigma_{7} + h\ n2\ \sigma_{7} + \\
& k\ n2\ \sigma_{7} + c2\ f\ \sigma_{8} - c\ f2\ \sigma_{8} - b2\ g\ \sigma_{8} - e2\ g\ \sigma_{8} + b\ g2\ \sigma_{8} + e\ g2\ \sigma_{8} + c2\ i\ \sigma_{8} - f2\ i\ \sigma_{8} - c\ i2\ \sigma_{8} + \\
& f\ i2\ \sigma_{8} + b2\ j\ \sigma_{8} + e2\ j\ \sigma_{8} + g2\ j\ \sigma_{8} - b\ j2\ \sigma_{8} - e\ j2\ \sigma_{8} - g\ j2\ \sigma_{8} - f2\ k\ \sigma_{8} - i2\ k\ \sigma_{8} + f\ k2\ \sigma_{8} + \\
& i\ k2\ \sigma_{8} + g2\ m\ \sigma_{8} - j2\ m\ \sigma_{8} - g\ m2\ \sigma_{8} + j\ m2\ \sigma_{8} + c2\ n\ \sigma_{8} + i2\ n\ \sigma_{8} + k2\ n\ \sigma_{8} - c\ n2\ \sigma_{8} - \\
& i\ n2\ \sigma_{8} - k\ n2\ \sigma_{8} + d2\ f\ \sigma_{9} - e2\ f\ \sigma_{9} - d\ f2\ \sigma_{9} + e\ f2\ \sigma_{9} - b2\ h\ \sigma_{9} - e2\ h\ \sigma_{9} + b\ h2\ \sigma_{9} + e\ h2\ \sigma_{9} + \\
& b2\ i\ \sigma_{9} + d2\ i\ \sigma_{9} - b\ i2\ \sigma_{9} - d\ i2\ \sigma_{9} + h2\ j\ \sigma_{9} - i2\ j\ \sigma_{9} - h\ j2\ \sigma_{9} + i\ j2\ \sigma_{9} - f2\ l\ \sigma_{9} - i2\ l\ \sigma_{9} + \\
& f\ l2\ \sigma_{9} + i\ l2\ \sigma_{9} + f2\ m\ \sigma_{9} + h2\ m\ \sigma_{9} - f\ m2\ \sigma_{9} - h\ m2\ \sigma_{9} + d2\ n\ \sigma_{9} - e2\ n\ \sigma_{9} + l2\ n\ \sigma_{9} - \\
& m2\ n\ \sigma_{9} - d\ n2\ \sigma_{9} + e\ n2\ \sigma_{9} - l\ n2\ \sigma_{9} + m\ n2\ \sigma_{9} - d2\ g\ \sigma_{10} + e2\ g\ \sigma_{10} + d\ g2\ \sigma_{10} - e\ g2\ \sigma_{10} + \\
& c2\ h\ \sigma_{10} - c\ h2\ \sigma_{10} - c2\ i\ \sigma_{10} - h2\ i\ \sigma_{10} + c\ i2\ \sigma_{10} + h\ i2\ \sigma_{10} - h2\ k\ \sigma_{10} + i2\ k\ \sigma_{10} + h\ k2\ \sigma_{10} - \\
& i\ k2\ \sigma_{10} + b2\ l\ \sigma_{10} + e2\ l\ \sigma_{10} + g2\ l\ \sigma_{10} - b\ l2\ \sigma_{10} - e\ l2\ \sigma_{10} - g\ l2\ \sigma_{10} - b2\ m\ \sigma_{10} - d2\ m\ \sigma_{10} - \\
& g2\ m\ \sigma_{10} - l2\ m\ \sigma_{10} + b\ m2\ \sigma_{10} + d\ m2\ \sigma_{10} + g\ m2\ \sigma_{10} + l\ m2\ \sigma_{10} + f2\ g\ \sigma_{11} - f\ g2\ \sigma_{11} - g2\ h\ \sigma_{11} + \\
& g\ h2\ \sigma_{11} - c2\ j\ \sigma_{11} - h2\ j\ \sigma_{11} + c\ j2\ \sigma_{11} + h\ j2\ \sigma_{11} + b2\ k\ \sigma_{11} + d2\ k\ \sigma_{11} + j2\ k\ \sigma_{11} - b\ k2\ \sigma_{11} - \\
& d\ k2\ \sigma_{11} - j\ k2\ \sigma_{11} - c2\ l\ \sigma_{11} + f2\ l\ \sigma_{11} - k2\ l\ \sigma_{11} + c\ l2\ \sigma_{11} - f\ l2\ \sigma_{11} + k\ l2\ \sigma_{11} - b2\ n\ \sigma_{11} - \\
& d2\ n\ \sigma_{11} - g2\ n\ \sigma_{11} - l2\ n\ \sigma_{11} + b\ n2\ \sigma_{11} + d\ n2\ \sigma_{11} + g\ n2\ \sigma_{11} + l\ n2\ \sigma_{11} - f2\ g\ \sigma_{12} + f\ g2\ \sigma_{12} + \\
& g2\ i\ \sigma_{12} - g\ i2\ \sigma_{12} + c2\ j\ \sigma_{12} + i2\ j\ \sigma_{12} - c\ j2\ \sigma_{12} - i\ j2\ \sigma_{12} - b2\ k\ \sigma_{12} - e2\ k\ \sigma_{12} - j2\ k\ \sigma_{12} + \\
& b\ k2\ \sigma_{12} + e\ k2\ \sigma_{12} + j\ k2\ \sigma_{12} + c2\ m\ \sigma_{12} - f2\ m\ \sigma_{12} + k2\ m\ \sigma_{12} - c\ m2\ \sigma_{12} + f\ m2\ \sigma_{12} - k\ m2\ \sigma_{12} + \\
& b2\ n\ \sigma_{12} + e2\ n\ \sigma_{12} + g2\ n\ \sigma_{12} + m2\ n\ \sigma_{12} - b\ n2\ \sigma_{12} - e\ n2\ \sigma_{12} - g\ n2\ \sigma_{12} - m\ n2\ \sigma_{12} - f2\ h\ \sigma_{13} + \\
& f\ h2\ \sigma_{13} + f2\ i\ \sigma_{13} + h2\ i\ \sigma_{13} - f\ i2\ \sigma_{13} - h\ i2\ \sigma_{13} + d2\ j\ \sigma_{13} - e2\ j\ \sigma_{13} - d\ j2\ \sigma_{13} + e\ j2\ \sigma_{13} - \\
& b2\ l\ \sigma_{13} - e2\ l\ \sigma_{13} - j2\ l\ \sigma_{13} + b\ l2\ \sigma_{13} + e\ l2\ \sigma_{13} + j\ l2\ \sigma_{13} + b2\ m\ \sigma_{13} + d2\ m\ \sigma_{13} + j2\ m\ \sigma_{13} + \\
& l2\ m\ \sigma_{13} - b\ m2\ \sigma_{13} - d\ m2\ \sigma_{13} - j\ m2\ \sigma_{13} - l\ m2\ \sigma_{13} + h2\ n\ \sigma_{13} - i2\ n\ \sigma_{13} - h\ n2\ \sigma_{13} + i\ n2\ \sigma_{13}
\end{aligned}
$$

Collect and organize the coefficients:

```
Collect[b2 d σ2 - b d2 σ2 - b2 e σ2 - d2 e σ2 + b e2 σ2 + d e2 σ2 + b2 c σ3 - b c2 σ3 - c2 d σ3 +
  c d2 σ3 - b2 f σ3 - d2 f σ3 + b f2 σ3 + d f2 σ3 - b2 c σ4 + b c2 σ4 + c2 e σ4 - c e2 σ4 + b2 f σ4 +
  e2 f σ4 - b f2 σ4 - e f2 σ4 - b2 d σ5 + b d2 σ5 + b2 e σ5 + d2 e σ5 - b e2 σ5 - d e2 σ5 + c2 d σ6 -
  c d2 σ6 - c2 e σ6 + c e2 σ6 + b2 h σ6 + e2 h σ6 + g2 h σ6 - b h2 σ6 - e h2 σ6 - g h2 σ6 - b2 i σ6 -
  d2 i σ6 - g2 i σ6 + b i2 σ6 + d i2 σ6 + g i2 σ6 - d2 k σ6 + e2 k σ6 + d k2 σ6 - e k2 σ6 + c2 l σ6 +
  i2 l σ6 + k2 l σ6 - c l2 σ6 - i l2 σ6 - k l2 σ6 - c2 m σ6 - h2 m σ6 - k2 m σ6 + c m2 σ6 + h m2 σ6 +
  k m2 σ6 - c2 f σ7 + c f2 σ7 + b2 g σ7 + d2 g σ7 - b g2 σ7 - d g2 σ7 - c2 h σ7 + f2 h σ7 + c h2 σ7 -
  f h2 σ7 - b2 j σ7 - d2 j σ7 - g2 j σ7 + b j2 σ7 + d j2 σ7 + g j2 σ7 + f2 k σ7 + h2 k σ7 - f k2 σ7 -
  h k2 σ7 - g2 l σ7 + j2 l σ7 + g l2 σ7 - j l2 σ7 - c2 n σ7 - h2 n σ7 - k2 n σ7 + c n2 σ7 + h n2 σ7 +
  k n2 σ7 + c2 f σ8 - c f2 σ8 - b2 g σ8 - e2 g σ8 + b g2 σ8 + e g2 σ8 + c2 i σ8 - f2 i σ8 - c i2 σ8 +
  f i2 σ8 + b2 j σ8 + e2 j σ8 + g2 j σ8 - b j2 σ8 - e j2 σ8 - g j2 σ8 - f2 k σ8 - i2 k σ8 + f k2 σ8 +
  i k2 σ8 + g2 m σ8 - j2 m σ8 - g m2 σ8 + j m2 σ8 + c2 n σ8 + i2 n σ8 + k2 n σ8 - c n2 σ8 -
  i n2 σ8 - k n2 σ8 + d2 f σ9 - e2 f σ9 - d f2 σ9 + e f2 σ9 - b2 h σ9 - e2 h σ9 + b h2 σ9 + e h2 σ9 +
  b2 i σ9 + d2 i σ9 - b i2 σ9 - d i2 σ9 + h2 j σ9 - i2 j σ9 - h j2 σ9 + i j2 σ9 - f2 l σ9 - i2 l σ9 +
  f l2 σ9 + i l2 σ9 + f2 m σ9 + h2 m σ9 - f m2 σ9 - h m2 σ9 + d2 n σ9 - e2 n σ9 + l2 n σ9 -
  m2 n σ9 - d n2 σ9 + e n2 σ9 - l n2 σ9 + m n2 σ9 - d2 g σ10 + e2 g σ10 + d g2 σ10 - e g2 σ10 +
  c2 h σ10 - c h2 σ10 - c2 i σ10 - h2 i σ10 + c i2 σ10 + h i2 σ10 - h2 k σ10 + i2 k σ10 + h k2 σ10 -
  i k2 σ10 + b2 l σ10 + e2 l σ10 + g2 l σ10 - b l2 σ10 - e l2 σ10 - g l2 σ10 - b2 m σ10 - d2 m σ10 -
  g2 m σ10 - l2 m σ10 + b m2 σ10 + d m2 σ10 + g m2 σ10 + l m2 σ10 + f2 g σ11 - f g2 σ11 - g2 h σ11 +
  g h2 σ11 - c2 j σ11 - h2 j σ11 + c j2 σ11 + h j2 σ11 + b2 k σ11 + d2 k σ11 + j2 k σ11 - b k2 σ11 -
  d k2 σ11 - j k2 σ11 - c2 l σ11 + f2 l σ11 - k2 l σ11 + c l2 σ11 - f l2 σ11 + k l2 σ11 - b2 n σ11 -
  d2 n σ11 - g2 n σ11 - l2 n σ11 + b n2 σ11 + d n2 σ11 + g n2 σ11 + l n2 σ11 - f2 g σ12 + f g2 σ12 +
  g2 i σ12 - g i2 σ12 + c2 j σ12 + i2 j σ12 - c j2 σ12 - i j2 σ12 - b2 k σ12 - e2 k σ12 - j2 k σ12 +
  b k2 σ12 + e k2 σ12 + j k2 σ12 + c2 m σ12 - f2 m σ12 + k2 m σ12 - c m2 σ12 + f m2 σ12 - k m2 σ12 +
  b2 n σ12 + e2 n σ12 + g2 n σ12 + m2 n σ12 - b n2 σ12 - e n2 σ12 - g n2 σ12 - m n2 σ12 - f2 h σ13 +
  f h2 σ13 + f2 i σ13 + h2 i σ13 - f i2 σ13 - h i2 σ13 + d2 j σ13 - e2 j σ13 - d j2 σ13 + e j2 σ13 -
  b2 l σ13 - e2 l σ13 - j2 l σ13 + b l2 σ13 + e l2 σ13 + j l2 σ13 + b2 m σ13 + d2 m σ13 + j2 m σ13 +
  l2 m σ13 - b m2 σ13 - d m2 σ13 - j m2 σ13 - l m2 σ13 + h2 n σ13 - i2 n σ13 - h n2 σ13 + i n2 σ13,
 {σ0, σ1, σ2, σ3, σ4, σ5, σ6, σ7, σ8, σ9, σ10, σ11, σ12, σ13}]
```

