14-Sets Algebra

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Abstract

A novel 14-D algebra based upon Kuratowski Monoid's Closure and Complement operators is uncovered. 14×14 Matrix Representation, Conditions for inverse elements, 0-Commutator computed using *Mathematica* 9.0 . Almost Random Almost Incompressible large-length coefficients are found to have inverse. Matrix representation can have the form of a Stochastic Matrix, and its infinite power has a limit with two 1-valued eigenvalues and determinant 0. An approximate Logarithmic algorithm devised for mapping Kuratowski's topological operators to 182 dimensional Lie Algebra of 14×14 Stochastic Matrices, thus a concept of pre-geometry is brought forth.

Preliminaries

Let X be a topological space. Denote A^{-} closure of a the set $A \subseteq X$ and A^{c} the Complement. A widely known fact due to K. Kuratowski [1] states that at most 14 distinct operations can be formed by such compositions of the two operators!

Kuratowski Operations:

$\sigma_0(A) = A$ (The Identity), $\sigma_1(A) = A^{c}$ (The Complement),
$\sigma_2(A) = A^{-}$ (The Closure), $\sigma_3(A) = A^{c-}$,
$\sigma_4(A) = A^{-c}$, $\sigma_5(A) = A^{c-c}$ (The Interior),
$\sigma_6(A) = A^{-c-}$, $\sigma_7(A) = A^{c-c-}$,
$\sigma_8(A) = A^{-c-c}$, $\sigma_9(A) = A^{c-c-c}$,
$\sigma_{10}(A) = A^{-c-c-}$, $\sigma_{11}(A) = A^{c-c-c-}$,
$\sigma_{12}(A) = A^{-c-c-c}$, $\sigma_{13}(A) = A^{c-c-c-c}$,

Cancellation rules:

$A^{-c-} = A^{-c-c-c-}$ $\quad$ $A^{c-c-} = A^{c-c-c-c-}$

The corresponding Cayley table for Kuratowski Monoid [1], also see [2] section 1:

$$\left(\begin{array}{cccccccccccccc}
\sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_1 & \sigma_0 & \sigma_4 & \sigma_5 & \sigma_2 & \sigma_3 & \sigma_8 & \sigma_9 & \sigma_6 & \sigma_7 & \sigma_{12} & \sigma_{13} & \sigma_{10} & \sigma_{11} \\
\sigma_2 & \sigma_3 & \sigma_2 & \sigma_3 & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_3 & \sigma_2 & \sigma_6 & \sigma_7 & \sigma_2 & \sigma_3 & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_4 & \sigma_5 & \sigma_4 & \sigma_5 & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_5 & \sigma_4 & \sigma_8 & \sigma_9 & \sigma_4 & \sigma_5 & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} \\
\sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_7 & \sigma_6 & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} \\
\sigma_9 & \sigma_8 & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 \\
\sigma_{11} & \sigma_{10} & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} & \sigma_{10} & \sigma_{11} & \sigma_6 & \sigma_7 & \sigma_6 & \sigma_7 & \sigma_{10} & \sigma_{11} \\
\sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 \\
\sigma_{13} & \sigma_{12} & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13} & \sigma_{12} & \sigma_{13} & \sigma_8 & \sigma_9 & \sigma_8 & \sigma_9 & \sigma_{12} & \sigma_{13}
\end{array}\right)$$

Methodology: Symbolic Computation

Kuratowski Monoid [1] has been around since 1922, yet no new results nor extensions have been found so far. This is not due to the scantiness of its structures, rather it is almost impossible to hand compute or prove any new results or extensions for this monoid due to the size issue.

By application of *Mathematica* 9.0 the author not only was able to extend the Kuratowski Monoid [1], but also using Monte-Carlo like random processes was able to uncover theorems and their corresponding proofs.

As the reader can see in this treatise, the algebraic structures are not trivial, sizeable and the theorems are not evident nor can easily be guessed by unaided human mind.

What follows of methodologies and code empower the researcher in three ways:

1. S/he needs not be a PhD genius algebraist to figure out the theorems and proofs. Indeed undergraduate students or researchers from other fields e.g. computer sciences can take part in such investigations.
2. Larger algebras can be investigated and the only limits would be the CPU and memory, which can easily be remedied in most cases
3. Learning curve is far less steep than traditional paper and pencil algebra, indeed an investigation can be carried out in record time.

Reference [2] and [3] provides all the code and computations.

1. Basis $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}\}$

Imagine σ_i as vector basis for a vector space over a (commutative) Field e.g. $\mathbb{R}$ or $\mathbb{C}$:

$$v = \sum_{i=0}^{13} a_i\,\sigma_i \qquad \text{(EQ 1.1)}$$

We call such a vector 14-Sets Vector.