Here are the coefficients per Kuratowski operators:

(b2 d - b d2 - b2 e - d2 e + b e2 + d e2) σ_2 + (b2 c - b c2 - c2 d + c d2 - b2 f - d2 f + b f2 + d f2) σ_3 +
(-b2 c + b c2 + c2 e - c e2 + b2 f + e2 f - b f2 - e f2) σ_4 +
(-b2 d + b d2 + b2 e + d2 e - b e2 - d e2) σ_5 +
(c2 d - c d2 - c2 e + c e2 + b2 h + e2 h + g2 h - b h2 - e h2 - g h2 -
b2 i - d2 i - g2 i + b i2 + d i2 + g i2 - d2 k + e2 k + d k2 - e k2 + c2 l +
i2 l + k2 l - c l2 - i l2 - k l2 - c2 m - h2 m - k2 m + c m2 + h m2 + k m2) σ_6 +
(-c2 f + c f2 + b2 g + d2 g - b g2 - d g2 - c2 h + f2 h + c h2 - f h2 - b2 j -
d2 j - g2 j + b j2 + d j2 + g j2 + f2 k + h2 k - f k2 - h k2 - g2 l +
j2 l + g l2 - j l2 - c2 n - h2 n - k2 n + c n2 + h n2 + k n2) σ_7 +
(c2 f - c f2 - b2 g - e2 g + b g2 + e g2 + c2 i - f2 i - c i2 + f i2 + b2 j + e2 j + g2 j - b j2 - e j2 -
g j2 - f2 k - i2 k + f k2 + i k2 + g2 m - j2 m - g m2 + j m2 + c2 n + i2 n + k2 n - c n2 - i n2 - k n2)
σ_8 + (d2 f - e2 f - d f2 + e f2 - b2 h - e2 h + b h2 + e h2 + b2 i + d2 i - b i2 -
d i2 + h2 j - i2 j - h j2 + i j2 - f2 l - i2 l + f l2 + i l2 + f2 m + h2 m -
f m2 - h m2 + d2 n - e2 n + l2 n - m2 n - d n2 + e n2 - l n2 + m n2) σ_9 +
(-d2 g + e2 g + d g2 - e g2 + c2 h - c h2 - c2 i - h2 i + c i2 + h i2 - h2 k + i2 k + h k2 - i k2 +
b2 l + e2 l + g2 l - b l2 - e l2 - g l2 - b2 m - d2 m - g2 m - l2 m + b m2 + d m2 + g m2 + l m2) σ_{10} +
(f2 g - f g2 - g2 h + g h2 - c2 j - h2 j + c j2 + h j2 + b2 k + d2 k + j2 k - b k2 - d k2 - j k2 -
c2 l + f2 l - k2 l + c l2 - f l2 + k l2 - b2 n - d2 n - g2 n - l2 n + b n2 + d n2 + g n2 + l n2) σ_{11} +
(-f2 g + f g2 + g2 i - g i2 + c2 j + i2 j - c j2 - i j2 - b2 k - e2 k - j2 k + b k2 + e k2 + j k2 +
c2 m - f2 m + k2 m - c m2 + f m2 - k m2 + b2 n + e2 n + g2 n + m2 n - b n2 - e n2 - g n2 - m n2) σ_{12} +
(-f2 h + f h2 + f2 i + h2 i - f i2 - h i2 + d2 j - e2 j - d j2 + e j2 - b2 l - e2 l - j2 l + b l2 +
e l2 + j l2 + b2 m + d2 m + j2 m + l2 m - b m2 - d m2 - j m2 - l m2 + h2 n - i2 n - h n2 + i n2) σ_{13}

Set all coefficients equal to zero and solve:

```
Solve[{b2 d - b d2 - b2 e - d2 e + b e2 + d e2, b2 c - b c2 - c2 d + c d2 - b2 f - d2 f + b f2 + d f2,
    -b2 c + b c2 + c2 e - c e2 + b2 f + e2 f - b f2 - e f2, -b2 d + b d2 + b2 e + d2 e - b e2 - d e2,
    c2 d - c d2 - c2 e + c e2 + b2 h + e2 h + g2 h - b h2 - e h2 - g h2 -
     b2 i - d2 i - g2 i + b i2 + d i2 + g i2 - d2 k + e2 k + d k2 - e k2 + c2 l +
     i2 l + k2 l - c l2 - i l2 - k l2 - c2 m - h2 m - k2 m + c m2 + h m2 + k m2,
    -c2 f + c f2 + b2 g + d2 g - b g2 - d g2 - c2 h + f2 h + c h2 - f h2 - b2 j - d2 j - g2 j + b j2 + d j2 +
     g j2 + f2 k + h2 k - f k2 - h k2 - g2 l + j2 l + g l2 - j l2 - c2 n - h2 n - k2 n + c n2 + h n2 + k n2,
    c2 f - c f2 - b2 g - e2 g + b g2 + e g2 + c2 i - f2 i - c i2 + f i2 + b2 j + e2 j + g2 j - b j2 - e j2 -
     g j2 - f2 k - i2 k + f k2 + i k2 + g2 m - j2 m - g m2 + j m2 + c2 n + i2 n + k2 n - c n2 - i n2 - k n2,
    d2 f - e2 f - d f2 + e f2 - b2 h - e2 h + b h2 + e h2 + b2 i + d2 i - b i2 - d i2 +
     h2 j - i2 j - h j2 + i j2 - f2 l - i2 l + f l2 + i l2 + f2 m + h2 m -
     f m2 - h m2 + d2 n - e2 n + l2 n - m2 n - d n2 + e n2 - l n2 + m n2,
    -d2 g + e2 g + d g2 - e g2 + c2 h - c h2 - c2 i - h2 i + c i2 + h i2 - h2 k + i2 k + h k2 - i k2 +
     b2 l + e2 l + g2 l - b l2 - e l2 - g l2 - b2 m - d2 m - g2 m - l2 m + b m2 + d m2 + g m2 + l m2,
    f2 g - f g2 - g2 h + g h2 - c2 j - h2 j + c j2 + h j2 + b2 k + d2 k + j2 k - b k2 - d k2 - j k2 -
     c2 l + f2 l - k2 l + c l2 - f l2 + k l2 - b2 n - d2 n - g2 n - l2 n + b n2 + d n2 + g n2 + l n2,
    -f2 g + f g2 + g2 i - g i2 + c2 j + i2 j - c j2 - i j2 - b2 k - e2 k - j2 k + b k2 + e k2 + j k2 +
     c2 m - f2 m + k2 m - c m2 + f m2 - k m2 + b2 n + e2 n + g2 n + m2 n - b n2 - e n2 - g n2 - m n2,
    -f2 h + f h2 + f2 i + h2 i - f i2 - h i2 + d2 j - e2 j - d j2 + e j2 - b2 l - e2 l - j2 l + b l2 +
     e l2 + j l2 + b2 m + d2 m + j2 m + l2 m - b m2 - d m2 - j m2 - l m2 + h2 n - i2 n - h n2 + i n2} ==
  ConstantArray[0, 12], {b2, c2, d2, e2, f2, g2, h2, i2, j2, k2, l2, m2, n2}]
```

```
{{e2 → -d2 (-b-e)/(b+d) - b2 (d-e)/(b+d), f2 → c2 - b2 (c-f)/(b+d) - d2 (c-f)/(b+d),
  j2 → g2 + (i2 (-(c + h + k)^2 + (b + d + g + l)^2) (-(g - j) (b + d + g + l) + (c + h + k) (c - f + k - n))) /
       (((c + h + k)^2 - (b + d + g + l)^2) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m))) -
     (c2 (d g - e g - c h + f h + c i - f i - d j + e j - h k + i k + g l - j l - g m + j m + h n - i n)) /
      (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
     (k2 (d g - e g - c h + f h + c i - f i - d j + e j - h k + i k + g l - j l - g m + j m + h n - i n)) /
      (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
     (h2 (c^2 - c f - b g - e g - g^2 + c i - f i + b j + e j + g j + 2 c k - f k + i k + k^2 - g m + j m - c n - i n - k n)) /
      (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m),
  l2 → -b2 - d2 - g2 - i2 (-(c+h+k)^2 + (b+d+g+l)^2)/((-c-i-k) (b+d+g+l) - (c+h+k) (-b-e-g-m)) - (h2 (-(-c - i - k) (c + h + k) + (b + d + g + l) (-b - e - g - m))) /
      ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)) - c2 (-(h-i) (c+h+k) + (b+d+g+l) (d-e+l-m))/((-c-i-k) (b+d+g+l) - (c+h+k) (-b-e-g-m)) - k2 (-(h-i) (c+h+k) + (b+d+g+l) (d-e+l-m))/((-c-i-k) (b+d+g+l) - (c+h+k) (-b-e-g-m)),
  m2 → -b2 (b+e)/(b+d) - d2 (b+e)/(b+d) - g2 - i2 ((c+h+k)/(b+d+g+l) - ((-(c + h + k)^2 + (b + d + g + l)^2) (-b - e - g - m)) /
          ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))) -
     h2 ((-c-i-k)/(b+d+g+l) - ((-(-c - i - k) (c + h + k) + (b + d + g + l) (-b - e - g - m)) (-b - e - g - m)) /
          ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))) -
     c2 ((h-i)/(b+d+g+l) - ((-(h - i) (c + h + k) + (b + d + g + l) (d - e + l - m)) (-b - e - g - m)) /
          ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))) -
     k2 ((h-i)/(b+d+g+l) - ((-(h - i) (c + h + k) + (b + d + g + l) (d - e + l - m)) (-b - e - g - m)) /
          ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))),
  n2 → -b2 (-c+f)/(b+d) - d2 (-c+f)/(b+d) - (i2 (b c + c d - b f - d f - f g - g h + c j + h j + b k + d k + j k + c l - f l + k l - b n - d n - g n - l n)) /
      (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
     (h2 (-b c - c e + b f + e f + f g + g i - c j - i j - b k - e k - j k - c m + f m - k m + b n + e n + g n + m n)) /
      (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
     (k2 (-d f + e f + b h + e h - b i - d i + h j - i j - f l - i l + f m + h m - d n + e n - l n + m n)) /
      (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
     (c2 (c d - c e - d f + e f - g h + g i + h sj - i j + d k - e k + c l - f l + k l - c m + f m - k m - d n + e n - l n + m n)) /
      (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m)}}
```

(EQ 14.1)