Let’s define such two vectors v1 and v2 :

$$\mathrm{v1} = a\,\sigma_0 + b\,\sigma_1 + c\,\sigma_2 + d\,\sigma_3 + e\,\sigma_4 + f\,\sigma_5 + g\,\sigma_6 + h\,\sigma_7 + i\,\sigma_8 + j\,\sigma_9 + k\,\sigma_{10} + l\,\sigma_{11} + m\,\sigma_{12} + n\,\sigma_{13}$$
$$\mathrm{v2} = \mathrm{a2}\,\sigma_0 + \mathrm{b2}\,\sigma_1 + \mathrm{c2}\,\sigma_2 + \mathrm{d2}\,\sigma_3 + \mathrm{e2}\,\sigma_4 + \mathrm{f2}\,\sigma_5 + \mathrm{g2}\,\sigma_6 + \mathrm{h2}\,\sigma_7 + \mathrm{i2}\,\sigma_8 + \mathrm{j2}\,\sigma_9 + \mathrm{k2}\,\sigma_{10} + \mathrm{l2}\,\sigma_{11} + \mathrm{m2}\,\sigma_{12} + \mathrm{n2}\,\sigma_{13}$$

Calculate their multiplicative product using the Cayley table above, [2] section 2:

$$\begin{aligned}
&(a\,\mathrm{a2} + b\,\mathrm{b2})\,\sigma_0 + (\mathrm{a2}\,b + a\,\mathrm{b2})\,\sigma_1 + \\
&\quad (\mathrm{a2}\,c + a\,\mathrm{c2} + c\,\mathrm{c2} + \mathrm{b2}\,d + b\,\mathrm{e2} + d\,\mathrm{e2})\,\sigma_2 + (\mathrm{b2}\,c + \mathrm{a2}\,d + a\,\mathrm{d2} + c\,\mathrm{d2} + b\,\mathrm{f2} + d\,\mathrm{f2})\,\sigma_3 + \\
&\quad (b\,\mathrm{c2} + \mathrm{a2}\,e + \mathrm{c2}\,e + a\,\mathrm{e2} + \mathrm{b2}\,f + \mathrm{e2}\,f)\,\sigma_4 + (b\,\mathrm{d2} + \mathrm{b2}\,e + \mathrm{d2}\,e + \mathrm{a2}\,f + a\,\mathrm{f2} + f\,\mathrm{f2})\,\sigma_5 + \\
&\quad (\mathrm{c2}\,d + c\,\mathrm{e2} + \mathrm{a2}\,g + \mathrm{c2}\,g + a\,\mathrm{g2} + c\,\mathrm{g2} + \mathrm{b2}\,h + \mathrm{e2}\,h + \mathrm{g2}\,h + b\,\mathrm{i2} + d\,\mathrm{i2} + g\,\mathrm{i2} + \mathrm{e2}\,k + \mathrm{g2}\,k + d\,\mathrm{k2} + \\
&\qquad g\,\mathrm{k2} + \mathrm{c2}\,l + \mathrm{i2}\,l + \mathrm{k2}\,l + c\,\mathrm{m2} + h\,\mathrm{m2} + k\,\mathrm{m2})\,\sigma_6 + (d\,\mathrm{d2} + c\,\mathrm{f2} + \mathrm{b2}\,g + \mathrm{d2}\,g + \mathrm{a2}\,h + \mathrm{f2}\,h + a\,\mathrm{h2} + \\
&\qquad c\,\mathrm{h2} + h\,\mathrm{h2} + b\,\mathrm{j2} + d\,\mathrm{j2} + g\,\mathrm{j2} + \mathrm{f2}\,k + \mathrm{h2}\,k + \mathrm{d2}\,l + \mathrm{j2}\,l + d\,\mathrm{l2} + g\,\mathrm{l2} + l\,\mathrm{l2} + c\,\mathrm{n2} + h\,\mathrm{n2} + k\,\mathrm{n2})\,\sigma_7 + \\
&\quad (e\,\mathrm{e2} + \mathrm{c2}\,f + b\,\mathrm{g2} + e\,\mathrm{g2} + \mathrm{a2}\,i + \mathrm{c2}\,i + a\,\mathrm{i2} + f\,\mathrm{i2} + i\,\mathrm{i2} + \mathrm{b2}\,j + \mathrm{e2}\,j + \mathrm{g2}\,j + f\,\mathrm{k2} + i\,\mathrm{k2} + \\
&\qquad \mathrm{e2}\,m + \mathrm{g2}\,m + e\,\mathrm{m2} + j\,\mathrm{m2} + m\,\mathrm{m2} + \mathrm{c2}\,n + \mathrm{i2}\,n + \mathrm{k2}\,n)\,\sigma_8 + \\
&\quad (\mathrm{d2}\,f + e\,\mathrm{f2} + b\,\mathrm{h2} + e\,\mathrm{h2} + \mathrm{b2}\,i + \mathrm{d2}\,i + \mathrm{a2}\,j + \mathrm{f2}\,j + \mathrm{h2}\,j + a\,\mathrm{j2} + f\,\mathrm{j2} + i\,\mathrm{j2} + f\,\mathrm{l2} + i\,\mathrm{l2} + \\
&\qquad \mathrm{f2}\,m + \mathrm{h2}\,m + \mathrm{d2}\,n + \mathrm{j2}\,n + \mathrm{l2}\,n + e\,\mathrm{n2} + j\,\mathrm{n2} + m\,\mathrm{n2})\,\sigma_9 + \\
&\quad (\mathrm{e2}\,g + d\,\mathrm{g2} + g\,\mathrm{g2} + \mathrm{c2}\,h + c\,\mathrm{i2} + h\,\mathrm{i2} + \mathrm{a2}\,k + \mathrm{c2}\,k + \mathrm{i2}\,k + a\,\mathrm{k2} + c\,\mathrm{k2} + h\,\mathrm{k2} + k\,\mathrm{k2} + \mathrm{b2}\,l + \mathrm{e2}\,l + \\
&\qquad \mathrm{g2}\,l + b\,\mathrm{m2} + d\,\mathrm{m2} + g\,\mathrm{m2} + l\,\mathrm{m2})\,\sigma_{10} + (\mathrm{f2}\,g + \mathrm{d2}\,h + d\,\mathrm{h2} + g\,\mathrm{h2} + c\,\mathrm{j2} + h\,\mathrm{j2} + \mathrm{b2}\,k + \\
&\qquad \mathrm{d2}\,k + \mathrm{j2}\,k + \mathrm{a2}\,l + \mathrm{f2}\,l + \mathrm{h2}\,l + a\,\mathrm{l2} + c\,\mathrm{l2} + h\,\mathrm{l2} + k\,\mathrm{l2} + b\,\mathrm{n2} + d\,\mathrm{n2} + g\,\mathrm{n2} + l\,\mathrm{n2})\,\sigma_{11} + \\
&\quad (f\,\mathrm{g2} + \mathrm{e2}\,i + \mathrm{g2}\,i + e\,\mathrm{i2} + \mathrm{c2}\,j + \mathrm{i2}\,j + b\,\mathrm{k2} + e\,\mathrm{k2} + j\,\mathrm{k2} + \mathrm{a2}\,m + \mathrm{c2}\,m + \mathrm{i2}\,m + \mathrm{k2}\,m + a\,\mathrm{m2} + f\,\mathrm{m2} + \\
&\qquad i\,\mathrm{m2} + \mathrm{b2}\,n + \mathrm{e2}\,n + \mathrm{g2}\,n + \mathrm{m2}\,n)\,\sigma_{12} + (f\,\mathrm{h2} + \mathrm{f2}\,i + \mathrm{h2}\,i + \mathrm{d2}\,j + e\,\mathrm{j2} + j\,\mathrm{j2} + b\,\mathrm{l2} + e\,\mathrm{l2} + \\
&\qquad j\,\mathrm{l2} + \mathrm{b2}\,m + \mathrm{d2}\,m + \mathrm{j2}\,m + \mathrm{l2}\,m + \mathrm{a2}\,n + \mathrm{f2}\,n + \mathrm{h2}\,n + a\,\mathrm{n2} + f\,\mathrm{n2} + i\,\mathrm{n2} + n\,\mathrm{n2})\,\sigma_{13}
\end{aligned}$$