Let’s test the solutions to see if we get **0** commutation, u1 an arbitrary set of coefficients while u2 constrained as the latter solution EQ 14.1:

```
u1 = coeff;
u2 = coeff /. {a → a2, b → b2, c → c2, d → d2, e → e2,
    f → f2, g → g2, h → h2, i → i2, j → j2, k → k2, l → l2, m → m2, n → n2};
u2 = u2 /. {e2 → -(d2 (-b - e))/(b + d) - (b2 (d - e))/(b + d), f2 → c2 - (b2 (c - f))/(b + d) - (d2 (c - f))/(b + d), j2 → g2 +
      (i2 (-(c + h + k)^2 + (b + d + g + l)^2) (-(g - j) (b + d + g + l) + (c + h + k) (c - f + k - n))) /
       (((c + h + k)^2 - (b + d + g + l)^2) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m))) -
      (c2 (d g - e g - c h + f h + c i - f i - d j + e j - h k + i k + g l - j l - g m + j m + h n - i n)) /
       (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
      (k2 (d g - e g - c h + f h + c i - f i - d j + e j - h k + i k + g l - j l - g m + j m + h n - i n)) /
       (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
      (h2 (c^2 - c f - b g - e g - g^2 + c i - f i + b j + e j + g j + 2 c k -
```

```
        f k + i k + k^2 - g m + j m - c n - i n - k n)) /
     (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m),
 l2 → -b2 - d2 - g2 - (i2 (-(c + h + k)^2 + (b + d + g + l)^2))/((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)) -
    (h2 (-(-c - i - k) (c + h + k) + (b + d + g + l) (-b - e - g - m))) /
     ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)) -
    (c2 (-(h - i) (c + h + k) + (b + d + g + l) (d - e + l - m)))/((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)) -
    (k2 (-(h - i) (c + h + k) + (b + d + g + l) (d - e + l - m)))/((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)), m2 → -(b2 (b + e))/(b + d) -
    (d2 (b + e))/(b + d) - g2 - i2 ((c + h + k)/(b + d + g + l) - ((-(c + h + k)^2 + (b + d + g + l)^2) (-b - e - g - m)) /
        ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))) -
    h2 ((-c - i - k)/(b + d + g + l) - ((-(-c - i - k) (c + h + k) + (b + d + g + l) (-b - e - g - m)) (-b - e - g -
           m)) / ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))) -
    c2 ((h - i)/(b + d + g + l) - ((-(h - i) (c + h + k) + (b + d + g + l) (d - e + l - m)) (-b - e - g - m)) /
        ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))) -
    k2 ((h - i)/(b + d + g + l) - ((-(h - i) (c + h + k) + (b + d + g + l) (d - e + l - m)) (-b - e - g - m)) /
        ((b + d + g + l) ((-c - i - k) (b + d + g + l) - (c + h + k) (-b - e - g - m)))),
 n2 → -(b2 (-c + f))/(b + d) - (d2 (-c + f))/(b + d) - (i2 (b c + c d - b f - d f - f g - g h + c j +
        h j + b k + d k + j k + c l - f l + k l - b n - d n - g n - l n)) /
     (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
    (h2 (-b c - c e + b f + e f + f g + g i - c j - i j - b k - e k -
        j k - c m + f m - k m + b n + e n + g n + m n)) /
     (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
    (k2 (-d f + e f + b h + e h - b i - d i + h j - i j - f l - i l + f m + h m - d n + e n - l n + m n)) /
     (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m) -
    (c2 (c d - c e - d f + e f - g h + g i + h j - i j + d k - e k + c l -
        f l + k l - c m + f m - k m - d n + e n - l n + m n)) /
     (c d - c e - b h - e h - g h + b i + d i + g i + d k - e k + c l + i l + k l - c m - h m - k m)};

Simplify[Expand[u2.u1 - u1.u2]] // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

By the above computation one can easily convince:

Lemma 14.1: *Commutator for a 14-Sets vector representing a discrete topological space i.e. first two coefficients not zero followed by all 0 coefficients, again has to be discrete.*

Proof: modify the first two lines of the above code as follows:

```
coeff = MatrixRep14Sets[{a, b, c, d, e, f, g, h, i, j, k, l, m, n}];
u1 = MatrixRep14Sets[{a, b, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}];
```

And you get these as the first 2 columns of u2:

{a2,b2,0,0,0,0,0,0,0,0,0,0,0,0}

{b2,a2,0,0,0,0,0,0,0,0,0,0,0,0}

In other words the Commutator solution is another discrete space.

By definition in section 17, feeling the same space requires the corresponding 14-Sets vectors to commute or:

Corollary 14.1: *Discrete topological spaces have the same feeling for space.*

By Lemma 14.1 a natural conclusion:

Corollary 14.2: *Discrete and non-discrete spaces do not have the same feel.*

By similar simple calculations:

Corollary 14.3: *Assuming v is a stochastic 14-Sets vector, v^{∞} and v commute. Therefore v^{∞} and v feel the same space!*

15. Pre-Geometry: Lie Algebra and Logarithm

Lie group $M(n, \mathbb{R})$ is endowed with the Lie Algebra [2]:

$$L_{k,l}^{i,j} = \delta_{i,k}\,\delta_{j,l} - \delta_{j,k}\,\delta_{j,l} \qquad \text{EQ 15.1}$$

Or all members of $M(14, \mathbb{R})$ can be written as the exponential sum:

$$e^{\sum_{i,j=1}^{14} \alpha_{i,j} L^{i,j}} \qquad \alpha_{i,j} \in \mathbb{R} \text{ or } \mathbb{C} \qquad \text{EQ 15.2}$$

We omit the k, l subscripts and indeed place an integer underneath the symbol to indicate the dimension n:

$\underset{14}{\mathbf{L}}^{1,2}$ **// MatrixForm**

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Note that $L^{i,i} = 0$ matrix:

```
L^{2,2}_{14} // MatrixForm
```

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

```
L^{1,3}_{14} // MatrixForm
```

$$\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Lemma 15.1: *Sum of the columns of the Lie Algebra elements add up to 0*:

```
sumColumn[L^{1,3}_{14}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The dimension of this Lie Algebra is:

$n^2 - n$ EQ 15.3

For the case n = 14, the dimension is $14^2 - 14 = 14 \times 13 = 182$. Therefore this dimension is quite large in comparison to dimension n of $M(n, \mathbb{R})$ i.e. 14.

By fixing the index j for a particular column and varying i we can generate any column of numbers we wish, by solving for individual entries (i, j) at each row i:

$$^{1}_{14}\alpha$$

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}\}$$

Their linear combination in the Lie Algebra:

$$\mathbf{Style}\Big[\Big(\alpha_1 * \underset{14}{\mathbf{L}}^{1,1} + \alpha_2 * \underset{14}{\mathbf{L}}^{2,1} + \alpha_3 * \underset{14}{\mathbf{L}}^{3,1} + \alpha_4 * \underset{14}{\mathbf{L}}^{4,1} + \alpha_5 * \underset{14}{\mathbf{L}}^{5,1} + \alpha_6 * \underset{14}{\mathbf{L}}^{6,1} + \alpha_7 * \underset{14}{\mathbf{L}}^{7,1} + \alpha_8 * \underset{14}{\mathbf{L}}^{8,1} + \alpha_9 * \underset{14}{\mathbf{L}}^{9,1} + \alpha_{10} * \underset{14}{\mathbf{L}}^{10,1} + \alpha_{11} * \underset{14}{\mathbf{L}}^{11,1} + \alpha_{12} * \underset{14}{\mathbf{L}}^{12,1} + \alpha_{13} * \underset{14}{\mathbf{L}}^{13,1} + \alpha_{14} * \underset{14}{\mathbf{L}}^{14,1}\Big) \;//\; \mathbf{MatrixForm},\ \mathbf{FontSize} \to 9\Big]$$

$$\begin{pmatrix}
-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Exponentiate to get the counterpart in Lie Group:

mat =

$$e^{\left(\alpha_1*L^{1,1}_{14}+\alpha_2*L^{2,1}_{14}+\alpha_3*L^{3,1}_{14}+\alpha_4*L^{4,1}_{14}+\alpha_5*L^{5,1}_{14}+\alpha_6*L^{6,1}_{14}+\alpha_7*L^{7,1}_{14}+\alpha_8*L^{8,1}_{14}+\alpha_9*L^{9,1}_{14}+\alpha_{10}*L^{10,1}_{14}+\alpha_{11}*L^{11,1}_{14}+\alpha_{12}*L^{12,1}_{14}+\alpha_{13}*L^{13,1}_{14}+\alpha_{14}*L^{14,1}_{14}\right)};$$

```
Style[mat // MatrixForm, FontSize → 7]
```

$$\begin{pmatrix}
e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_2}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_3}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_4}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_5}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_6}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_7}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_8}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_9}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_{10}}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_{11}}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_{12}}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_{13}}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}}\left(-1+e^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}}\right)\alpha_{14}}{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Set $e^{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}} = \text{sumU}$ and therefore the other entries become:

$$\alpha_i = a_i \frac{\text{sumU}}{\left(1-e^{-\text{sumU}}\right)} \quad , \quad i \neq 1 \qquad \text{EQ 15.4}$$

Repeat this for all the remaining columns to get all α_i per 14 columns and form a matrix:

```
cfs = {a, b, c, d, e, f, g, h, i, j, k, l, m , n};
matREP = MatrixRep14Sets[cfs];
Style[matREP // MatrixForm, FontSize → 7]
```

a	b	0	0	0	0	0	0	0	0	0	0	0	0
b	a	0	0	0	0	0	0	0	0	0	0	0	0
c	d	a + c	0	b + d	0	0	0	0	0	0	0	0	0
d	c	0	a + c	0	b + d	0	0	0	0	0	0	0	0
e	f	b + e	0	a + f	0	0	0	0	0	0	0	0	0
f	e	0	b + e	0	a + f	0	0	0	0	0	0	0	0
g	h	d + g + l	0	c + h + k	0	a + c + h + k	0	b + d + g + l	0	d + g + l	0	c + h + k	0
h	g	0	d + g + l	0	c + h + k	0	a + c + h + k	0	b + d + g + l	0	d + g + l	0	c + h + k
i	j	f + i + n	0	e + j + m	0	b + e + j + m	0	a + f + i + n	0	f + i + n	0	e + j + m	0
j	i	0	f + i + n	0	e + j + m	0	b + e + j + m	0	a + f + i + n	0	f + i + n	0	e + j + m
k	l	h + k	0	g + l	0	d + g + l	0	c + h + k	0	a + c + h + k	0	b + d + g + l	0
l	k	0	h + k	0	g + l	0	d + g + l	0	c + h + k	0	a + c + h + k	0	b + d + g + l
m	n	j + m	0	i + n	0	f + i + n	0	e + j + m	0	b + e + j + m	0	a + f + i + n	0
n	m	0	j + m	0	i + n	0	f + i + n	0	e + j + m	0	b + e + j + m	0	a + f + i + n

```
Llist =.;
Llist[u_] := Table[L^{w,u}_{14}, {w, 1, 14}] ;

logMat = {};
Table[
  sumU = -Log[matREP[[u]][[u]]];
  logMat = Insert[logMat,
    Table[If[w ≠ u, matREP[[w]][[u]] * sumU / (-matREP[[u]][[u]] + 1), 0],
     {w, 1, 14}], u], {u, 1, 14}];
Style[logMat // MatrixForm, FontSize → 5];
vsum = ConstantArray[0, {14, 14}];
Table[vsum = vsum + (logMat[[u]].Llist[u]), {u, 1, 14}];

Style[vsum // MatrixForm, FontSize → 5]
```

```
b Log[a]/(1-a) + c Log[a]/(1-a) + d Log[a]/(1-a) + e Log[a]/(1-a) + f Log[a]/(1-a) + g Log[a]/(1-a) + h Log[a]/(1-a) + i Log[a]/(1-a) + j Log[a]/(1-a) + k Log[a]/(1-a) + l Log[a]/(1-a) + m Log[a]/(1-a) + n Log[a]/(1-a)        -b Log[a]/(1-a)
-b Log[a]/(1-a)        b Log[a]/(1-a) + c Log[a]/(1-a) + d Log[a]/(1-a) + e Log[a]/(1-a) + f Log[a]/(1-a) + g Log[a]/(1-a) + h Log[a]/(1-a) + i Log[a]/(1-a) + j Log[
-c Log[a]/(1-a)        -d Log[a]/(1-a)
-d Log[a]/(1-a)        -c Log[a]/(1-a)
-e Log[a]/(1-a)        -f Log[a]/(1-a)
-f Log[a]/(1-a)        -e Log[a]/(1-a)
-g Log[a]/(1-a)        -h Log[a]/(1-a)
-h Log[a]/(1-a)        -g Log[a]/(1-a)
-i Log[a]/(1-a)        -j Log[a]/(1-a)
-j Log[a]/(1-a)        -i Log[a]/(1-a)
-k Log[a]/(1-a)        -l Log[a]/(1-a)
-l Log[a]/(1-a)        -k Log[a]/(1-a)
-m Log[a]/(1-a)        -n Log[a]/(1-a)
-n Log[a]/(1-a)        -m Log[a]/(1-a)
```

Columns add to 0 so belongs to the Lie Algebra:

```
Simplify[sumColumn[vsum]]
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The above matrix serves as an approximation for Logarithm of a 14-Sets algebra member.

Example 15.1:

Get a set of 14 random coefficient member of the 14-Sets algebra and compute its matrix representation.

```
rands = Join[Sort[ RandomReal[{0, 1}, 13], Less], {1}];
cfs = Join[{rands[[1]]}, Table[rands[[w]] - rands[[w - 1]], {w, 2, 14}]];
matREP = MatrixRep14Sets[cfs];
sumColumn[matREP]
Style[matREP // MatrixForm, FontSize → 7]
sumColumn[matREP]
Norm[matREP]
```

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

0.10016	0.100453	0	0	0	0	0	0	0	0	0	0	0	0
0.100453	0.10016	0	0	0	0	0	0	0	0	0	0	0	0
0.0147574	0.0379001	0.114918	0	0.138353	0	0	0	0	0	0	0	0	0
0.0379001	0.0147574	0	0.114918	0	0.138353	0	0	0	0	0	0	0	0
0.236013	0.0410796	0.336465	0	0.14124	0	0	0	0	0	0	0	0	0
0.0410796	0.236013	0	0.336465	0	0.14124	0	0	0	0	0	0	0	0
0.0454851	0.00374176	0.109013	0	0.0487453	0	0.148905	0	0.209465	0	0.109013	0	0.0487453	0
0.00374176	0.0454851	0	0.109013	0	0.0487453	0	0.148905	0	0.209465	0	0.109013	0	0.0487453
0.156704	0.0272065	0.248862	0	0.392768	0	0.49322	0	0.349022	0	0.248862	0	0.392768	0
0.0272065	0.156704	0	0.248862	0	0.392768	0	0.49322	0	0.349022	0	0.248862	0	0.392768
0.0302462	0.0256277	0.0339879	0	0.0711127	0	0.109013	0	0.0487453	0	0.148905	0	0.209465	0
0.0256277	0.0302462	0	0.0339879	0	0.0711127	0	0.109013	0	0.0487453	0	0.148905	0	0.209465
0.129548	0.0510777	0.156755	0	0.207782	0	0.248862	0	0.392768	0	0.49322	0	0.349022	0
0.0510777	0.129548	0	0.156755	0	0.207782	0	0.248862	0	0.392768	0	0.49322	0	0.349022

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

1.28014

```
Llist =.;
Llist[u_] := Table[L_14^(w,u), {w, 1, 14}] ;

logMat = {};
Table[
  sumU = -Log[matREP[[u]][[u]]];
  logMat = Insert[logMat,
    Table[If[w ≠ u, matREP[[w]][[u]] * sumU / (-matREP[[u]][[u]] + 1), 0],
     {w, 1, 14}], u], {u, 1, 14}];
Style[logMat // MatrixForm, FontSize → 5];
vsum = ConstantArray[0, {14, 14}];
Table[vsum = vsum + (logMat[[u]].Llist[u]), {u, 1, 14}];
final = N[e^vsum, 100];
Style[final // MatrixForm, FontSize → 7]
sumColumn[final]
Norm[matREP - final] / Norm[matREP]
```

0.103483	0.0260118	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.0260118	0.103483	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.0173233	0.0144825	0.131252	0.	0.0420062	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.0144825	0.0173233	0.	0.131252	0.	0.0420062	0.	0.	0.	0.	0.	0.	0.	0.
0.0799697	0.0272139	0.109563	0.	0.158727	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.0272139	0.0799697	0.	0.109563	0.	0.158727	0.	0.	0.	0.	0.	0.	0.	0.
0.0562262	0.0181639	0.0929437	0.	0.0798615	0.	0.213732	0.	0.1076	0.	0.0872154	0.	0.0681712	0.
0.0181639	0.0562262	0.	0.0929437	0.	0.0798615	0.	0.213732	0.	0.1076	0.	0.0872154	0.	0.0681712
0.229101	0.0764875	0.333692	0.	0.360723	0.	0.391005	0.	0.510317	0.	0.308048	0.	0.313911	0.
0.0764875	0.229101	0.	0.333692	0.	0.360723	0.	0.391005	0.	0.510317	0.	0.308048	0.	0.313911
0.0506246	0.0236815	0.0666162	0.	0.0744944	0.	0.0872154	0.	0.0681712	0.	0.213732	0.	0.1076	0.
0.0236815	0.0506246	0.	0.0666162	0.	0.0744944	0.	0.0872154	0.	0.0681712	0.	0.213732	0.	0.1076
0.198504	0.0787265	0.265933	0.	0.284188	0.	0.308048	0.	0.313911	0.	0.391005	0.	0.510317	0.
0.0787265	0.198504	0.	0.265933	0.	0.284188	0.	0.308048	0.	0.313911	0.	0.391005	0.	0.510317

{1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

0.279809

We obtain a Logarithm with error about 0.3%.

The above performs better than the regular Taylor Series expansions:

```
x =.;
y =.;
series = Series[Log[y + 1], {y, 0, 100}]
```

$$y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}-\frac{y^6}{6}+\frac{y^7}{7}-\frac{y^8}{8}+\frac{y^9}{9}-\frac{y^{10}}{10}+\frac{y^{11}}{11}-\frac{y^{12}}{12}+\frac{y^{13}}{13}-\frac{y^{14}}{14}+\frac{y^{15}}{15}-\frac{y^{16}}{16}+$$
$$\frac{y^{17}}{17}-\frac{y^{18}}{18}+\frac{y^{19}}{19}-\frac{y^{20}}{20}+\frac{y^{21}}{21}-\frac{y^{22}}{22}+\frac{y^{23}}{23}-\frac{y^{24}}{24}+\frac{y^{25}}{25}-\frac{y^{26}}{26}+\frac{y^{27}}{27}-\frac{y^{28}}{28}+\frac{y^{29}}{29}-\frac{y^{30}}{30}+\frac{y^{31}}{31}-$$
$$\frac{y^{32}}{32}+\frac{y^{33}}{33}-\frac{y^{34}}{34}+\frac{y^{35}}{35}-\frac{y^{36}}{36}+\frac{y^{37}}{37}-\frac{y^{38}}{38}+\frac{y^{39}}{39}-\frac{y^{40}}{40}+\frac{y^{41}}{41}-\frac{y^{42}}{42}+\frac{y^{43}}{43}-\frac{y^{44}}{44}+\frac{y^{45}}{45}-$$
$$\frac{y^{46}}{46}+\frac{y^{47}}{47}-\frac{y^{48}}{48}+\frac{y^{49}}{49}-\frac{y^{50}}{50}+\frac{y^{51}}{51}-\frac{y^{52}}{52}+\frac{y^{53}}{53}-\frac{y^{54}}{54}+\frac{y^{55}}{55}-\frac{y^{56}}{56}+\frac{y^{57}}{57}-\frac{y^{58}}{58}+\frac{y^{59}}{59}-$$
$$\frac{y^{60}}{60}+\frac{y^{61}}{61}-\frac{y^{62}}{62}+\frac{y^{63}}{63}-\frac{y^{64}}{64}+\frac{y^{65}}{65}-\frac{y^{66}}{66}+\frac{y^{67}}{67}-\frac{y^{68}}{68}+\frac{y^{69}}{69}-\frac{y^{70}}{70}+\frac{y^{71}}{71}-\frac{y^{72}}{72}+\frac{y^{73}}{73}-$$
$$\frac{y^{74}}{74}+\frac{y^{75}}{75}-\frac{y^{76}}{76}+\frac{y^{77}}{77}-\frac{y^{78}}{78}+\frac{y^{79}}{79}-\frac{y^{80}}{80}+\frac{y^{81}}{81}-\frac{y^{82}}{82}+\frac{y^{83}}{83}-\frac{y^{84}}{84}+\frac{y^{85}}{85}-\frac{y^{86}}{86}+\frac{y^{87}}{87}-$$
$$\frac{y^{88}}{88}+\frac{y^{89}}{89}-\frac{y^{90}}{90}+\frac{y^{91}}{91}-\frac{y^{92}}{92}+\frac{y^{93}}{93}-\frac{y^{94}}{94}+\frac{y^{95}}{95}-\frac{y^{96}}{96}+\frac{y^{97}}{97}-\frac{y^{98}}{98}+\frac{y^{99}}{99}-\frac{y^{100}}{100}+O[y]^{101}$$

$$\text{series} = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \frac{y^7}{7} - \frac{y^8}{8} + \frac{y^9}{9} - \frac{y^{10}}{10} + \frac{y^{11}}{11} - \frac{y^{12}}{12} + \frac{y^{13}}{13} - \frac{y^{14}}{14} + \frac{y^{15}}{15} - \frac{y^{16}}{16} + \frac{y^{17}}{17} - \frac{y^{18}}{18} + \frac{y^{19}}{19} - \frac{y^{20}}{20} + \frac{y^{21}}{21} - \frac{y^{22}}{22} + \frac{y^{23}}{23} - \frac{y^{24}}{24} + \frac{y^{25}}{25} - \frac{y^{26}}{26} + \frac{y^{27}}{27} - \frac{y^{28}}{28} + \frac{y^{29}}{29} - \frac{y^{30}}{30} + \frac{y^{31}}{31} - \frac{y^{32}}{32} + \frac{y^{33}}{33} - \frac{y^{34}}{34} + \frac{y^{35}}{35} - \frac{y^{36}}{36} + \frac{y^{37}}{37} - \frac{y^{38}}{38} + \frac{y^{39}}{39} - \frac{y^{40}}{40} + \frac{y^{41}}{41} - \frac{y^{42}}{42} + \frac{y^{43}}{43} - \frac{y^{44}}{44} + \frac{y^{45}}{45} - \frac{y^{46}}{46} + \frac{y^{47}}{47} - \frac{y^{48}}{48} + \frac{y^{49}}{49} - \frac{y^{50}}{50} + \frac{y^{51}}{51} - \frac{y^{52}}{52} + \frac{y^{53}}{53} - \frac{y^{54}}{54} + \frac{y^{55}}{55} - \frac{y^{56}}{56} + \frac{y^{57}}{57} - \frac{y^{58}}{58} + \frac{y^{59}}{59} - \frac{y^{60}}{60} + \frac{y^{61}}{61} - \frac{y^{62}}{62} + \frac{y^{63}}{63} - \frac{y^{64}}{64} + \frac{y^{65}}{65} - \frac{y^{66}}{66} + \frac{y^{67}}{67} - \frac{y^{68}}{68} + \frac{y^{69}}{69} - \frac{y^{70}}{70} + \frac{y^{71}}{71} - \frac{y^{72}}{72} + \frac{y^{73}}{73} - \frac{y^{74}}{74} + \frac{y^{75}}{75} - \frac{y^{76}}{76} + \frac{y^{77}}{77} - \frac{y^{78}}{78} + \frac{y^{79}}{79} - \frac{y^{80}}{80} + \frac{y^{81}}{81} - \frac{y^{82}}{82} + \frac{y^{83}}{83} - \frac{y^{84}}{84} + \frac{y^{85}}{85} - \frac{y^{86}}{86} + \frac{y^{87}}{87} - \frac{y^{88}}{88} + \frac{y^{89}}{89} - \frac{y^{90}}{90} + \frac{y^{91}}{91} - \frac{y^{92}}{92} + \frac{y^{93}}{93} - \frac{y^{94}}{94} + \frac{y^{95}}{95} - \frac{y^{96}}{96} + \frac{y^{97}}{97} - \frac{y^{98}}{98} + \frac{y^{99}}{99} - \frac{y^{100}}{100};$$

```
x = matREP - IdentityMatrix[14];
log = N[series /. {y → x}, 100];
Style[MatrixExp[log] // MatrixForm, FontSize → 7]
Norm[MatrixExp[log] - matREP] / Norm[matREP]
sumColumn[log]
sumColumn[MatrixExp[log]]
```

0.10062	0.00960215	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.00960215	0.10062	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.00340771	0.0044596	0.117241	0.	0.0166418	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.0044596	0.00340771	0.	0.117241	0.	0.0166418	0.	0.	0.	0.	0.	0.	0.	0.
0.0260149	0.00664335	0.0372475	0.	0.143728	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.00664335	0.0260149	0.	0.0372475	0.	0.143728	0.	0.	0.	0.	0.	0.	0.	0.
0.0108511	0.00248078	0.021288	0.	0.0171187	0.	0.159959	0.	0.049553	0.	0.0235558	0.	0.0228599	0.
0.00248078	0.0108511	0.	0.021288	0.	0.0171187	0.	0.159959	0.	0.049553	0.	0.0235558	0.	0.0228599
0.0460842	0.0123927	0.0708486	0.	0.0956991	0.	0.111619	0.	0.382769	0.	0.0781642	0.	0.129547	0.
0.0123927	0.0460842	0.	0.0708486	0.	0.0956991	0.	0.111619	0.	0.382769	0.	0.0781642	0.	0.129547
0.00905782	0.00504355	0.0118504	0.	0.0176941	0.	0.0235558	0.	0.0228599	0.	0.159959	0.	0.049553	0.
0.00504355	0.00905782	0.	0.0118504	0.	0.0176941	0.	0.0235558	0.	0.0228599	0.	0.159959	0.	0.049553
0.0396147	0.015532	0.0530502	0.	0.0677391	0.	0.0781642	0.	0.129547	0.	0.111619	0.	0.382769	0.
0.015532	0.0396147	0.	0.0530502	0.	0.0677391	0.	0.0781642	0.	0.129547	0.	0.111619	0.	0.382769

0.601914

{-1.45389, -1.45389, -1.36877, -1.36877, -1.19134, -1.19134, -1.17781, -1.17781, -0.483555, -0.483555, -1.17781, -1.17781, -0.483555, -0.483555}

{0.291804, 0.291804, 0.311525, 0.311525, 0.358621, 0.358621, 0.373298, 0.373298, 0.584729, 0.584729, 0.373298, 0.373298, 0.584729, 0.584729}

We obtain a Logarithm with error about 0.6%.

Remark 15.1: *Taylor series expansion also fails to produce a member of Lie Algebra since the column sums are not 0. Therefore the subsequent exponentiation does not have the column sums of 1.*

Logarithm of Closure Operator

We obtain an approximation of the Boundary Operator's Logarithm in terms of a variable z and then take the limit of the z approaching Infinity and thus obtain a Logarithm as a limit process:

```
cfs = {0, 0, z, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0};
matREP = MatrixRep14Sets[cfs];
sumColumn[matREP]
Style[matREP // MatrixForm, FontSize → 9]
```

```
{z, z, z, z, z, z, z, z, z, z, z, z, z, z}
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ z&0&z&0&0&0&0&0&0&0&0&0&0&0 \\ 0&z&0&z&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&z&0&z&0&0&0&0&0&z&0 \\ 0&0&0&0&0&z&0&z&0&0&0&0&0&z \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&z&0&z&0&0&0 \\ 0&0&0&0&0&0&0&0&0&z&0&z&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Approximate the Logarithm:

```
Llist =.;
Llist[u_] := Table[L^{w,u}_{14}, {w, 1, 14}] ;
LlistHOLD[u_] :=
  Table[Superscript["L", ToString[w] <> "," <> ToString[u]], {w, 1, 14}] ;

logMat = {};
Table[
  sumU = -Log[matREP[[u]][[u]]];
  logMat = Insert[logMat, Table[If[w ≠ u, matREP[[w]][[u]], 0], {w, 1, 14}], u],
  {u, 1, 14}];

vsum = ConstantArray[0, {14, 14}];
vsumHOLD = 0;
Table[vsum = vsum + (logMat[[u]].Llist[u]), {u, 1, 14}];
Table[vsumHOLD = vsumHOLD + (logMat[[u]].LlistHOLD[u]), {u, 1, 14}];

vsumHOLD
final = e^vsum;
Style[final // MatrixForm, FontSize → 8]
Simplify[sumColumn[final]]
```

$z\,L^{11,9} + z\,L^{12,10} + z\,L^{3,1} + z\,L^{4,2} + z\,L^{7,13} + z\,L^{7,5} + z\,L^{8,14} + z\,L^{8,6}$

$$\begin{pmatrix}
e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z}
\end{pmatrix}$$

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

The Boundary Operator Logarithm is the 8 element linear combination while z approaches Infinity:

$z\,L^{11,9} + z\,L^{12,10} + z\,L^{3,1} + z\,L^{4,2} + z\,L^{7,13} + z\,L^{7,5} + z\,L^{8,14} + z\,L^{8,6}$

Remark 15.2: *This version of the algorithm is based upon the fact from EQ 15.4*

$$\lim_{\text{sumU}\to 0} \frac{\text{sumU}}{\left(1-e^{-\text{sumU}}\right)} = 1 \qquad \text{EQ 15.5}$$

Take the limit:

```
lim = Limit[final, z → Infinity];
Style[lim // MatrixForm, FontSize → 10]

(*{0,0,1,0,0,0,0,0,0,0,0,0,0,0} is the Closure operator *)
MatrixRep14Sets[{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}] === lim
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

```
True
```

The Logarithm of the Boundary Operator exists as a limit process for z approaching Infinity:

```
logComplement = vsum;
Style[logComplement // MatrixForm, FontSize → 8]
```

$$\begin{pmatrix} -z&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&-z&0&0&0&0&0&0&0&0&0&0&0&0 \\ z&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&z&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&-z&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&-z&0&0&0&0&0&0&0&0 \\ 0&0&0&0&z&0&0&0&0&0&0&0&z&0 \\ 0&0&0&0&0&z&0&0&0&0&0&0&0&z \\ 0&0&0&0&0&0&0&0&-z&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&-z&0&0&0&0 \\ 0&0&0&0&0&0&0&0&z&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&z&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&-z&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-z \end{pmatrix}$$

Logarithm of Complement Operator

We obtain an approximation of the Complement Operator's Logarithm in terms of a variable z and then take the limit of the z approaching i and thus obtain a Logarithm as an exact calculation:

```
cfs = {0, Log[z], 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0};
matREP = MatrixRep14Sets[cfs];
sumColumn[matREP]
Style[matREP // MatrixForm, FontSize → 8]
```

```
{Log[z], Log[z], Log[z], Log[z], Log[z], Log[z],
 Log[z], Log[z], Log[z], Log[z], Log[z], Log[z], Log[z], Log[z]}
```

0	Log[z]	0	0	0	0	0	0	0	0	0	0	0	0
Log[z]	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	Log[z]	0	0	0	0	0	0	0	0	0
0	0	0	0	0	Log[z]	0	0	0	0	0	0	0	0
0	0	Log[z]	0	0	0	0	0	0	0	0	0	0	0
0	0	0	Log[z]	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	Log[z]	0	0	0	0	0
0	0	0	0	0	0	0	0	0	Log[z]	0	0	0	0
0	0	0	0	0	0	Log[z]	0	0	0	0	0	0	0
0	0	0	0	0	0	0	Log[z]	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	Log[z]	0
0	0	0	0	0	0	0	0	0	0	0	0	0	Log[z]
0	0	0	0	0	0	0	0	0	0	Log[z]	0	0	0
0	0	0	0	0	0	0	0	0	0	0	Log[z]	0	0

Approximate the Logarithm:

```
Llist =.;
Llist[u_] := Table[L^{w,u}_{14}, {w, 1, 14}] ;
LlistHOLD[u_] :=
  Table[Superscript["L", ToString[w] <> "," <> ToString[u]], {w, 1, 14}] ;

logMat = {};
Table[
  sumU = -Log[matREP[[u]][[u]]];
  logMat = Insert[logMat, Table[If[w ≠ u, matREP[[w]][[u]], 0], {w, 1, 14}], u],
  {u, 1, 14}];

vsum = ConstantArray[0, {14, 14}];
vsumHOLD = 0;
Table[vsum = vsum + (logMat[[u]].Llist[u]), {u, 1, 14}];
Table[vsumHOLD = vsumHOLD + (logMat[[u]].LlistHOLD[u]), {u, 1, 14}];

vsumHOLD
vsumHOLD /. {z → ⅈ}
final = ⅇ^vsum;
Style[final // MatrixForm, FontSize → 8]
Simplify[sumColumn[final]]
```

$$\mathrm{Log}[z]\, L^{10,8} + \mathrm{Log}[z]\, L^{11,13} + \mathrm{Log}[z]\, L^{1,2} + \mathrm{Log}[z]\, L^{12,14} + \mathrm{Log}[z]\, L^{13,11} + \mathrm{Log}[z]\, L^{14,12} + \mathrm{Log}[z]\, L^{2,1} + \mathrm{Log}[z]\, L^{3,5} + \mathrm{Log}[z]\, L^{4,6} + \mathrm{Log}[z]\, L^{5,3} + \mathrm{Log}[z]\, L^{6,4} + \mathrm{Log}[z]\, L^{7,9} + \mathrm{Log}[z]\, L^{8,10} + \mathrm{Log}[z]\, L^{9,7}$$

$$\frac{1}{2} i \pi L^{10,8} + \frac{1}{2} i \pi L^{11,13} + \frac{1}{2} i \pi L^{1,2} + \frac{1}{2} i \pi L^{12,14} + \frac{1}{2} i \pi L^{13,11} + \frac{1}{2} i \pi L^{14,12} + \frac{1}{2} i \pi L^{2,1} + \frac{1}{2} i \pi L^{3,5} + \frac{1}{2} i \pi L^{4,6} + \frac{1}{2} i \pi L^{5,3} + \frac{1}{2} i \pi L^{6,4} + \frac{1}{2} i \pi L^{7,9} + \frac{1}{2} i \pi L^{8,10} + \frac{1}{2} i \pi L^{9,7}$$

$$\begin{pmatrix}
\frac{1+z^2}{2z^2} & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1+z^2}{2z^2} & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2}
\end{pmatrix}$$

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

The Logarithm of Complement Operator is the following 14 member linear combination:

$$\frac{1}{2} i \pi L^{10,8} + \frac{1}{2} i \pi L^{11,13} + \frac{1}{2} i \pi L^{1,2} + \frac{1}{2} i \pi L^{12,14} + \frac{1}{2} i \pi L^{13,11} + \frac{1}{2} i \pi L^{14,12} + \frac{1}{2} i \pi L^{2,1} + \frac{1}{2} i \pi L^{3,5} + \frac{1}{2} i \pi L^{4,6} + \frac{1}{2} i \pi L^{5,3} + \frac{1}{2} i \pi L^{6,4} + \frac{1}{2} i \pi L^{7,9} + \frac{1}{2} i \pi L^{8,10} + \frac{1}{2} i \pi L^{9,7}$$

Take the limit $z \longrightarrow i$:

```
lim = Limit[final, z → i];
Style[lim // MatrixForm, FontSize → 10]
MatrixRep14Sets[{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}] === lim
```

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

```
True
```

Therefore the Logarithm of the Complement Operator is:

```
logComplement = Limit[vsum, z → i];
Style[logComplement // MatrixForm, FontSize → 10]
```

$$\begin{pmatrix}
-\frac{i\pi}{2} & \frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{i\pi}{2} & -\frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{i\pi}{2} & 0 & \frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{i\pi}{2} & 0 & \frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i\pi}{2} & 0 & -\frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{i\pi}{2} & 0 & -\frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{i\pi}{2} & 0 & \frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i\pi}{2} & 0 & \frac{i\pi}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i\pi}{2} & 0 & -\frac{i\pi}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i\pi}{2} & 0 & -\frac{i\pi}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i\pi}{2} & 0 & \frac{i\pi}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i\pi}{2} & 0 & \frac{i\pi}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i\pi}{2} & 0 & -\frac{i\pi}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i\pi}{2} & 0 & -\frac{i\pi}{2}
\end{pmatrix}$$

Remark 15.3: *Perhaps there is a Real Logarithm of the Complement Operator, the author currently does not know of it.*

16. 14-Sets Hopf Algebra

Coproduct defined as a simple tensor product, where *g* belongs to the 14-Sets group:

$\Delta(g) = g \otimes g$