From this product we can compute the 14×14 matrix representation, [2] section 3:

$$v = \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & d & a+c & 0 & b+d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & c & 0 & a+c & 0 & b+d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & f & b+e & 0 & a+f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f & e & 0 & b+e & 0 & a+f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g & h & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l & 0 & d+g+l & 0 & c+h+k & 0 \\
h & g & 0 & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l & 0 & d+g+l & 0 & c+h+k \\
i & j & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n & 0 & f+i+n & 0 & e+j+m & 0 \\
j & i & 0 & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n & 0 & f+i+n & 0 & e+j+m \\
k & l & h+k & 0 & g+l & 0 & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l & 0 \\
l & k & 0 & h+k & 0 & g+l & 0 & d+g+l & 0 & c+h+k & 0 & a+c+h+k & 0 & b+d+g+l \\
m & n & j+m & 0 & i+n & 0 & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n & 0 \\
n & m & 0 & j+m & 0 & i+n & 0 & f+i+n & 0 & e+j+m & 0 & b+e+j+m & 0 & a+f+i+n
\end{pmatrix}$$

(EQ 1.2)

In order for this product to form a group, the matrix representation must have an inverse i.e. non-zero determinant, and the conditions for determinant being 0, see [2] section 7:

$$\Big\{\{a \to -b\}, \{a \to -b\}, \{a \to -b\}, \{a \to b\}, \{a \to b\}, \{a \to b\}, \Big\{a \to \tfrac{1}{2}\left(-c - f - \sqrt{\triangle}\right)\Big\},$$
$$\Big\{a \to \tfrac{1}{2}\left(-c - f - \sqrt{\triangle}\right)\Big\}, \Big\{a \to \tfrac{1}{2}\left(-c - f + \sqrt{\triangle}\right)\Big\}, \Big\{a \to \tfrac{1}{2}\left(-c - f + \sqrt{\triangle}\right)\Big\},$$
$$\{a \to -b - c - d - e - f - g - h - i - j - k - l - m - n\}, \{a \to -b - c - d - e - f - g - h - i - j - k - l - m - n\},$$
$$\{a \to b - c + d + e - f + g - h - i + j - k + l + m - n\}, \{a \to b - c + d + e - f + g - h - i + j - k + l + m - n\}\Big\}$$

where $\triangle = 4 b^2 + c^2 + 4 b d + 4 b e + 4 d e - 2 c f + f^2$ (EQ 1.3)

Remark 1.1: $\{a \to -b\}$ *means if* $a = -b$ *then the determinant is 0. Each {...} is a single criterion by itself, ‘,’ stands for ‘or’.*

When such 14-Sets vectors form an algebra we call the result 14-Sets Algebra.

Corollary 1.1: *Matrix representation (EQ 1.2) applies to both multiplication and addition of 14-Sets vectors.*

Random Coefficient Theorem:

Theorem 1.1: 14-Sets vectors with (almost-incompressible) almost-random coefficients have inverse.

Proof: We model the concept of randomness by Kolmogorov complexity (Appendix A). By constraints of (EQ 1.2) an irreversible element has to have a pattern to its coefficients and therefore could not be as random as possible since its Kolmogorov complexity will be considerably less than the length of strings of its coefficients, recall Theorem A.2 and $C(x) \geqslant x$ if x (concatenated coefficients) were incompressible . Therefore all random or incompressible vectors have inverse.