```
Notation[ Δ (c_) ⟹ Distribute[c_ ⊗ c_] ]
Notation[ Δ c_ ⟹ Distribute[c_ ⊗ c_] ]
(* ⋄ is left as blank in most math
  books but we need to have an explicit bi-product *)
Notation[ (s_ a_ ⊗ c_) ⋄ (t_ b_ ⊗ d_) ⟹ (s_ * t_) (a_ ⊙ b_) ⊗ (c_ ⊙ d_) ]
Notation[ (a_ x_) ⊗ (b_ y_) ⟹ a_ * b_ (x_ ⊗ y_) ]
(* Kludge, Mathematica is sensitive to the format of the expressions *)
Notation[ x_/a_ ⊗ y_/b_ ⟹ (1 / (a_ * b_)) * (x_ ⊗ y_) ]
Notation[ x_/a_ ⊗ y_ ⟹ (1 / (a_)) * (x_ ⊗ y_) ]
Notation[ x_ ⊗ y_/b_ ⟹ (1 / (b_)) * (x_ ⊗ y_) ]
(*Kludge: Mathematica did not multiply two Reals *)
Notation[ (a_ a2_ x_) ⊗ (b_ b2_ y_) ⟹ a_ * b_ * a2_ * b2_ (x_ ⊗ y_) ]
```

Let’s actually compute a sample of the Coproduct:

$$v = a\,\sigma_0 + b\,\sigma_1 + c\,\sigma_2 + d\,\sigma_3 + e\,\sigma_4 + f\,\sigma_5 + g\,\sigma_6 + h\,\sigma_7 + i\,\sigma_8 + j\,\sigma_9 + k\,\sigma_{10} + l\,\sigma_{11} + m\,\sigma_{12} + n\,\sigma_{13};$$

$$\Delta\,(v)$$

$(a\,\sigma_0)\otimes(a\,\sigma_0) + (a\,\sigma_0)\otimes(b\,\sigma_1) + (a\,\sigma_0)\otimes(c\,\sigma_2) + (a\,\sigma_0)\otimes(d\,\sigma_3) + (a\,\sigma_0)\otimes(e\,\sigma_4) +$
$(a\,\sigma_0)\otimes(f\,\sigma_5) + (a\,\sigma_0)\otimes(g\,\sigma_6) + (a\,\sigma_0)\otimes(h\,\sigma_7) + (a\,\sigma_0)\otimes(i\,\sigma_8) + (a\,\sigma_0)\otimes(j\,\sigma_9) +$
$(a\,\sigma_0)\otimes(k\,\sigma_{10}) + (a\,\sigma_0)\otimes(l\,\sigma_{11}) + (a\,\sigma_0)\otimes(m\,\sigma_{12}) + (a\,\sigma_0)\otimes(n\,\sigma_{13}) + (b\,\sigma_1)\otimes(a\,\sigma_0) +$
$(b\,\sigma_1)\otimes(b\,\sigma_1) + (b\,\sigma_1)\otimes(c\,\sigma_2) + (b\,\sigma_1)\otimes(d\,\sigma_3) + (b\,\sigma_1)\otimes(e\,\sigma_4) + (b\,\sigma_1)\otimes(f\,\sigma_5) +$
$(b\,\sigma_1)\otimes(g\,\sigma_6) + (b\,\sigma_1)\otimes(h\,\sigma_7) + (b\,\sigma_1)\otimes(i\,\sigma_8) + (b\,\sigma_1)\otimes(j\,\sigma_9) + (b\,\sigma_1)\otimes(k\,\sigma_{10}) +$
$(b\,\sigma_1)\otimes(l\,\sigma_{11}) + (b\,\sigma_1)\otimes(m\,\sigma_{12}) + (b\,\sigma_1)\otimes(n\,\sigma_{13}) + (c\,\sigma_2)\otimes(a\,\sigma_0) + (c\,\sigma_2)\otimes(b\,\sigma_1) +$
$(c\,\sigma_2)\otimes(c\,\sigma_2) + (c\,\sigma_2)\otimes(d\,\sigma_3) + (c\,\sigma_2)\otimes(e\,\sigma_4) + (c\,\sigma_2)\otimes(f\,\sigma_5) + (c\,\sigma_2)\otimes(g\,\sigma_6) +$
$(c\,\sigma_2)\otimes(h\,\sigma_7) + (c\,\sigma_2)\otimes(i\,\sigma_8) + (c\,\sigma_2)\otimes(j\,\sigma_9) + (c\,\sigma_2)\otimes(k\,\sigma_{10}) + (c\,\sigma_2)\otimes(l\,\sigma_{11}) +$
$(c\,\sigma_2)\otimes(m\,\sigma_{12}) + (c\,\sigma_2)\otimes(n\,\sigma_{13}) + (d\,\sigma_3)\otimes(a\,\sigma_0) + (d\,\sigma_3)\otimes(b\,\sigma_1) + (d\,\sigma_3)\otimes(c\,\sigma_2) +$
$(d\,\sigma_3)\otimes(d\,\sigma_3) + (d\,\sigma_3)\otimes(e\,\sigma_4) + (d\,\sigma_3)\otimes(f\,\sigma_5) + (d\,\sigma_3)\otimes(g\,\sigma_6) + (d\,\sigma_3)\otimes(h\,\sigma_7) +$
$(d\,\sigma_3)\otimes(i\,\sigma_8) + (d\,\sigma_3)\otimes(j\,\sigma_9) + (d\,\sigma_3)\otimes(k\,\sigma_{10}) + (d\,\sigma_3)\otimes(l\,\sigma_{11}) + (d\,\sigma_3)\otimes(m\,\sigma_{12}) +$
$(d\,\sigma_3)\otimes(n\,\sigma_{13}) + (e\,\sigma_4)\otimes(a\,\sigma_0) + (e\,\sigma_4)\otimes(b\,\sigma_1) + (e\,\sigma_4)\otimes(c\,\sigma_2) + (e\,\sigma_4)\otimes(d\,\sigma_3) +$
$(e\,\sigma_4)\otimes(e\,\sigma_4) + (e\,\sigma_4)\otimes(f\,\sigma_5) + (e\,\sigma_4)\otimes(g\,\sigma_6) + (e\,\sigma_4)\otimes(h\,\sigma_7) + (e\,\sigma_4)\otimes(i\,\sigma_8) +$
$(e\,\sigma_4)\otimes(j\,\sigma_9) + (e\,\sigma_4)\otimes(k\,\sigma_{10}) + (e\,\sigma_4)\otimes(l\,\sigma_{11}) + (e\,\sigma_4)\otimes(m\,\sigma_{12}) + (e\,\sigma_4)\otimes(n\,\sigma_{13}) +$
$(f\,\sigma_5)\otimes(a\,\sigma_0) + (f\,\sigma_5)\otimes(b\,\sigma_1) + (f\,\sigma_5)\otimes(c\,\sigma_2) + (f\,\sigma_5)\otimes(d\,\sigma_3) + (f\,\sigma_5)\otimes(e\,\sigma_4) +$
$(f\,\sigma_5)\otimes(f\,\sigma_5) + (f\,\sigma_5)\otimes(g\,\sigma_6) + (f\,\sigma_5)\otimes(h\,\sigma_7) + (f\,\sigma_5)\otimes(i\,\sigma_8) + (f\,\sigma_5)\otimes(j\,\sigma_9) +$
$(f\,\sigma_5)\otimes(k\,\sigma_{10}) + (f\,\sigma_5)\otimes(l\,\sigma_{11}) + (f\,\sigma_5)\otimes(m\,\sigma_{12}) + (f\,\sigma_5)\otimes(n\,\sigma_{13}) + (g\,\sigma_6)\otimes(a\,\sigma_0) +$
$(g\,\sigma_6)\otimes(b\,\sigma_1) + (g\,\sigma_6)\otimes(c\,\sigma_2) + (g\,\sigma_6)\otimes(d\,\sigma_3) + (g\,\sigma_6)\otimes(e\,\sigma_4) + (g\,\sigma_6)\otimes(f\,\sigma_5) +$
$(g\,\sigma_6)\otimes(g\,\sigma_6) + (g\,\sigma_6)\otimes(h\,\sigma_7) + (g\,\sigma_6)\otimes(i\,\sigma_8) + (g\,\sigma_6)\otimes(j\,\sigma_9) + (g\,\sigma_6)\otimes(k\,\sigma_{10}) +$
$(g\,\sigma_6)\otimes(l\,\sigma_{11}) + (g\,\sigma_6)\otimes(m\,\sigma_{12}) + (g\,\sigma_6)\otimes(n\,\sigma_{13}) + (h\,\sigma_7)\otimes(a\,\sigma_0) + (h\,\sigma_7)\otimes(b\,\sigma_1) +$
$(h\,\sigma_7)\otimes(c\,\sigma_2) + (h\,\sigma_7)\otimes(d\,\sigma_3) + (h\,\sigma_7)\otimes(e\,\sigma_4) + (h\,\sigma_7)\otimes(f\,\sigma_5) + (h\,\sigma_7)\otimes(g\,\sigma_6) +$
$(h\,\sigma_7)\otimes(h\,\sigma_7) + (h\,\sigma_7)\otimes(i\,\sigma_8) + (h\,\sigma_7)\otimes(j\,\sigma_9) + (h\,\sigma_7)\otimes(k\,\sigma_{10}) + (h\,\sigma_7)\otimes(l\,\sigma_{11}) +$
$(h\,\sigma_7)\otimes(m\,\sigma_{12}) + (h\,\sigma_7)\otimes(n\,\sigma_{13}) + (i\,\sigma_8)\otimes(a\,\sigma_0) + (i\,\sigma_8)\otimes(b\,\sigma_1) + (i\,\sigma_8)\otimes(c\,\sigma_2) +$
$(i\,\sigma_8)\otimes(d\,\sigma_3) + (i\,\sigma_8)\otimes(e\,\sigma_4) + (i\,\sigma_8)\otimes(f\,\sigma_5) + (i\,\sigma_8)\otimes(g\,\sigma_6) + (i\,\sigma_8)\otimes(h\,\sigma_7) +$
$(i\,\sigma_8)\otimes(i\,\sigma_8) + (i\,\sigma_8)\otimes(j\,\sigma_9) + (i\,\sigma_8)\otimes(k\,\sigma_{10}) + (i\,\sigma_8)\otimes(l\,\sigma_{11}) + (i\,\sigma_8)\otimes(m\,\sigma_{12}) +$
$(i\,\sigma_8)\otimes(n\,\sigma_{13}) + (j\,\sigma_9)\otimes(a\,\sigma_0) + (j\,\sigma_9)\otimes(b\,\sigma_1) + (j\,\sigma_9)\otimes(c\,\sigma_2) + (j\,\sigma_9)\otimes(d\,\sigma_3) +$
$(j\,\sigma_9)\otimes(e\,\sigma_4) + (j\,\sigma_9)\otimes(f\,\sigma_5) + (j\,\sigma_9)\otimes(g\,\sigma_6) + (j\,\sigma_9)\otimes(h\,\sigma_7) + (j\,\sigma_9)\otimes(i\,\sigma_8) +$
$(j\,\sigma_9)\otimes(j\,\sigma_9) + (j\,\sigma_9)\otimes(k\,\sigma_{10}) + (j\,\sigma_9)\otimes(l\,\sigma_{11}) + (j\,\sigma_9)\otimes(m\,\sigma_{12}) + (j\,\sigma_9)\otimes(n\,\sigma_{13}) +$
$(k\,\sigma_{10})\otimes(a\,\sigma_0) + (k\,\sigma_{10})\otimes(b\,\sigma_1) + (k\,\sigma_{10})\otimes(c\,\sigma_2) + (k\,\sigma_{10})\otimes(d\,\sigma_3) + (k\,\sigma_{10})\otimes(e\,\sigma_4) +$
$(k\,\sigma_{10})\otimes(f\,\sigma_5) + (k\,\sigma_{10})\otimes(g\,\sigma_6) + (k\,\sigma_{10})\otimes(h\,\sigma_7) + (k\,\sigma_{10})\otimes(i\,\sigma_8) + (k\,\sigma_{10})\otimes(j\,\sigma_9) +$
$(k\,\sigma_{10})\otimes(k\,\sigma_{10}) + (k\,\sigma_{10})\otimes(l\,\sigma_{11}) + (k\,\sigma_{10})\otimes(m\,\sigma_{12}) + (k\,\sigma_{10})\otimes(n\,\sigma_{13}) +$
$(l\,\sigma_{11})\otimes(a\,\sigma_0) + (l\,\sigma_{11})\otimes(b\,\sigma_1) + (l\,\sigma_{11})\otimes(c\,\sigma_2) + (l\,\sigma_{11})\otimes(d\,\sigma_3) + (l\,\sigma_{11})\otimes(e\,\sigma_4) +$
$(l\,\sigma_{11})\otimes(f\,\sigma_5) + (l\,\sigma_{11})\otimes(g\,\sigma_6) + (l\,\sigma_{11})\otimes(h\,\sigma_7) + (l\,\sigma_{11})\otimes(i\,\sigma_8) + (l\,\sigma_{11})\otimes(j\,\sigma_9) +$
$(l\,\sigma_{11})\otimes(k\,\sigma_{10}) + (l\,\sigma_{11})\otimes(l\,\sigma_{11}) + (l\,\sigma_{11})\otimes(m\,\sigma_{12}) + (l\,\sigma_{11})\otimes(n\,\sigma_{13}) +$
$(m\,\sigma_{12})\otimes(a\,\sigma_0) + (m\,\sigma_{12})\otimes(b\,\sigma_1) + (m\,\sigma_{12})\otimes(c\,\sigma_2) + (m\,\sigma_{12})\otimes(d\,\sigma_3) + (m\,\sigma_{12})\otimes(e\,\sigma_4) +$
$(m\,\sigma_{12})\otimes(f\,\sigma_5) + (m\,\sigma_{12})\otimes(g\,\sigma_6) + (m\,\sigma_{12})\otimes(h\,\sigma_7) + (m\,\sigma_{12})\otimes(i\,\sigma_8) + (m\,\sigma_{12})\otimes(j\,\sigma_9) +$
$(m\,\sigma_{12})\otimes(k\,\sigma_{10}) + (m\,\sigma_{12})\otimes(l\,\sigma_{11}) + (m\,\sigma_{12})\otimes(m\,\sigma_{12}) + (m\,\sigma_{12})\otimes(n\,\sigma_{13}) +$