□

By direct computation, see [2]:

Corollary 1.2: *The following single criterion makes the 14-Sets Vectors commutative:*

$$\Big\{\Big\{e2 \to -\frac{d2\,(-b-e)}{b+d} - \frac{b2\,(d-e)}{b+d},\ f2 \to c2 - \frac{b2\,(c-f)}{b+d} - \frac{d2\,(c-f)}{b+d},$$

$$j2 \to g2 + \Big(i2\,\big(-(c+h+k)^2 + (b+d+g+l)^2\big)\,(-(g-j)\,(b+d+g+l) + (c+h+k)\,(c-f+k-n))\Big)\Big/$$

$$\Big(\big((c+h+k)^2 - (b+d+g+l)^2\big)\,((-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m))\Big) -$$

$$(c2\,(d\,g - e\,g - c\,h + f\,h + c\,i - f\,i - d\,j + e\,j - h\,k + i\,k + g\,l - j\,l - g\,m + j\,m + h\,n - i\,n))\,/$$

$$(c\,d - c\,e - b\,h - e\,h - g\,h + b\,i + d\,i + g\,i + d\,k - e\,k + c\,l + i\,l + k\,l - c\,m - h\,m - k\,m) -$$

$$(k2\,(d\,g - e\,g - c\,h + f\,h + c\,i - f\,i - d\,j + e\,j - h\,k + i\,k + g\,l - j\,l - g\,m + j\,m + h\,n - i\,n))\,/$$

$$(c\,d - c\,e - b\,h - e\,h - g\,h + b\,i + d\,i + g\,i + d\,k - e\,k + c\,l + i\,l + k\,l - c\,m - h\,m - k\,m) -$$

$$\Big(h2\,\big(c^2 - c\,f - b\,g - e\,g - g^2 + c\,i - f\,i + b\,j + e\,j + g\,j + 2\,c\,k - f\,k + i\,k + k^2 - g\,m + j\,m - c\,n - i\,n - k\,n\big)\Big)\Big/$$

$$(c\,d - c\,e - b\,h - e\,h - g\,h + b\,i + d\,i + g\,i + d\,k - e\,k + c\,l + i\,l + k\,l - c\,m - h\,m - k\,m),$$

$$l2 \to -b2 - d2 - g2 - \frac{i2\,\big(-(c+h+k)^2 + (b+d+g+l)^2\big)}{(-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)} - (h2\,(-(-c-i-k)\,(c+h+k) + (b+d+g+l)\,(-b-e-g-m)))\,/$$

$$((-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)) -$$

$$\frac{c2\,(-(h-i)\,(c+h+k) + (b+d+g+l)\,(d-e+l-m))}{(-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)} - \frac{k2\,(-(h-i)\,(c+h+k) + (b+d+g+l)\,(d-e+l-m))}{(-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)},$$

$$m2 \to -\frac{b2\,(b+e)}{b+d} - \frac{d2\,(b+e)}{b+d} - g2 - i2\,\Big(\frac{c+h+k}{b+d+g+l} - \Big(\big(-(c+h+k)^2 + (b+d+g+l)^2\big)\,(-b-e-g-m)\Big)\Big/$$

$$((b+d+g+l)\,((-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)))\Big) -$$

$$h2\,\Big(\frac{-c-i-k}{b+d+g+l} - ((-(-c-i-k)\,(c+h+k) + (b+d+g+l)\,(-b-e-g-m))\,(-b-e-g-m))\,/$$

$$((b+d+g+l)\,((-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)))\Big) -$$

$$c2\,\Big(\frac{h-i}{b+d+g+l} - ((-(h-i)\,(c+h+k) + (b+d+g+l)\,(d-e+l-m))\,(-b-e-g-m))\,/$$

$$((b+d+g+l)\,((-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)))\Big) -$$

$$k2\,\Big(\frac{h-i}{b+d+g+l} - ((-(h-i)\,(c+h+k) + (b+d+g+l)\,(d-e+l-m))\,(-b-e-g-m))\,/$$

$$((b+d+g+l)\,((-c-i-k)\,(b+d+g+l) - (c+h+k)\,(-b-e-g-m)))\Big),$$

$$n2 \to -\frac{b2\,(-c+f)}{b+d} - \frac{d2\,(-c+f)}{b+d} - (i2\,(b\,c + c\,d - b\,f - d\,f - f\,g - g\,h + c\,j + h\,j + b\,k + d\,k + j\,k + c\,l - f\,l + k\,l - b\,n - d\,n - g\,n - l\,n))\,/$$

$$(c\,d - c\,e - b\,h - e\,h - g\,h + b\,i + d\,i + g\,i + d\,k - e\,k + c\,l + i\,l + k\,l - c\,m - h\,m - k\,m) -$$

$$(h2\,(-b\,c - c\,e + b\,f + e\,f + f\,g + g\,i - c\,j - i\,j - b\,k - e\,k - j\,k - c\,m + f\,m - k\,m + b\,n + e\,n + g\,n + m\,n))\,/$$

$$(c\,d - c\,e - b\,h - e\,h - g\,h + b\,i + d\,i + g\,i + d\,k - e\,k + c\,l + i\,l + k\,l - c\,m - h\,m - k\,m) -$$

$$(k2\,(-d\,f + e\,f + b\,h + e\,h - b\,i - d\,i + h\,j - i\,j - f\,l - i\,l + f\,m + h\,m - d\,n + e\,n - l\,n + m\,n))\,/$$

$$(c\,d - c\,e - b\,h - e\,h - g\,h + b\,i + d\,i + g\,i + d\,k - e\,k + c\,l + i\,l + k\,l - c\,m - h\,m - k\,m) -$$

$$(c2\,(c\,d - c\,e - d\,f + e\,f - g\,h + g\,i + h\,sj - i\,j + d\,k - e\,k + c\,l - f\,l + k\,l - c\,m + f\,m - k\,m - d\,n + e\,n - l\,n + m\,n))\,/$$

$$(c\,d - c\,e - b\,h - e\,h - g\,h + b\,i + d\,i + g\,i + d\,k - e\,k + c\,l + i\,l + k\,l - c\,m - h\,m - k\,m)\Big\}\Big\}$$