$(n\,\sigma_{13})\otimes(a\,\sigma_0) + (n\,\sigma_{13})\otimes(b\,\sigma_1) + (n\,\sigma_{13})\otimes(c\,\sigma_2) + (n\,\sigma_{13})\otimes(d\,\sigma_3) + (n\,\sigma_{13})\otimes(e\,\sigma_4) +$
$(n\,\sigma_{13})\otimes(f\,\sigma_5) + (n\,\sigma_{13})\otimes(g\,\sigma_6) + (n\,\sigma_{13})\otimes(h\,\sigma_7) + (n\,\sigma_{13})\otimes(i\,\sigma_8) + (n\,\sigma_{13})\otimes(j\,\sigma_9) +$
$(n\,\sigma_{13})\otimes(k\,\sigma_{10}) + (n\,\sigma_{13})\otimes(l\,\sigma_{11}) + (n\,\sigma_{13})\otimes(m\,\sigma_{12}) + (n\,\sigma_{13})\otimes(n\,\sigma_{13})$

Simplify the coefficients:

$$a^2\,\sigma_0\otimes\sigma_0 + a\,b\,\sigma_0\otimes\sigma_1 + a\,c\,\sigma_0\otimes\sigma_2 + a\,d\,\sigma_0\otimes\sigma_3 + a\,e\,\sigma_0\otimes\sigma_4 + a\,f\,\sigma_0\otimes\sigma_5 + a\,g\,\sigma_0\otimes\sigma_6 + a\,h\,\sigma_0\otimes\sigma_7 +$$
$$a\,i\,\sigma_0\otimes\sigma_8 + a\,j\,\sigma_0\otimes\sigma_9 + a\,k\,\sigma_0\otimes\sigma_{10} + a\,l\,\sigma_0\otimes\sigma_{11} + a\,m\,\sigma_0\otimes\sigma_{12} + a\,n\,\sigma_0\otimes\sigma_{13} + a\,b\,\sigma_1\otimes\sigma_0 +$$
$$b^2\,\sigma_1\otimes\sigma_1 + b\,c\,\sigma_1\otimes\sigma_2 + b\,d\,\sigma_1\otimes\sigma_3 + b\,e\,\sigma_1\otimes\sigma_4 + b\,f\,\sigma_1\otimes\sigma_5 + b\,g\,\sigma_1\otimes\sigma_6 + b\,h\,\sigma_1\otimes\sigma_7 +$$
$$b\,i\,\sigma_1\otimes\sigma_8 + b\,j\,\sigma_1\otimes\sigma_9 + b\,k\,\sigma_1\otimes\sigma_{10} + b\,l\,\sigma_1\otimes\sigma_{11} + b\,m\,\sigma_1\otimes\sigma_{12} + b\,n\,\sigma_1\otimes\sigma_{13} + a\,c\,\sigma_2\otimes\sigma_0 +$$
$$b\,c\,\sigma_2\otimes\sigma_1 + c^2\,\sigma_2\otimes\sigma_2 + c\,d\,\sigma_2\otimes\sigma_3 + c\,e\,\sigma_2\otimes\sigma_4 + c\,f\,\sigma_2\otimes\sigma_5 + c\,g\,\sigma_2\otimes\sigma_6 + c\,h\,\sigma_2\otimes\sigma_7 +$$
$$c\,i\,\sigma_2\otimes\sigma_8 + c\,j\,\sigma_2\otimes\sigma_9 + c\,k\,\sigma_2\otimes\sigma_{10} + c\,l\,\sigma_2\otimes\sigma_{11} + c\,m\,\sigma_2\otimes\sigma_{12} + c\,n\,\sigma_2\otimes\sigma_{13} + a\,d\,\sigma_3\otimes\sigma_0 +$$
$$b\,d\,\sigma_3\otimes\sigma_1 + c\,d\,\sigma_3\otimes\sigma_2 + d^2\,\sigma_3\otimes\sigma_3 + d\,e\,\sigma_3\otimes\sigma_4 + d\,f\,\sigma_3\otimes\sigma_5 + d\,g\,\sigma_3\otimes\sigma_6 + d\,h\,\sigma_3\otimes\sigma_7 +$$
$$d\,i\,\sigma_3\otimes\sigma_8 + d\,j\,\sigma_3\otimes\sigma_9 + d\,k\,\sigma_3\otimes\sigma_{10} + d\,l\,\sigma_3\otimes\sigma_{11} + d\,m\,\sigma_3\otimes\sigma_{12} + d\,n\,\sigma_3\otimes\sigma_{13} + a\,e\,\sigma_4\otimes\sigma_0 +$$
$$b\,e\,\sigma_4\otimes\sigma_1 + c\,e\,\sigma_4\otimes\sigma_2 + d\,e\,\sigma_4\otimes\sigma_3 + e^2\,\sigma_4\otimes\sigma_4 + e\,f\,\sigma_4\otimes\sigma_5 + e\,g\,\sigma_4\otimes\sigma_6 + e\,h\,\sigma_4\otimes\sigma_7 +$$
$$e\,i\,\sigma_4\otimes\sigma_8 + e\,j\,\sigma_4\otimes\sigma_9 + e\,k\,\sigma_4\otimes\sigma_{10} + e\,l\,\sigma_4\otimes\sigma_{11} + e\,m\,\sigma_4\otimes\sigma_{12} + e\,n\,\sigma_4\otimes\sigma_{13} + a\,f\,\sigma_5\otimes\sigma_0 +$$
$$b\,f\,\sigma_5\otimes\sigma_1 + c\,f\,\sigma_5\otimes\sigma_2 + d\,f\,\sigma_5\otimes\sigma_3 + e\,f\,\sigma_5\otimes\sigma_4 + f^2\,\sigma_5\otimes\sigma_5 + f\,g\,\sigma_5\otimes\sigma_6 + f\,h\,\sigma_5\otimes\sigma_7 +$$
$$f\,i\,\sigma_5\otimes\sigma_8 + f\,j\,\sigma_5\otimes\sigma_9 + f\,k\,\sigma_5\otimes\sigma_{10} + f\,l\,\sigma_5\otimes\sigma_{11} + f\,m\,\sigma_5\otimes\sigma_{12} + f\,n\,\sigma_5\otimes\sigma_{13} + a\,g\,\sigma_6\otimes\sigma_0 +$$
$$b\,g\,\sigma_6\otimes\sigma_1 + c\,g\,\sigma_6\otimes\sigma_2 + d\,g\,\sigma_6\otimes\sigma_3 + e\,g\,\sigma_6\otimes\sigma_4 + f\,g\,\sigma_6\otimes\sigma_5 + g^2\,\sigma_6\otimes\sigma_6 + g\,h\,\sigma_6\otimes\sigma_7 +$$
$$g\,i\,\sigma_6\otimes\sigma_8 + g\,j\,\sigma_6\otimes\sigma_9 + g\,k\,\sigma_6\otimes\sigma_{10} + g\,l\,\sigma_6\otimes\sigma_{11} + g\,m\,\sigma_6\otimes\sigma_{12} + g\,n\,\sigma_6\otimes\sigma_{13} + a\,h\,\sigma_7\otimes\sigma_0 +$$
$$b\,h\,\sigma_7\otimes\sigma_1 + c\,h\,\sigma_7\otimes\sigma_2 + d\,h\,\sigma_7\otimes\sigma_3 + e\,h\,\sigma_7\otimes\sigma_4 + f\,h\,\sigma_7\otimes\sigma_5 + g\,h\,\sigma_7\otimes\sigma_6 + h^2\,\sigma_7\otimes\sigma_7 +$$
$$h\,i\,\sigma_7\otimes\sigma_8 + h\,j\,\sigma_7\otimes\sigma_9 + h\,k\,\sigma_7\otimes\sigma_{10} + h\,l\,\sigma_7\otimes\sigma_{11} + h\,m\,\sigma_7\otimes\sigma_{12} + h\,n\,\sigma_7\otimes\sigma_{13} + a\,i\,\sigma_8\otimes\sigma_0 +$$
$$b\,i\,\sigma_8\otimes\sigma_1 + c\,i\,\sigma_8\otimes\sigma_2 + d\,i\,\sigma_8\otimes\sigma_3 + e\,i\,\sigma_8\otimes\sigma_4 + f\,i\,\sigma_8\otimes\sigma_5 + g\,i\,\sigma_8\otimes\sigma_6 + h\,i\,\sigma_8\otimes\sigma_7 +$$
$$i^2\,\sigma_8\otimes\sigma_8 + i\,j\,\sigma_8\otimes\sigma_9 + i\,k\,\sigma_8\otimes\sigma_{10} + i\,l\,\sigma_8\otimes\sigma_{11} + i\,m\,\sigma_8\otimes\sigma_{12} + i\,n\,\sigma_8\otimes\sigma_{13} + a\,j\,\sigma_9\otimes\sigma_0 +$$
$$b\,j\,\sigma_9\otimes\sigma_1 + c\,j\,\sigma_9\otimes\sigma_2 + d\,j\,\sigma_9\otimes\sigma_3 + e\,j\,\sigma_9\otimes\sigma_4 + f\,j\,\sigma_9\otimes\sigma_5 + g\,j\,\sigma_9\otimes\sigma_6 + h\,j\,\sigma_9\otimes\sigma_7 +$$
$$i\,j\,\sigma_9\otimes\sigma_8 + j^2\,\sigma_9\otimes\sigma_9 + j\,k\,\sigma_9\otimes\sigma_{10} + j\,l\,\sigma_9\otimes\sigma_{11} + j\,m\,\sigma_9\otimes\sigma_{12} + j\,n\,\sigma_9\otimes\sigma_{13} + a\,k\,\sigma_{10}\otimes\sigma_0 +$$
$$b\,k\,\sigma_{10}\otimes\sigma_1 + c\,k\,\sigma_{10}\otimes\sigma_2 + d\,k\,\sigma_{10}\otimes\sigma_3 + e\,k\,\sigma_{10}\otimes\sigma_4 + f\,k\,\sigma_{10}\otimes\sigma_5 + g\,k\,\sigma_{10}\otimes\sigma_6 + h\,k\,\sigma_{10}\otimes\sigma_7 +$$
$$i\,k\,\sigma_{10}\otimes\sigma_8 + j\,k\,\sigma_{10}\otimes\sigma_9 + k^2\,\sigma_{10}\otimes\sigma_{10} + k\,l\,\sigma_{10}\otimes\sigma_{11} + k\,m\,\sigma_{10}\otimes\sigma_{12} + k\,n\,\sigma_{10}\otimes\sigma_{13} + a\,l\,\sigma_{11}\otimes\sigma_0 +$$
$$b\,l\,\sigma_{11}\otimes\sigma_1 + c\,l\,\sigma_{11}\otimes\sigma_2 + d\,l\,\sigma_{11}\otimes\sigma_3 + e\,l\,\sigma_{11}\otimes\sigma_4 + f\,l\,\sigma_{11}\otimes\sigma_5 + g\,l\,\sigma_{11}\otimes\sigma_6 + h\,l\,\sigma_{11}\otimes\sigma_7 +$$
$$i\,l\,\sigma_{11}\otimes\sigma_8 + j\,l\,\sigma_{11}\otimes\sigma_9 + k\,l\,\sigma_{11}\otimes\sigma_{10} + l^2\,\sigma_{11}\otimes\sigma_{11} + l\,m\,\sigma_{11}\otimes\sigma_{12} + l\,n\,\sigma_{11}\otimes\sigma_{13} + a\,m\,\sigma_{12}\otimes\sigma_0 +$$
$$b\,m\,\sigma_{12}\otimes\sigma_1 + c\,m\,\sigma_{12}\otimes\sigma_2 + d\,m\,\sigma_{12}\otimes\sigma_3 + e\,m\,\sigma_{12}\otimes\sigma_4 + f\,m\,\sigma_{12}\otimes\sigma_5 + g\,m\,\sigma_{12}\otimes\sigma_6 + h\,m\,\sigma_{12}\otimes\sigma_7 +$$
$$i\,m\,\sigma_{12}\otimes\sigma_8 + j\,m\,\sigma_{12}\otimes\sigma_9 + k\,m\,\sigma_{12}\otimes\sigma_{10} + l\,m\,\sigma_{12}\otimes\sigma_{11} + m^2\,\sigma_{12}\otimes\sigma_{12} + m\,n\,\sigma_{12}\otimes\sigma_{13} +$$
$$a\,n\,\sigma_{13}\otimes\sigma_0 + b\,n\,\sigma_{13}\otimes\sigma_1 + c\,n\,\sigma_{13}\otimes\sigma_2 + d\,n\,\sigma_{13}\otimes\sigma_3 + e\,n\,\sigma_{13}\otimes\sigma_4 + f\,n\,\sigma_{13}\otimes\sigma_5 + g\,n\,\sigma_{13}\otimes\sigma_6 +$$
$$h\,n\,\sigma_{13}\otimes\sigma_7 + i\,n\,\sigma_{13}\otimes\sigma_8 + j\,n\,\sigma_{13}\otimes\sigma_9 + k\,n\,\sigma_{13}\otimes\sigma_{10} + l\,n\,\sigma_{13}\otimes\sigma_{11} + m\,n\,\sigma_{13}\otimes\sigma_{12} + n^2\,\sigma_{13}\otimes\sigma_{13}$$

By direct computation the following property of the Coproduct can be verified:

$$\Delta(h\,g) = \Delta(h)\diamond\Delta(g)$$

where $\odot$ is 14-Sets product and $\diamond$ as the tensor product of algebras:

$$(a\otimes c)\diamond(b\otimes d) = (a\odot b)\otimes(c\odot d)$$

Remark 16.1: *In most places $\diamond$ left as blank.*

Let’s look at some short example:

$\Delta\ (\sigma_0)$

$\sigma_0\otimes\sigma_0$

$\Delta\ (a\ \sigma_0 + b\ \sigma_1)$