(EQ 1.4)

A 14-Sets vector is discrete (i.e. specifies a discrete topological space) if it is of the form:

$a\,\sigma_0 + b\,\sigma_1$ (EQ 1.5)

By direct symbolic computations, see [3]:

Corollary 1.3: *Discrete 14-Sets vectors commute.*

Corollary 1.4: *Discrete and non-discrete 14-Sets vectors do not commute.*

2. Stochastic Matrix

This matrix representation is a Left Stochastic if:

$a+b+c+d+e+f+g+h+i+j+k+l+m+n = 1$ (EQ 2.1)

and all these coefficients are assumed greater than or equal to 0.

We call these 14-Sets Stochastic Vectors or matrices.

Remark 2.1: *14-Sets Stochastic Vectors are closed under the multiplication of (EQ 1.2) representation but not addition.*

Zeros of the determinant (EQ 1.3) are simplified in case due to the stochastic condition:

$$\left\{\{a\to -b\}, \{a\to -b\}, \{a\to -b\}, \{a\to b\}, \{a\to b\}, \{a\to b\}, \left\{a\to \frac{1}{2}\left(-c-f-\sqrt{\triangle}\right)\right\}, \left\{a\to \frac{1}{2}\left(-c-f-\sqrt{\triangle}\right)\right\}, \left\{a\to \frac{1}{2}\left(-c-f+\sqrt{\triangle}\right)\right\}, \left\{a\to \frac{1}{2}\left(-c-f+\sqrt{\triangle}\right)\right\}\right\}$$

where $\triangle = 4b^2+c^2+4bd+4be+4de-2cf+f^2$

(EQ 2.2)

Therefore:

Lemma 2.1: *First 6 coefficients (a, b, c, d, e, f) determine if the 14-Sets Stochastic matrix has an inverse. The other 8 coefficients play no role.*

The following is a list of eigenvalues, see [2] section 4:

$$\left\{1, 1, a-b, a-b, a-b, a+b, a+b, a+b, \frac{1}{2}\left(2a+c+f-\sqrt{\triangle}\right), \frac{1}{2}\left(2a+c+f-\sqrt{\triangle}\right), \frac{1}{2}\left(2a+c+f+\sqrt{\triangle}\right), \frac{1}{2}\left(2a+c+f+\sqrt{\triangle}\right), 1-2b-2d-2e-2g-2j-2l-2m, 1-2b-2d-2e-2g-2j-2l-2m\right\}$$

where $\triangle = 4b^2+c^2+4bd+4be+4de-2cf+f^2$ (EQ 2.3)

Remark 2.2: *Often the eigenvector* {*1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1*} *might not correspond to the above 1-valued eigenvalues, but can be obtained through the linear combination of their corresponding eigenvectors!*

Let v be a 14-Sets stochastic vector/matrix of the form (EQ 1.2), then we define its infinite power as

$v^\infty = \lim_{n\longrightarrow\infty} v^n$.

By direct symbolic computation [3]:

Theorem 2.1: *If v is a discrete 14-Sets stochastic vector (EQ 1.5), so is v^∞.*

This is a powerful result: <u>A discrete topological space stochastically cannot be become non-discrete</u>.

<u>Or passage of time (or iterations of a process) cannot morph a topological space non-discrete</u>. They are incurably discrete topological structures.

Theorem 2.2: *Infinite power of a 14-Sets Stochastic matrix with all positive entries and non-zero first 6*

coefficients, has a limit which is Stochastic, with two 1-valued Eigenvalues. Moreover all 6 coefficients for the infinite power limit are 0, and the remaining 8 coefficients are repetition of the first 4:

$$v^{\infty} = \left(\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & \beta & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 \\
\beta & \alpha & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta \\
\omega & \eta & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 \\
\eta & \omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega \\
\alpha & \beta & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 \\
\beta & \alpha & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta \\
\omega & \eta & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 \\
\eta & \omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega & 0 & \eta+\omega
\end{array}\right)$$

where $\alpha+\beta+\omega+\eta = \frac{1}{2}$, $\omega = \frac{\alpha-2\alpha^2-2\alpha\beta}{2(\alpha+\beta)}$,

or

$$v^{\infty} = \left(\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & \beta & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 \\
\beta & \alpha & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta \\
-\frac{\alpha(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & -\frac{\beta(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 \\
-\frac{\beta(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & -\frac{\alpha(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta \\
\alpha & \beta & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 \\
\beta & \alpha & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta & 0 & \alpha+\beta \\
-\frac{\alpha(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & -\frac{\beta(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 \\
-\frac{\beta(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & -\frac{\alpha(-1+2\alpha+2\beta)}{2(\alpha+\beta)} & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta & 0 & \frac{1}{2}-\alpha-\beta
\end{array}\right)$$

and $\alpha, \beta, \omega, \eta \leqslant \frac{1}{8}$.

Proof*:* It is well-known fact that the infinite powers of non-negative stochastic matrices converge. In our specific case, for the first 6 coefficients being non-zero, we can estimate the first column of v^n by the following upper-bounds assuming:

$$v = a\sigma_0 + b\sigma_1 + c\sigma_2 + d\sigma_3 + e\sigma_4 + f\sigma_5 + g\sigma_6 + h\sigma_7 + i\sigma_8 + j\sigma_9 + k\sigma_{10} + l\sigma_{11} + m\sigma_{12} + n\sigma_{13}$$

With all positive coefficients, its power v^n can be approximated from above, coefficient by coefficient, by X while assuming $0 < x \leqslant 1$:

$$X = \frac{x}{14}\sigma_0 + \frac{x}{14}\sigma_1 + \frac{x}{14}\sigma_2 + \frac{x}{14}\sigma_3 + \frac{x}{14}\sigma_4 + \frac{x}{14}\sigma_5 + \frac{x}{14}\sigma_6 + \frac{x}{14}\sigma_7 + \frac{x}{14}\sigma_8 + \frac{x}{14}\sigma_9 + \frac{x}{14}\sigma_{10} + \frac{x}{14}\sigma_{11} + \frac{x}{14}\sigma_{12} + \frac{x}{14}\sigma_{13}$$

Where X^n is computed symbolically, see
[3]:

$$X^n = \frac{1}{2} 7^{-n} x^n \sigma_0 + \frac{1}{2} 7^{-n} x^n \sigma_1 + \frac{1}{2} 7^{-n} (-1+2^n) x^n \sigma_2 + \frac{1}{2} 7^{-n} (-1+2^n) x^n \sigma_3 + \frac{1}{2} 7^{-n} (-1+2^n) x^n \sigma_4 + \frac{1}{2} 7^{-n} (-1+2^n) x^n \sigma_5 + \frac{1}{2} 7^{-n} (-1+2^n) x^n \sigma_6 + \frac{1}{8} 7^{-n} (-1+7^n-2n) x^n \sigma_7 + \frac{1}{8} 7^{-n} (-1+7^n-2n) x^n \sigma_8 + \frac{1}{8} 7^{-n} (-1+7^n-2n) x^n \sigma_9 + \frac{1}{8} 7^{-n} (-1+7^n-2n) x^n \sigma_{10} + \frac{1}{8} 7^{-n} \left(3-2^{2+n}+7^n+2n\right) x^n \sigma_{11} + \frac{1}{8} 7^{-n} \left(3-2^{2+n}+7^n+2n\right) x^n \sigma_{12} + \frac{1}{8} 7^{-n} \left(3-2^{2+n}+7^n+2n\right) x^n \sigma_{13}$$

Or

$$(v^n)_{s,1} \leqslant \frac{1}{2} 7^{-n} x^n \quad s = 1, 2$$

$$(v^n)_{s,1} \leqslant \frac{1}{2} 7^{-n} (-1+2^n) x^n \quad s = 3, 4, 5, 6$$

$$(v^n)_{s,1} \leqslant \frac{1}{8} 7^{-n} (-1+7^n-2n) x^n \quad s = 7, 8, 9, 10$$

$$(v^n)_{s,1} \leqslant \frac{1}{8} 7^{-n} \left(3-2^{2+n}+7^n+2n\right) x^n \quad s = 11, 12, 13, 14$$

Basically X, X^n serve as ‘order of magnitude’ vector to approximate the coefficients for v^n from above. This only works for the positive x i.e. the coefficients are replaced by the x and product of X^n is computed according to the 14-Sets algebra rules but if x approaches 1 then the corresponding coefficients of X^n become upper-bounds to coefficients of v^n. $\frac{1}{14}$ serves as a normalizer to make sure X, X^n are both Stochastic as x approaches 1.

It is obvious

$$(v^n)_{s,1} \leqslant \frac{1}{2} 7^{-n} x^n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, x \longrightarrow 1 \quad s = 1, 2$$
$$(v^n)_{s,1} \leqslant \frac{1}{2} 7^{-n} (-1+2^n) x^n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, x \longrightarrow 1 \quad s = 3, 4, 5, 6$$

This establishes the first 6 coefficients of v^∞ are 0.

For the remaining coeffi-
cients:

$$(v^n)_{s,1} \leqslant \frac{1}{8} 7^{-n} (-1+7^n-2n) x^n \longrightarrow \frac{1}{8} \quad \text{as} \quad n \longrightarrow \infty, x \longrightarrow 1 \quad s = 7, 8, 9, 10$$
$$(v^n)_{s,1} \leqslant \frac{1}{8} 7^{-n} (-1+7^n-2n) x^n \longrightarrow \frac{1}{8} \quad \text{as} \quad n \longrightarrow \infty, x \longrightarrow 1 \quad s = 11, 12, 13, 14$$

This establishes that the coefficients 7-14 are generally non-zero with upper-bound of $\frac{1}{8}$.
By the same symbolic computations it is easy to show:

$$(v^n)_{s,1} - (v^n)_{s+4,1} \longrightarrow 0 \text{ as } \quad n \longrightarrow \infty, x \longrightarrow 1 \quad s = 7, 8, 9, 10$$

For example for $n = 16$:

$(v^{16})_{7,1} - (v^{16})_{11,1} = -g^{16} + 15\, g^{15}\, h - 105\, g^{14}\, h^2 + 455\, g^{13}\, h^3 - 1365\, g^{12}\, h^4 + 3003\, g^{11}\, h^5 + \cdots$

Has combined powers adding up to 16 (in general add up to n) and each coefficient being positive and less than 1 makes the difference $(v^n)_{7,1} - (v^n)_{11,1}$ tends to 0 as n increases.