```
(a σ0) ⊗ (a σ0) + (a σ0) ⊗ (b σ1) + (b σ1) ⊗ (a σ0) + (b σ1) ⊗ (b σ1)
```

Simplify:

$a^2\, \sigma_0 \otimes \sigma_0 + a\, b\, \sigma_0 \otimes \sigma_1 + a\, b\, \sigma_1 \otimes \sigma_0 + b^2\, \sigma_1 \otimes \sigma_1$

Compute the Coproduct:

```
Δ (σ0 ⊙ σ5)
```

$\sigma_5 \otimes \sigma_5$

See if it is same as this tensor product of Coproduct:

```
(Δ (σ0)) ⋄ (Δ (σ5))
Off[Syntax::sntxf] (* this warnning occurs but there is no error!!! *)
(σ0 ⊗ σ0) ⋄ (σ5 ⊗ σ5)
```

$\sigma_5 \otimes \sigma_5$

Do the same for this Coproduct:

```
Δ ((a σ0 + b σ1) ⊙ (a2 σ0 + b2 σ1))
((a a2 + b b2) σ0) ⊗ ((a a2 + b b2) σ0) + ((a a2 + b b2) σ0) ⊗ ((a2 b + a b2) σ1) +
  ((a2 b + a b2) σ1) ⊗ ((a a2 + b b2) σ0) + ((a2 b + a b2) σ1) ⊗ ((a2 b + a b2) σ1)
```

$(a\, a2 + b\, b2)^2\, \sigma_0 \otimes \sigma_0 + (a2\, b + a\, b2)\, (a\, a2 + b\, b2)\, \sigma_0 \otimes \sigma_1 +$
$(a2\, b + a\, b2)\, (a\, a2 + b\, b2)\, \sigma_1 \otimes \sigma_0 + (a2\, b + a\, b2)^2\, \sigma_1 \otimes \sigma_1$

It is same as this tensor product:

```
Off[Syntax::sntxf] (* this warnning occurs but there is no error!!! *)
Distribute[(Δ (a σ0 + b σ1)) ⋄ (Δ (a2 σ0 + b2 σ1))]
```

```
((a σ0) ⊗ (a σ0)) ⋄ ((a2 σ0) ⊗ (a2 σ0)) + ((a σ0) ⊗ (a σ0)) ⋄ ((a2 σ0) ⊗ (b2 σ1)) +
  ((a σ0) ⊗ (a σ0)) ⋄ ((b2 σ1) ⊗ (a2 σ0)) + ((a σ0) ⊗ (a σ0)) ⋄ ((b2 σ1) ⊗ (b2 σ1)) +
  ((a σ0) ⊗ (b σ1)) ⋄ ((a2 σ0) ⊗ (a2 σ0)) + ((a σ0) ⊗ (b σ1)) ⋄ ((a2 σ0) ⊗ (b2 σ1)) +
  ((a σ0) ⊗ (b σ1)) ⋄ ((b2 σ1) ⊗ (a2 σ0)) + ((a σ0) ⊗ (b σ1)) ⋄ ((b2 σ1) ⊗ (b2 σ1)) +
  ((b σ1) ⊗ (a σ0)) ⋄ ((a2 σ0) ⊗ (a2 σ0)) + ((b σ1) ⊗ (a σ0)) ⋄ ((a2 σ0) ⊗ (b2 σ1)) +
  ((b σ1) ⊗ (a σ0)) ⋄ ((b2 σ1) ⊗ (a2 σ0)) + ((b σ1) ⊗ (a σ0)) ⋄ ((b2 σ1) ⊗ (b2 σ1)) +
  ((b σ1) ⊗ (b σ1)) ⋄ ((a2 σ0) ⊗ (a2 σ0)) + ((b σ1) ⊗ (b σ1)) ⋄ ((a2 σ0) ⊗ (b2 σ1)) +
  ((b σ1) ⊗ (b σ1)) ⋄ ((b2 σ1) ⊗ (a2 σ0)) + ((b σ1) ⊗ (b σ1)) ⋄ ((b2 σ1) ⊗ (b2 σ1))
```

```
Simplify[(a a2 σ0) ⊗ (a a2 σ0) + (a a2 σ0) ⊗ (b b2 σ0) + (a a2 σ0) ⊗ (a2 b σ1) +
   (a a2 σ0) ⊗ (a b2 σ1) + (b b2 σ0) ⊗ (a a2 σ0) + (b b2 σ0) ⊗ (b b2 σ0) +
   (b b2 σ0) ⊗ (a2 b σ1) + (b b2 σ0) ⊗ (a b2 σ1) + (a2 b σ1) ⊗ (a a2 σ0) +
   (a2 b σ1) ⊗ (b b2 σ0) + (a2 b σ1) ⊗ (a2 b σ1) + (a2 b σ1) ⊗ (a b2 σ1) +
   (a b2 σ1) ⊗ (a a2 σ0) + (a b2 σ1) ⊗ (b b2 σ0) + (a b2 σ1) ⊗ (a2 b σ1) + (a b2 σ1) ⊗ (a b2 σ1)]
```

$$(a\,a2 + b\,b2)^2\, \sigma_0 \otimes \sigma_0 + (a2\,b + a\,b2)\,((a\,a2 + b\,b2)\, \sigma_0 \otimes \sigma_1 + (a\,a2 + b\,b2)\, \sigma_1 \otimes \sigma_0 + (a2\,b + a\,b2)\, \sigma_1 \otimes \sigma_1)$$

Let’s show $\Delta(1) = 1 \otimes 1$:

$$\Delta\left(\underset{14\ \text{Sets}}{1}\right)$$

$\sigma_0 \otimes \sigma_0$

$$\underset{14\ \text{Sets}}{1} \otimes \underset{14\ \text{Sets}}{1}$$

$\sigma_0 \otimes \sigma_0$

In summary, this Coproduct allows for 14-Sets algebra to become a Hopf Algebra.

17. 14-D Vector: Feel of the Space

The 14-Sets algebra and computations are used to model how space is felt.

Remark 17.1: *Concept of Category is used by Alfred North Whitehead in place of Axioms and Definitions. This forgotten approach is the most fruitful. Categories, similar to the mathematical counterpart, are over-generalizations of* <u>*ordinary concepts stretched beyond their normal meanings*</u>*. For example author used the word Here or There below, they are the here and there we feel all the time but stretched almost to a new meaning e.g. a 14-D vector which has no reverse.*

Category of Memory

Every being feels a space, even before becoming (coming to existence), during the being, while ceasing and after cessation i.e. non-being.

Every being is a memory. When remembering and when forgetting space is felt. When space felt something was remembered or forgotten.

Category of Existence

Feeling of the space.

There.

Here.

Organism has one vector of feeling for space. (not counting Here and There and other general vectors)

Dis-organism has multiple or infinite vectors of feelings for space.

Example: Human being is an organism, no matter how many cells it is comprised of and how many limbs and organs it operates with, feels one locality of space thus we afford to say “I am here”. On the contrary the electron is a Dis-organism, it passes through two slits on a metal barrier at the same time! electron feels the space by multiple vectors.

Category of Explanation

Space is felt due to the transfer of Information or knowledge from ‘there to here’, or transfer of erasing or forgetting of information or knowledge from there to here. Feeling of space and flux of information or knowledge always appear as a pair.

In general, Feelings are ‘vectors’; for they feel what is there and transform it into what is here [3]. This vector, feeling the spatial attributes, is the 14-D vector as a member of the 14-Sets algebra.

Things do not reside within some space, they feel the space and one such feeling is that of an interior thus we say it is inside a space.

Two endpoints of the said vector are:

1) tail or ‘there’
2) head or ‘here’

all the while the vector transform what is ‘there’ into ‘here’. Just like a memory, copy from one address to another, or copy+erase from one location to another or just erase i.e. from here to here.

‘here’ and ‘there’ cannot be erased.

One feeling of the space followed by another is the binary product in the 14-Sets algebra of two of its vectors.

An ordered sequence of feeling vectors, finite or infinite, is called a Complex Feeling. They are felt one after the other, while preserving the order.

Accumulative multiplication of sequence of feelings in a Complex Feeling is called Evolution and accumulative vectors/feelings are said evolving.

If such an infinite sequence of Complex Feeling has an accumulative limit, we say its limit emerged, or Complex Feeling is emergent.

Two vectors A and B are said to feel the same space if their commutator is 0 :

$$[A, B] = 0$$

A feeling is Persistent if it has no reverse (determinant 0). See Also: Corollary 14.1-3.

Appendix A

We assume all strings and programs are binary coded.

Definition A.1: The Kolmogorov Complexity $C_{\mathcal{U}}(x)$ of a string x with respect to a universal computer (Turing Machine) $\mathcal{U}$ is defined as

$$C_{\mathcal{U}}(x) = \min_{p:\mathcal{U}(p) = x} l(p)$$

the minimum length program p in $\mathcal{U}$ which outputs x.

Theorem A.1 (Universality of the Kolmogorov Complexity): *If $\mathcal{U}$ is a universal computer, then for any other computer $\mathcal{A}$ and all strings x,*

$$C_{\mathcal{U}}(x) \leq C_{\mathcal{A}}(x) + c_{\mathcal{A}}$$

where the constant $c_{\mathcal{A}}$ does not depend on x.

Corollary A.1: $\lim_{l(x)\to\infty} \frac{C_{\mathcal{U}}(x) - C_{\mathcal{A}}(x)}{l(x)} = 0$ *for any two universal computers.*

Remark A.1: *Therefore we drop the universal computer subscript and simply write $C(x)$.*

Theorem A.2: *$C(x) \leqslant |x| + c$.*

A string x is called incompressible if $C(x) \geqslant x$.

Definition A.2: Self-delimiting string (or program) is a string or program which has its own length encoded as a part of itself i.e. a Turing machine reading Self-delimiting string knows exactly when to stop reading.

Definition A.3: The Conditional or Prefix Kolmogorov Complexity of self-delimiting string x given string y is

$$K(x \mid y) = \min_{p:\mathcal{U}(p,\, y) = x} l(p)$$

The length of the shortest program that can compute both x and y and a way to tell them apart is

$$K(x,\, y) = \min_{p:\mathcal{U}(p) = x,y} l(p)$$

Remark A.2: *x, y can be thought of as concatenation of the strings with additional separation information.*

Theorem A.3: $K(x) \leq l(x) + 2 \log l(x) + O(1), \quad K(x \mid l(x)) \leq l(x) + O(1)$.

Theorem A.4: $K(x,\, y) \leq K(x) + K(y)$.

Theorem A.5: $K(f(x)) \leq K(x) + K(f)$, f *a* computble function

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