In other words

$$
\begin{aligned}
(v^\infty)_{7,1} &= (v^\infty)_{11,1} = \alpha \\
(v^\infty)_{8,1} &= (v^\infty)_{12,1} = \beta \\
(v^\infty)_{9,1} &= (v^\infty)_{13,1} = \omega \\
(v^\infty)_{10,1} &= (v^\infty)_{14,1} = \eta
\end{aligned}
$$

(EQ 2.2) by direct computation establishes the two 1-valued eigenvalues, and being stochastic matrix columns should sum to 1 or for the first column of v^∞:

$$0 + 0 + 0 + 0 + 0 + 0 + \alpha + \beta + \omega + \eta + \alpha + \beta + \omega + \eta = 1$$

Therefore

$$\alpha + \beta + \omega + \eta = \frac{1}{2}.$$

if we replace $\eta = -(\alpha + \beta + \omega) + \frac{1}{2}$ and form equations for $v^\infty \cdot v^\infty = v^\infty$ there is a solution for ω :

$\omega = \frac{\alpha - 2\,\alpha^2 - 2\,\alpha\,\beta}{2\,(\alpha + \beta)}$. see [3]

□

Remark 2.3: *This result will not hold for negative coefficients, nor for non-stochastic matrices.*

Since v^∞ has all 0 rows:

Corollary 2.1: det $v^\infty = 0$ *(no assumption for the determinant of v).*

By direct symbolic calculation [3] performed on v^∞:

Corollary 2.1: *Given the first 6 non-zero coefficients, the characteristic polynomial of* v^∞ *is* $\lambda^{12}\,(\lambda - 1)^2$. *With the following list of eigenvalues and corresponding eigenvectors:*

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1}

$$\Big\{\Big\{0, 0, 0, 0, 0, 0, 0, -\frac{1-2\alpha-2\beta}{2(\alpha+\beta)}, 0, 0, 0, 0, 0, 1\Big\}, \Big\{0, 0, 0, 0, 0, 0, -\frac{1-2\alpha-2\beta}{2(\alpha+\beta)}, 0, 0, 0, 0, 0, 1, 0\Big\},$$
$$\{0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 1, 0, 0, 0\}, \Big\{0, 0, 0, 0, 0, 0, 0, -\frac{1-2\alpha-2\beta}{2(\alpha+\beta)}, 0, 1, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, 0, 0, 0, 0, -\frac{1-2\alpha-2\beta}{2(\alpha+\beta)}, 0, 1, 0, 0, 0, 0, 0\Big\}, \{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \Big\{\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1\Big\}, \Big\{\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0\Big\}\Big\}$$

Generally:

$$v^{\infty} \cdot v^{\infty} = v \cdot v^{\infty} = v^{\infty} \cdot v = (v^{\infty})^{n} = (v^{\infty})^{\infty} \; .$$

By direct symbolic computation and solving systems of equations [3]:

Theorem 2.3: If ${v_1}^{\infty} \neq {v_2}^{\infty} \Longleftrightarrow {v_1}^{\infty} \cdot {v_2}^{\infty} \neq {v_2}^{\infty} \cdot {v_1}^{\infty}$.

Therefore these infinite powers are governed by these set of rules:

$$\begin{cases} A \cdot B \neq B \cdot A & A = v^{\infty}, \;\; B = w^{\infty} \;\; , \; v \neq w \\ A^{n} = A & n = 1, \; 2, \; \cdots \; \infty \\ \prod_{\mu} A_{\mu} \neq 1 & \end{cases}$$

3. Pre-Geometry: Lie Algebra and Logarithm

Lie group of Stochastic matrices of dimension n or $M(n, \mathbb{R})$ is endowed with the Lie Algebra [4]:

$$L_{k,l}^{i,j} = \delta_{i,k}\, \delta_{j,l} - \delta_{j,k}\, \delta_{j,l} \qquad \text{(EQ 3.1)}$$

Or all members of $M(14, \mathbb{R})$ can be written as the exponential sum:

$$e^{\sum_{i,j=1}^{14} \alpha_{i,j} L^{i,j}} \qquad \alpha_{i,j} \in \mathbb{R} \text{ or } \mathbb{C} \qquad \text{(EQ 3.2)}$$

and so in specific all 14-Sets Stochastic vectors.

Therefore the Lie Algebra above in cooperation with the exponential map provides a pre-geometry for the topological operators and spaces they represent.

Lemma 3.1: *Sum of the columns of the Lie Algebra elements add up to 0.*

The dimension of this Lie Algebra is [4]:

$n^2 - n$ (EQ 3.3)

For the case n = 14, the dimension is $14^2 - 14 = 182$. Therefore this dimension is quite large in comparison to dimension n of $M(n, \mathbb{R})$ i.e. 14.

Corollary 3.1: *Dimension of Lie Algebra, or pre-geometry, for 14-Sets Stochastic vectors is 182.*

Example 3.1

The Closure Operator Logarithm is the 8 element linear combination while z approaches infinity, see [2] section 15:

$$\lim_{z\to\infty} e^{z L^{11,9}+z L^{12,10}+z L^{3,1}+z L^{4,2}+z L^{7,13}+z L^{7,5}+z L^{8,14}+z L^{8,6}} = \sigma_2$$

$$z L^{11,9} + z L^{12,10} + z L^{3,1} + z L^{4,2} + z L^{7,13} + z L^{7,5} + z L^{8,14} + z L^{8,6} = \begin{pmatrix}
-z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\
0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z
\end{pmatrix}$$

$$e^{z L^{11,9}+z L^{12,10}+z L^{3,1}+z L^{4,2}+z L^{7,13}+z L^{7,5}+z L^{8,14}+z L^{8,6}} =$$

$$\begin{pmatrix}
e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z}(-1+e^{z}) & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-z}
\end{pmatrix}$$

Example 3.2

We obtain an approximation of the Complement Operator's Logarithm in terms of a variable z and then take the limit of the z approaching i and thus obtain a Logarithm as an exact calculation, see [2] section 15:

$$\lim_{z\to i} e^{\frac{\left(z\pi L^{10,8}+z\pi L^{11,13}+z\pi L^{1,2}+z\pi L^{12,14}+z\pi L^{13,11}+z\pi L^{14,12}+z\pi L^{2,1}+z\pi L^{3,5}+z\pi L^{4,6}+z\pi L^{5,3}+z\pi L^{6,4}+z\pi L^{7,9}+z\pi L^{8,10}+z\pi L^{9,7}\right)}{2}} = \sigma_1$$

$$\frac{1}{2}\left(z\,\pi\,L^{10,8}+z\,\pi\,L^{11,13}+z\,\pi\,L^{1,2}+z\,\pi\,L^{12,14}+z\,\pi\,L^{13,11}+z\,\pi\,L^{14,12}+z\,\pi\,L^{2,1}+z\,\pi\,L^{3,5}+z\,\pi\,L^{4,6}+z\,\pi\,L^{5,3}+z\,\pi\,L^{6,4}+z\,\pi\,L^{7,9}+z\,\pi\,L^{8,10}+z\,\pi\,L^{9,7}\right)=$$

$$\begin{pmatrix}
-\frac{z\pi}{2} & \frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{z\pi}{2} & -\frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{z\pi}{2} & 0 & \frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{z\pi}{2} & 0 & \frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{z\pi}{2} & 0 & -\frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{z\pi}{2} & 0 & -\frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{z\pi}{2} & 0 & \frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{z\pi}{2} & 0 & \frac{z\pi}{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{z\pi}{2} & 0 & -\frac{z\pi}{2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{z\pi}{2} & 0 & -\frac{z\pi}{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{z\pi}{2} & 0 & \frac{z\pi}{2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{z\pi}{2} & 0 & \frac{z\pi}{2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{z\pi}{2} & 0 & -\frac{z\pi}{2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{z\pi}{2} & 0 & -\frac{z\pi}{2}
\end{pmatrix}$$

$$e^{\frac{\left(z\,\pi\,L^{10,8}+z\,\pi\,L^{11,13}+z\,\pi\,L^{1,2}+z\,\pi\,L^{12,14}+z\,\pi\,L^{13,11}+z\,\pi\,L^{14,12}+z\,\pi\,L^{2,1}+z\,\pi\,L^{3,5}+z\,\pi\,L^{4,6}+z\,\pi\,L^{5,3}+z\,\pi\,L^{6,4}+z\,\pi\,L^{7,9}+z\,\pi\,L^{8,10}+z\,\pi\,L^{9,7}\right)}{2}}=$$

$$\begin{pmatrix}
\frac{1+z^2}{2z^2} & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-1+z^2}{2z^2} & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+z^2}{2z^2} & 0 & \frac{-1+z^2}{2z^2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+z^2}{2z^2} & 0 & \frac{1+z^2}{2z^2}
\end{pmatrix}$$

Appendix A

We assume all strings and programs are binary coded.

Definition A.1: The Kolmogorov Complexity $C_{\mathcal{U}}(x)$ of a string x with respect to a universal computer (Turing Machine) $\mathcal{U}$ is defined as

$$C_{\mathcal{U}}(x) = \min_{p:\mathcal{U}(p)=x} l(p)$$

the minimum length program p in $\mathcal{U}$ which outputs x.

Theorem A.1 (Universality of the Kolmogorov Complexity): *If $\mathcal{U}$ is a universal computer, then for any other computer $\mathcal{A}$ and all strings x,*

$$C_{\mathcal{U}}(x) \leq C_{\mathcal{A}}(x) + c_{\mathcal{A}}$$

where the constant $c_{\mathcal{A}}$ does not depend on x.

Corollary A.1: $\lim_{l(x)\to\infty} \frac{C_{\mathcal{U}}(x) - C_{\mathcal{A}}(x)}{l(x)} = 0$ *for any two universal computers.*

Remark A.1: *Therefore we drop the universal computer subscript and simply write $C(x)$.*

Theorem A.2: *$C(x) \leqslant |x| + c$.*

A string x is called incompressible if $C(x) \geqslant x$.

Definition A.2: Self-delimiting string (or program) is a string or program which has its own length encoded as a part of itself i.e. a Turing machine reading Self-delimiting string knows exactly when to stop reading.

Definition A.3: The Conditional or Prefix Kolmogorov Complexity of self-delimiting string x given string y is

$$K(x \mid y) = \min_{p:\mathcal{U}(p,\, y)=x} l(p)$$

The length of the shortest program that can compute both x and y and a way to tell them apart is

$$K(x,\, y) = \min_{p:\mathcal{U}(p)=x,y} l(p)$$

Remark A.2: *x, y can be thought of as concatenation of the strings with additional separation information.*

Theorem A.3: $K(x) \leq l(x) + 2\log l(x) + O(1), \quad K(x \mid l(x)) \leq l(x) + O(1)$.

Theorem A.4: $K(x,\, y) \leq K(x) + K(y)$.

Theorem A.5: $K(f(x)) \leq K(x) + K(f)$, f *a* computble function

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